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EXISTENTIALLY CLOSED LOCALLY COFINITE GROUPS

by FELIX LEINEN

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Let $\mathfrak{X}$ be a class of finite groups. Then a $c\mathfrak{X}$-group shall be a topological group which has a fundamental system of open neighbourhoods of the identity consisting of normal subgroups with $\mathfrak{X}$-factor groups and trivial intersection. In this note we study groups which are existentially closed (e.c.) with respect to the class $Lc\mathfrak{X}$ of all direct limits of $c\mathfrak{X}$-groups (where $\mathfrak{X}$ satisfies certain closure properties). We show that the so-called locally closed normal subgroups of an e.c. $Lc\mathfrak{X}$-group are totally ordered via inclusion. Moreover it turns out that every $\forall_2$-sentence, which is true for countable e.c. $L\mathfrak{X}$-groups, also holds for e.c. $Lc\mathfrak{X}$-groups. This allows it to transfer many known properties from e.c. $L\mathfrak{X}$-groups to e.c. $Lc\mathfrak{X}$-groups.

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1. Introduction

In this paper, $\mathfrak{X}$ will always denote a class of finite groups, which is closed with respect to subgroups, homomorphic images, and extensions. In particular, we write $\mathfrak{F}$, $\mathfrak{F}_\pi$, $\mathfrak{F}_\pi \cap \mathfrak{G}$, $\mathfrak{F}_p$ for the classes of all finite groups, finite π-groups, finite soluble π-groups, and finite p-groups (resp.); here π is a fixed set of primes. A great amount of information has been obtained about existentially closed (e.c.) groups in the class $L\mathfrak{X}$ of all locally $\mathfrak{X}$-groups, especially in the cases when $\mathfrak{X}=\mathfrak{F}$ or $\mathfrak{X}=\mathfrak{F}_p$ (see [7], [**14**, § **6**], [**21**], [**15**], [**17**], [**18**]). It was the original purpose of the present note to use this knowledge in studying e.c. locally residually $\mathfrak{X}$-groups (*$LR\mathfrak{X}$-groups*). Here we cannot expect results as nice as in the $L\mathfrak{X}$-case. The reason is that there exist $2^{\aleph_0}$ finitely generated (f.g.) $R\mathfrak{F}_p$-groups ([**5**], [**6**]), hence also $2^{\aleph_0}$ countable e.c. $LR\mathfrak{X}$-groups (see also Example 4.5), while we have unique countable e.c. objects in $L\mathfrak{F}$ and in $L\mathfrak{F}_p$. However, the close connection between the classes $L\mathfrak{X}$ and $LR\mathfrak{X}$ is demonstrated by the fact, that there exist countable e.c. $L\mathfrak{X}$-groups, which are e.c. in $LR\mathfrak{X}$ (use the argument of [**9**, Satz 3.5]). If $\mathfrak{X}=\mathfrak{F}$ or $\mathfrak{X}=\mathfrak{F}_p$, then every e.c. $L\mathfrak{X}$-group is e.c. in $LR\mathfrak{X}$ (copy the proofs of [**21**, Satz 6] and [**17**, Theorem 3.7]).

Unfortunately, the study of e.c. $LR\mathfrak{X}$-groups is considerably complicated by the fact, that it seems to be very hard to find general constructions in order to produce sufficiently interesting $LR\mathfrak{X}$-supergroups of given $LR\mathfrak{X}$-groups. Here we are only able to use some ad hoc arguments. For example, HNN-extensions and free products of $LR\mathfrak{F}$-groups stay in $LR\mathfrak{F}$, whenever the involved isomorphic subgroups are finite (see [**1**], [**2**]). The information obtained in this way is fairly weak.

Because of this situation, we put more emphasis on the profinite topologies living on each $R\mathfrak{X}$-group. This leads to the following refinement or our considerations. We replace

$R\mathfrak{X}$ by the class $c\mathfrak{X}$ of all co-$\mathfrak{X}$-groups in the sense of [**8**]. These arise as follows. Let $U \in R\mathfrak{X}$. Suppose that $\mathscr{R}$ is a *residual system* in U, i.e., a set of normal subgroups in U satisfying

(1) $U/N \in \mathfrak{X}$ for all $N \in \mathscr{R}$,

(2) for any $N_1, N_2 \in \mathscr{R}$ there exists $N_3 \in \mathscr{R}$ such that $N_3 \leqq N_1 \cap N_2$, and

(3) $\bigcap \mathscr{R} = 1$.

Then U becomes a Hausdorff topological space by requiring that $\mathscr{R}$ be a basis of neighbourhoods of the identity (the reader is referred to [**11**] or [**10**, §2] for the standard facts about topological groups). The topological group U is now called a $c\mathfrak{X}$-group. Thus, a $c\mathfrak{X}$-group is just an $R\mathfrak{X}$-group equipped with a certain fixed pro-$\mathfrak{X}$ topology. In general, different residual systems on the same $R\mathfrak{X}$-group lead to different $c\mathfrak{X}$-groups.

If $U \leqq V \in c\mathfrak{X}$, then U is a $c\mathfrak{X}$-group via the topology induced by V. For this reason, an *embedding* $\phi: U \to V$ of $c\mathfrak{X}$-groups shall be a group homeomorphism ϕ of U onto the $c\mathfrak{X}$-subgroup $U\phi$ of V. Note that embeddings are always continuous. In order to prove that a monomorphism $\phi: U \to V$ of $c\mathfrak{X}$-groups is an embedding, it suffices to find residual systems $\mathscr{R}_U$ and $\mathscr{R}_V$ which give the topologies on U resp. V such that, for every $N \in \mathscr{R}_U$ there exists some $M \in \mathscr{R}_V$ with $M \cap U\phi \leqq N\phi$, and such that for every $M \in \mathscr{R}_V$ there exists some $N \in \mathscr{R}_U$ with $N\phi \leqq M \cap U\phi$. Since $\mathfrak{X}$-groups can only carry the discrete pro-$\mathfrak{X}$ topology, every monomorphism $\phi: U \to V$, where $U \in \mathfrak{X}$ and $V \in c\mathfrak{X}$, is an embedding. An $Lc\mathfrak{X}$-group G is a direct limit of $c\mathfrak{X}$-groups with respect to embeddings. Note that G is in general not a topological group with respect to the direct limit topology, since multiplication in G need not be continuous [**3**, Appendix 2, I.9]. However, with each f.g. subgroup of G there is associated a unique topology. An *embedding* $\phi: G \to H$ of $Lc\mathfrak{X}$-groups shall be a group monomorphism ϕ such that, for each f.g. subgroup U of G, the restriction $\phi | U$ is a homeomorphism of U onto $U\phi$. Again, monomorphisms $\phi: G \to H$, where $G \in L\mathfrak{X}$ and $H \in Lc\mathfrak{X}$, are embeddings.

By the standard argument [**13**, Proposition 1.3], every $Lc\mathfrak{X}$-group G is embeddable into an e.c. $Lc\mathfrak{X}$-group of cardinality $\max\{\aleph_0, |G|\}$. As before, there exist countable e.c. $L\mathfrak{X}$-groups which are e.c. in $Lc\mathfrak{X}$. If $\mathfrak{X} = \mathfrak{F}$ or $\mathfrak{X} = \mathfrak{F}_p$, then every e.c. $L\mathfrak{X}$-group is e.c. in $Lc\mathfrak{X}$.

Clearly, the class $p\mathfrak{X}$ of all pro-$\mathfrak{X}$-groups is a subclass of $c\mathfrak{X}$. Note that continuous monomorphisms of $p\mathfrak{X}$-groups are already embeddings [**11**, Proposition 10($\mathscr{T}$)]. On the other hand, every $c\mathfrak{X}$-group U is a subgroup of its pro-$\mathfrak{X}$ completion $\hat{U}$ [**8**, Lemmata 2.5/6]. Let $\mathscr{L}$ be the local system of all f.g. subgroups in the $Lc\mathfrak{X}$-group G. If $X, Y \in \mathscr{L}$ satisfy $X \leqq Y$, then there exists an embedding α of $\hat{X}$ into $\hat{Y}$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{id} & Y \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ \hat{X} & \xrightarrow{\alpha} & \hat{Y} \end{array}$$

commutes [**8**, Theorem 2.1]. It follows, that G is embeddable into its $Lp\mathfrak{X}$-completion $\hat{G}=\varinjlim\{\hat{X}\mid X\in\mathscr{L}\}$. Consequently, every e.c. $Lp\mathfrak{X}$-group is e.c. in $Lc\mathfrak{X}$, and the study of e.c. $Lc\mathfrak{X}$-groups comprises the study of e.c. $Lp\mathfrak{X}$-groups. Note also, that every e.c. $Lc\mathfrak{X}$-group is e.c. in its $Lp\mathfrak{X}$-completion.

Our basic amalgamation technique within $Lc\mathfrak{X}$ uses suitable factor groups of free products with amalgamation. A combination with embeddings of $c\mathfrak{X}$-groups into cartesian products of $\mathfrak{X}$-groups then reduces the problem of solving finite systems of equations and inequalities over an e.c. $Lc\mathfrak{X}$-group G to solving them over certain $\mathfrak{X}$-sections of f.g. subgroups of G. As a valuable corollary we note, that every $\forall_2$-sentence, which holds in every countable e.c. $L\mathfrak{X}$-group, is also true in every e.c. $Lc\mathfrak{X}$-group.

From this, we can immediately carry over a lot of information from e.c. $L\mathfrak{X}$-groups to e.c. $Lc\mathfrak{X}$-groups. For example, it follows that every e.c. $Lc\mathfrak{F}$-group G is simple. And isomorphisms between finite subgroups of G are always induced by conjugation in G, while there can exist an element $g\in G$ of infinite order such that g is conjugate to g^n ($n\in\mathbb{Z}$) if and only if $n=\pm 1$ (see Section 6).

In the general case, we can carry over elementary properties like verbal completeness and triviality of centralizers of non-trivial normal subgroups. Also, unions and intersections of the so-called locally closed normal subgroups (see Section 3) of an e.c. $Lc\mathfrak{X}$-group G are totally ordered via inclusion. However, it remains open whether every normal subgroup of G is such a union or intersection. In Section 4 we construct examples which show, that the $L\mathfrak{F}$-residual of G can be a proper subgroup. The $L\mathfrak{F}$-radical and the factor group modulo the $L\mathfrak{F}$-residual can be treated by methods used for e.c. $L\mathfrak{X}$-groups (see Section 3).

The theory becomes much more satisfactory in the case when $\mathfrak{X}=\mathfrak{F}_p$ (see Section 5). Here we can use our corollary about $\forall_2$-sentences directly to show, that every e.c. $Lc\mathfrak{F}_p$-group has a unique chief series Σ, that the factors of Σ are central and cyclic of order p, and that the order type of Σ is a dense order without endpoints. If $K\trianglelefteq G$ satisfies $K\neq\langle g^G\rangle$ for all $g\in G$, then K is e.c. in G. However, it seems to be unlikely that such a K is in general e.c. in $Lc\mathfrak{F}_p$ (as one might expect from the theory of e.c. $L\mathfrak{F}_p$-groups). We will show, that every subnormal subgroup of G is already normal in G. Also, the results about embeddings of $L\mathfrak{F}_p$-groups into e.c. $L\mathfrak{F}_p$-groups [**15**, §3] and about conjugacy of finite subgroups of e.c. $L\mathfrak{F}_p$-groups [**17**, Theorem 6.1] remain true for e.c. $Lc\mathfrak{F}_p$-groups. As in the $Lc\mathfrak{F}$-case, conjugacy of infinite f.g. subgroups of G is more delicate. Although we can give a quite satisfactory necessary and sufficient condition (which shows for example, that an element $g\in G$ of infinite order is conjugate to g^n ($n\in\mathbb{Z}$) if and only if $n\equiv 1 \pmod p$), it remains open whether an isomorphism $\phi: A\to B$ between f.g. subgroups of G is induced by conjugation in G if and only if $a^{-1}\cdot a\phi\in N$ for all $a\in(A\cap M)-N$ and all chief factors M/N of G.

2. Constructions within Lc$\mathfrak{X}$

In the sequel, if $U\in c\mathfrak{X}$, then $N\trianglelefteq_0 U$ will denote that N is an open normal subgroup of U. The following amalgamation theorem is the foundation of this note.

Theorem 2.1. *An amalgam $G \cup H \mid U$ of $Lc\mathfrak{X}$-groups over the f.g. common subgroup U is embeddable into an $Lc\mathfrak{X}$-group if and only if there exist local systems $\mathscr{L}_G$ in G and $\mathscr{L}_H$ in H consisting of f.g. subgroups which contain U such that, for every pair $(X, Y) \in \mathscr{L}_G \times \mathscr{L}_H$, the topologies on X and Y are given by residual systems $\mathscr{R}_{X,Y}$ in X resp. $\mathscr{R}_{Y,X}$ in Y such that, for every $L \in \mathscr{R}_{X,Y}$ (resp. $L \in \mathscr{R}_{Y,X}$), there exist $M \in \mathscr{R}_{X,Y}$ and $N \in \mathscr{R}_{Y,X}$ satisfying*

(1) *$M \leqq L$ (resp. $N \leqq L$) and $M \cap U = L \cap U = N \cap U$, and*

(2) *the amalgam $X/M \cup Y/N \mid UL/L$ (where UM/M and UN/N are identified with UL/L via $uM \equiv uL \equiv uN$ for all $u \in U$) is contained in an $\mathfrak{X}$-group.*

Proof. Suppose first that the amalgam is contained in an $Lc\mathfrak{X}$-group W. Let $\mathscr{L}_W$, $\mathscr{L}_G$ and $\mathscr{L}_H$ be the local systems of all f.g. subgroups in W, G resp. H containing U. For every $V \in \mathscr{L}_W$, denote by $\mathscr{R}_V$ the residual system of all $L \trianglelefteq_0 V$. If $(X, Y) \in \mathscr{L}_G \times \mathscr{L}_H$, then let $\mathscr{R}_{X,Y} = \mathscr{R}_{\langle X,Y \rangle} \cap X$ and $\mathscr{R}_{Y,X} = \mathscr{R}_{\langle X,Y \rangle} \cap Y$. Since X and Y are $c\mathfrak{X}$-subgroups of $\langle X, Y \rangle$, the topologies on X and Y are given by the residual systems $\mathscr{R}_{X,Y}$ resp. $\mathscr{R}_{Y,X}$. Now every $M \in \mathscr{R}_{X,Y}$ (resp. $N \in \mathscr{R}_{Y,X}$) is induced by some $K \in \mathscr{R}_{\langle X,Y \rangle}$. Put $N = K \cap Y \in \mathscr{R}_{Y,X}$ (resp. $M = K \cap X \in \mathscr{R}_{X,Y}$). Then $M \cap U = N \cap U$, and the amalgam $X/M \cup Y/N \mid UM/M \equiv UN/N$ is embedded canonically in the $\mathfrak{X}$-group $\langle X, Y \rangle / K$.

Conversely, fix some $(X, Y) \in \mathscr{L}_G \times \mathscr{L}_H$ and regard the free product with amalgamation $F_{X,Y} = X *_U Y$. Let

$$\mathscr{R}^*_{X,Y} = \{K \trianglelefteq F_{X,Y} \mid F_{X,Y}/K \in \mathfrak{X}, \quad K \cap X \trianglelefteq_0 X, \quad K \cap Y \trianglelefteq_0 Y\}.$$

Denote epimorphic images modulo $\bigcap \mathscr{R}^*_{X,Y}$ by bars. Then $\bar{F}_{X,Y} \in c\mathfrak{X}$ with the topology given by the residual system $\bar{\mathscr{R}}^*_{X,Y}$. We will show:

An embedding of the amalgam $X \cup Y \mid U$ into $\bar{F}_{X,Y}$ is given by $z \to \bar{z}$ for all $z \in X \cup Y$. (2.1)

If $X \leqq V \in \mathscr{L}_G$ and $Y \leqq W \in \mathscr{L}_H$, then $\mathrm{id}: F_{X,Y} \to F_{V,W}$ induces an embedding of $\bar{F}_{X,Y}$ into $\bar{F}_{V,W}$. (2.2)

Once (2.1) and (2.2) are proved, the discovered embeddings form commuting diagrams

$$\begin{array}{ccc} X \cup Y \mid U & \xrightarrow{id} & V \cup W \mid U \\ \downarrow & & \downarrow \\ \bar{F}_{X,Y} & \longrightarrow & \bar{F}_{V,W} \end{array}$$

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and therefore, the amalgam $G \cup H \mid U$ is embeddable into the direct limit of the $c\mathfrak{X}$-groups $\bar{F}_{X,Y}$ with respect to the embeddings given in (2.2).

To see (2.1), suppose that $1 \neq x \in X$. Then $x \notin L$ for some $L \in \mathscr{R}_{X,Y}$. By hypothesis, there exist $M \in \mathscr{R}_{X,Y}$ and $N \in \mathscr{R}_{Y,X}$ with $M \leq L$ and $M \cap U = L \cap U = N \cap U$, and such that the amalgam $X/M \cup Y/N \mid UL/L$ is contained in an $\mathfrak{X}$-group P. Let K be the kernel of the canonical homomorphism $F_{X,Y} \to P$. Clearly, $K \trianglelefteq F_{X,Y}$ with $\mathfrak{X}$-factor group such that $K \cap X = M \trianglelefteq_0 X$ and $K \cap Y = N \trianglelefteq_0 Y$. Now $K \in \mathscr{R}^*_{X,Y}$ and $x \notin K$. It follows that $X \cap [\bigcap \mathscr{R}^*_{X,Y}] = 1$, and so the canonical map $\phi: X \to \bar{X}$ is a group isomorphism. By definition of $\mathscr{R}^*_{X,Y}$, the map ϕ is continuous. The above argument also shows that we can find for every $L \in \mathscr{R}_{X,Y}$ some $K \in \mathscr{R}^*_{X,Y}$ with $K \cap X \leq L$, whence ϕ is open.

In order to prove (2.2), it suffices to show:

$$\check{K} \cap F_{X,Y} \in \mathscr{R}^*_{X,Y} \quad \text{for every } \check{K} \in \mathscr{R}^*_{V,W}. \tag{2.3}$$

$$\text{For every } K \in \mathscr{R}^*_{X,Y} \text{ there exists } \check{K} \in \mathscr{R}^*_{V,W} \text{ such that } \check{K} \cap F_{X,Y} \leq K. \tag{2.4}$$

The assertion (2.3) is an immediate consequence of the definition of $\mathscr{R}^*_{X,Y}$ and $\mathscr{R}^*_{V,W}$, and of the fact that the topology on X resp. Y is induced by the topology on V resp. W. It remains to prove (2.4).

Let $K \in \mathscr{R}^*_{X,Y}$. Since $\mathscr{R}_{V,W} \cap X$ and $\mathscr{R}_{W,V} \cap Y$ are residual systems in X resp. Y which induce the topology on X resp. Y, there exist $L_1 \in \mathscr{R}_{V,W}$ and $L_2 \in \mathscr{R}_{W,V}$ such that $L_1 \cap X \leq K \cap X$ and $L_2 \cap Y \leq K \cap Y$. From hypothesis, we obtain $M_i \in \mathscr{R}_{V,W}$ and $N_i \in \mathscr{R}_{W,V}$ such that $M_1 \leq L_1$, $N_2 \leq L_2$, and $M_i \cap U = L_i \cap U = N_i \cap U$, and such that the amalgam $V/M_i \cup W/N_i \mid UL_i/L_i$ is contained in an $\mathfrak{X}$-group P_i. Plainly, $M \cap U = N \cap U$ for $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Denote epimorphic images modulo M resp. N by $\tilde{\ }$. Then an embedding α of the amalgam $\tilde{V} \cup \tilde{W} \mid \tilde{U}$ into $P_3 = P_1 \times P_2 \in \mathfrak{X}$ is given by $\tilde{v}\alpha = (vM_1, vM_2)$ for all $v \in V$ and $\tilde{w}\alpha = (wN_1, wN_2)$ for all $w \in W$. Put $Q = F_{X,Y}/K$, and regard $\tilde{X} \cup \tilde{Y} \mid \tilde{U}$ as an amalgam in $P_4 = Q \times P_3$ via $\tilde{z} \equiv (zK, \tilde{z}\alpha)$ for all $z \in X \cup Y$. We will now embed the amalgam $\tilde{V} \cup P_4 \cup \tilde{W}$, where $\tilde{V} \cap P_4 = \tilde{X}$ and $P_4 \cap \tilde{W} = \tilde{Y}$, into an $\mathfrak{X}$-group.

To this end, let $P_5 = Q \operatorname{Wr} P_3 \in \mathfrak{X}$, and denote by $\pi_1: P_4 \to P_3$ and $\pi_2: P_5 \to P_3$ the canonical projections. Because of $\tilde{X} \cap Q = 1$, we obtain from [**12**, Lemma 1] an embedding $\sigma: P_4 \to P_5$ satisfying $\sigma\pi_2 = \pi_1$ and $\sigma|_{\tilde{X}} = \pi_1|_{\tilde{X}} = \alpha|_{\tilde{X}}$. Lift σ to an embedding of the amalgam $\tilde{V} \cup P_4 \mid \tilde{X}$ into P_5 via $\sigma|_{\tilde{V}} = \alpha|_{\tilde{V}}$. Regard the amalgam $P_5 \cup \tilde{W}\sigma \mid \tilde{Y}\sigma$, where $\tilde{W}\sigma$ is an artificial copy of $\tilde{W}$. Denote the base group of P_5 by Ω, and put $P_6 = \Omega \operatorname{Wr} P_3 \in \mathfrak{X}$. Since $\tilde{Y}\sigma \cap \Omega = 1$, a further application of [**12**, Lemma 1] leads to an embedding $\tau: P_5 \to P_6$ such that $\sigma\tau|_{\tilde{Y}} = \sigma\pi_2|_{\tilde{Y}} = \pi_1|_{\tilde{Y}} = \alpha|_{\tilde{Y}}$. Extend τ to an embedding of the amalgam $P_5 \cup \tilde{W}\sigma \mid \tilde{Y}\sigma$ into P_6 via $\sigma\tau|_{\tilde{W}} = \alpha|_{\tilde{W}}$. Now $\sigma\tau$ embeds the amalgam $\tilde{V} \cup P_4 \cup \tilde{W}$ into the $\mathfrak{X}$-group P_6.

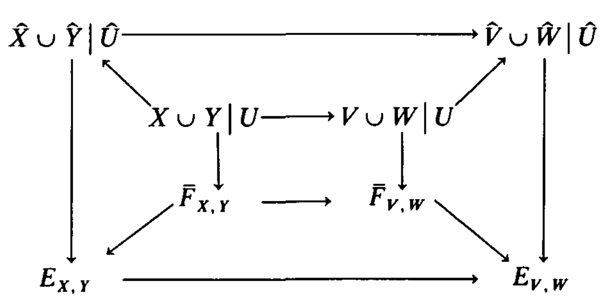
Let $\check{K}$ be the kernel of the group homomorphism $F_{V,W} \to P_6$ induced from $\tilde{\ }\sigma\tau$. Then $\check{K} \cap V = M$ and $\check{K} \cap W = N$, whence $\check{K} \in \mathscr{R}^*_{V,W}$. Moreover, by choice of P_4, we have $\check{K} \cap F_{X,Y} \leq K$. □

Remark 2.2. *Under the assumptions of Theorem 2.1, the amalgam* $G \cup H \mid U$ *is actually contained in an* $Lc\mathfrak{X}$*-group* W *with the following property: For all f.g.* $X \leqq G$ *and* $Y \leqq H$ *containing* U, *if* $M \trianglelefteq_0 X$ *and* $N \trianglelefteq_0 Y$ *are such that* $M \cap U = N \cap U$ *and such that the amalgam* $X/M \cup Y/N \mid UM/M \equiv UN/N$ *is contained in an* $\mathfrak{X}$*-group, then there exists* $K \trianglelefteq_0 \langle X, Y\rangle \leqq W$ *such that* $K \cap X = M$ *and* $K \cap Y = N$.

Proof. Since the amalgam is contained in an $Lc\mathfrak{X}$-group, we may assume that $\mathscr{L}_G$ and $\mathscr{L}_H$ in Theorem 2.1 are the local systems of all f.g. subgroups in G resp. H which contain U. For $(X, Y) \in \mathscr{L}_G \times \mathscr{L}_H$, define $F_{X,Y}$ and $\mathscr{R}^*_{X,Y}$ as in the proof of Theorem 2.1. If $M \trianglelefteq_0 X$ and $N \trianglelefteq_0 Y$ satisfy the above assumptions, then the argument in the proof of (2.1) shows that there exists $K \in \mathscr{R}^*_{X,Y}$ with $K \cap X = M$ and $K \cap Y = N$. □

Amalgams of $\mathfrak{F}$-groups are always contained in an $\mathfrak{F}$-group [22, Theorem 5.2]. Thus, if $\mathfrak{X} = \mathfrak{F}$, then the condition (2) in Theorem 2.1 becomes redundant. An amalgam $A \cup B \mid U$ of $\mathfrak{F}_p$-groups is contained in an $\mathfrak{F}_p$-group, if and only if there exist chief series in A and B which both induce the same chief series in U (see [12]). In particular, amalgams of $\mathfrak{F}_p$-groups over a common cyclic subgroup are always contained in an $\mathfrak{F}_p$-group. Moreover, every cyclic group U carries a unique pro-$\mathfrak{F}_p$ topology, and if $U \leqq V \in c\mathfrak{F}_p$, then the residual system of all $M \trianglelefteq_0 V$ induces the residual system of all $N \trianglelefteq_0 U$ in U. Therefore, Theorem 2.1 yields that every amalgam of $Lc\mathfrak{F}_p$-groups over a common cyclic subgroup is embeddable into an $Lc\mathfrak{F}_p$-group.

In the case when $\mathfrak{X} = \mathfrak{F}_p$, it is readily verified that the conditions in Theorem 2.1 are equivalent to the property that, for each pair $(X, Y) \in \mathscr{L}_G \times \mathscr{L}_H$, the topologies on X and Y are given by descending chains $\mathscr{M} = \{M_\alpha \mid \alpha \in \omega\}$ and $\mathscr{N} = \{N_\alpha \mid \alpha \in \omega\}$ of open normal subgroups $M_\alpha \trianglelefteq X$, $N_\alpha \trianglelefteq Y$ of index $\leqq p^\alpha$ such that $\bigcap \mathscr{M} = 1 = \bigcap \mathscr{N}$ and $\mathscr{M} \cap U = \mathscr{N} \cap U$. Therefore our criteria for amalgamation within $Lc\mathfrak{F}$ resp. $Lc\mathfrak{F}_p$ are in line with the criteria given in [23, Theorems 1.2 and 3.1] for the existence of the pro-$\mathfrak{F}$ resp. pro-$\mathfrak{F}_p$ amalgamated product. In fact, it follows from [8, Theorem 2.1], that the pro-$\mathfrak{X}$ completion $E_{X,Y}$ of the group $\bar{F}_{X,Y}$ in the proof of Theorem 2.1 is the pro-$\mathfrak{X}$ amalgamated product of $\hat{X} \cup \hat{Y} \mid \hat{U}$. In particular, the conditions in Theorem 2.1 ensure the existence of the pro-$\mathfrak{X}$ amalgamated product of $\hat{X} \cup \hat{Y} \mid \hat{U}$. Moreover, (2.1)/(2.2) and [8, Theorem 2.1] lead to commuting diagrams



of the canonical embeddings, and so $\hat{G} \cup \hat{H} \mid \hat{U}$ is embedded canonically in $\varinjlim E_{X,Y}$.

Theorem 2.1 allows us to deduce a necessary and sufficient criterion for the solvability of finite systems of equations and inequalities in e.c. $Lc\mathfrak{X}$-groups.

Theorem 2.3. *A finite system of equations and inequalities with coefficients $c_1,\dots,c_r$ in the e.c. $Lc\mathfrak{X}$-group G has a solution in G if and only if there exists a local system $\mathscr{L}$ in G consisting of f.g. subgroups which contain $U=\langle c_1,\dots,c_r\rangle$ such that, for every $X\in\mathscr{L}$, the topology on X is given by a residual system $\mathscr{R}_X$ such that, for every $M\in\mathscr{R}_X$, the system $\mathscr{S}/M$ (with coefficients $c_1M,\dots,c_rM$) has a solution in some $\mathfrak{X}$-group $W_{X,M}\geqq X/M$.*

Proof. Suppose that $g_1,\dots,g_s$ is a solution to $\mathscr{S}$ in G. Let $\mathscr{L}$ be the local system of all f.g. subgroups of G which contain $V=\langle U,g_1,\dots,g_s\rangle$. For $X\in\mathscr{L}$, denote by $\mathscr{R}_X$ the residual system of all $M\trianglelefteqq_0 X$ such that $w(c_1,\dots,c_r,g_1,\dots,g_s)\notin M$ for every inequality $w(c_1,\dots,c_r,x_1,\dots,x_s)\neq 1$ in $\mathscr{S}$. Clearly, if $M\in\mathscr{R}_X$. then $\mathscr{S}/M$ has the solution $g_1M,\dots,g_sM$ in $W_{X,M}=X/M$.

Conversely, let $\mathscr{M}=\{(X,M)\mid M\in\mathscr{R}_X, X\in\mathscr{L}\}$. Regard $H=\prod\{W_{X,M}\mid (X,M)\in\mathscr{M}\}$ as a $c\mathfrak{X}$-group under the product topology (where each $W_{X,M}$ carries the discrete topology). Denote by $K_{X,M}$ the obvious direct complement to $W_{X,M}$ in H. An embedding $\phi:U\to H$ is given by

$$u\phi=(uM)_{(X,M)\in\mathscr{M}}\quad\text{for all } u\in U,$$

since $K_{X,M}\cap U\phi=(M\cap U)\phi$ for all $(X,M)\in\mathscr{M}$. In the following, we suppress ϕ and regard U as a subgroup of H. Because H contains the componentwise solution to $\mathscr{S}$, and because G is e.c. in $Lc\mathfrak{X}$, it suffices to embed the amalgam $G\cup H\mid U$ into an $Lc\mathfrak{X}$-group. To this end, we will check the conditions of Theorem 2.1.

Put $\mathscr{L}_G=\mathscr{L}$, and let $\mathscr{L}_H$ be the local system in H of all f.g. subgroups containing U. For $(X,Y)\in\mathscr{L}_G\times\mathscr{L}_H$, choose $\mathscr{R}_{X,Y}=\mathscr{R}_X$ and $\mathscr{R}_{Y,X}=\{N\trianglelefteqq_0 Y\mid N\cap U\in\mathscr{R}_X\cap U\}$. If $L\trianglelefteqq_0 Y$, then there exists $M\in\mathscr{R}_X$ such that $M\cap U\leqq L\cap U$, whence $N=L\cap K_{X,M}\in\mathscr{R}_{Y,X}$ satisfies $N\leqq L$. This shows that $\mathscr{R}_{Y,X}$ is a residual system in Y which gives the topology on Y.

If $M\in\mathscr{R}_{X,Y}$, then $N=K_{X,M}\cap Y\in\mathscr{R}_{Y,X}$ satisfies $M\cap U=N\cap U$, and the amalgam $X/M\cup Y/N\mid UM/M\equiv UN/N$ can be embedded into $W_{X,M}$ via $xM\to xM$ for all $x\in X$ and $yK_{X,M}\to y_{X,M}$ for all $y\in Y$ (where $y_{X,M}$ denotes the component of $y\in H$ in $W_{X,M}$). Finally, regard some $L\in\mathscr{R}_{Y,X}$. Then $M\cap U=L\cap U$ for some $M\in\mathscr{R}_{X,Y}$, and $N=K_{X,M}\cap L\in\mathscr{R}_{Y,X}$ satisfies $N\leqq L$ and $L\cap U=N\cap U$. Denote epimorphic images modulo M resp. N by bars. It remains to embed the amalgam $\bar{X}\cup\bar{Y}\mid\bar{U}$ into an $\mathfrak{X}$-group. Identify $\bar{Y}$ with a subgroup of $P_1=W_{X,M}\times Y/L\in\mathfrak{X}$ via $\bar{y}\equiv(y_{X,M},yL)$ for all $y\in Y$. Since $\bar{U}\cap Y/L=1$, we obtain from [12, Lemma 1] an embedding $\sigma:P_1\to P_2=Y/L\operatorname{Wr}W_{X,M}\in\mathfrak{X}$ satisfying $\bar{u}\sigma=\bar{u}\in W_{X,M}$ for all $u\in U$. Extend σ to an embedding of the amalgam $\bar{X}\cup P_1\mid\bar{U}$ into P_2 via $\bar{x}\sigma=\bar{x}\in W_{X,M}$ for all $x\in X$. Then P_2 embeds in particular the amalgam $\bar{X}\cup\bar{Y}\mid\bar{U}$. □

Lemma 2.4. *Suppose that $\mathscr{R}=\mathscr{R}_1\cup\mathscr{R}_2$ is a residual system in the $R\mathfrak{X}$-group U. Then $\mathscr{R}_1$ or $\mathscr{R}_2$ is a residual system in U which gives the same topology on U as $\mathscr{R}$.*

Proof. It suffices to show that one of the sets $\mathscr{R}_1$ and $\mathscr{R}_2$ contains for every $N\in\mathscr{R}$ some M such that $M\leqq N$. Assume that there exists $N\in\mathscr{R}_1$ such that no $M\in\mathscr{R}_2$ is contained in N. Let $\tilde{\mathscr{R}}=\{M\in\mathscr{R}\,|\,M\leqq N\}$. Clearly $\tilde{\mathscr{R}}\subseteq\mathscr{R}_1$, and thus $\mathscr{R}_1$ has the desired property. □

Using the method of construction given in [**9**, Satz 3.5], every $\mathfrak{X}$-group can be embedded into a countable e.c. $L\mathfrak{X}$-group, which is e.c. in $Lc\mathfrak{X}$. It is therefore possible to formulate the following corollary.

Corollary 2.5. *The $\forall_2$-sentences, which hold in every e.c. $Lc\mathfrak{X}$-group, are precisely the $\forall_2$-sentences, which hold in those countable e.c. $L\mathfrak{X}$-group, which are e.c. in $Lc\mathfrak{X}$.*

Note that the classes $L\mathfrak{F}$ and $L\mathfrak{F}_p$ contain a unique countable e.c. group.

Proof. Let ψ be an $\forall_2$-sentence, i.e., let $\psi=\forall\bar{x}(\phi_1(\bar{x})\vee\cdots\vee\phi_n(\bar{x}))$ where $\phi_i(\bar{x})$ is a primitive formula for every i. Suppose that one of the countable e.c. $L\mathfrak{X}$-group E, which are e.c. in $Lc\mathfrak{X}$, satisfies ψ. Regard any e.c. $Lc\mathfrak{X}$-group G and some $\bar{c}$ from G. Denote by $\mathscr{L}$ the local system of all f.g. subgroups of G containing $\bar{c}$.

Fix some $X\in\mathscr{L}$. Let $\mathscr{R}_i$ be the set of all $M\trianglelefteq_0 X$ such that $\phi_i(\overline{cM})$ holds in some $\mathfrak{X}$-group containing X/M. If $M\trianglelefteq_0 X$, then $X/M\times E\in LcX$, and so the group E contains a copy of X/M. Since $E\models\psi$, it follows that M lies in some $\mathscr{R}_i$. Hence Lemma 2.4 yields that one of the $\mathscr{R}_i$ is a residual system in X which induces the topology on X.

This shows that $\mathscr{L}=\mathscr{L}_1\cup\cdots\cup\mathscr{L}_n$ where $\mathscr{L}_i$ is the set of all $X\in\mathscr{L}$ in which the topology is given by a residual system $\mathscr{R}_X$ such that, for every $M\in\mathscr{R}_X$, there exists an $\mathfrak{X}$-group $W\geqq X/M$ with $W\models\phi_i(\overline{\mathrm{cM}})$. By [**14**, Lemma 1.A.10], one of the $\mathscr{L}_i$ is a local system in G. Now Theorem 2.3 applied to this local system yields that $G\models\phi_i(\bar{c})$. This shows that $G\models\psi$. □

As a second embedding technique within $Lc\mathfrak{X}$, we will adopt the construction of [**18**, §2] as far as possible.

Construction 2.6. *Let $\bar{\ }:G\to H$ be a homomorphism with kernel N. For every f.g. $X\leqq G$, let $W_X=X\,\mathrm{Wr}\,H$ (unrestricted wreath product). Fix some f.g. $U\leqq G$, and choose left transversals R of $U\cap N$ in U, and T of $\bar{U}$ in H. Then a group monomorphism*

$$\tau:G\longrightarrow W=\bigcup\{W_X\,|\,X\leqq G\ f.g.\}$$

is given by $g\tau=f_g\cdot\bar{g}$, where the function $f_g:H\to\langle g,U\rangle$ is defined via $(t_1\bar{r}_1)f_g=r_1gr_2^{-1}$ whenever $t_i\in T$, $r_i\in R$ satisfy $t_2\bar{r}_2=t_1\bar{r}_1\bar{g}$.

Theorem 2.7. *Adopt the notation of Construction 2.6. Let $\mathscr{L}_G$ and $\mathscr{L}_H$ be the local systems of all f.g. subgroups in G resp. H. Suppose that $G\in Lc\mathfrak{X}$, $\mathrm{H}\in L\mathfrak{X}$, and that*

$N \cap X \trianglelefteq_0 X$ for all $X \in \mathscr{L}_G$. Regard the base group Ω_X of W_X as a c$\mathfrak{X}$-group under the product topology. The groups $W_{X,Y} = \Omega_X \times Y$, where $(X, Y) \in \mathscr{L}_G \rtimes \mathscr{L}_H$, form a local system in W. Each $W_{X,Y}$ is a c$\mathfrak{X}$-group under the topology given by the residual system

$$\mathscr{R}_{X,Y} = \{core_{W_{X,Y}}(K) \mid K \trianglelefteq_0 \Omega_X\} = \{L \trianglelefteq_0 \Omega_X \mid L \trianglelefteq W_{X,Y}\}.$$

With these topologies, $\tau: G \to W$ becomes an embedding of Lc$\mathfrak{X}$-groups.

Proof. If $(X_i, Y_i) \in \mathscr{L}_G \times \mathscr{L}_H$ with $X_1 \leqq X_2$ and $Y_1 \leqq Y_2$, then the topology on X_2 induces the topology on X_1, and so the product topology on Ω_{X_2} induces the product topology on Ω_{X_1}. Therefore, $id: W_{X_1,Y_1} \to W_{X_2,Y_2}$ is an embedding. This shows that W is an $Lc\mathfrak{X}$-group.

Now, fix some $X \in \mathscr{L}_G$ with $U \leqq X$. It remains to show that $\tau|_X: X \to X\tau \leqq W_{X,\bar{X}}$ is a homeomorphism. To this end, we consider the residual system $\mathscr{R}_X = \{M \trianglelefteq_0 X \mid M \leqq N \cap X\}$ in X. If $x \in N \cap X$, then $x\tau = f_x$ where $(t\bar{r})f_x = rxr^{-1}$ for all $t \in T$, $r \in R$. Therefore,

$$L_M = \{f: H \to X \mid (\bar{X})f \subseteq M\} \in \mathscr{R}_{X,\bar{X}}$$

satisfies $L_M \cap X\tau = M\tau$ for every $M \in \mathscr{R}_X$, and thus $\tau|_X$ is open. To see that $\tau|_X$ is continuous, let $L \in \mathscr{R}_{X,\bar{X}}$. Then there exist $h_1, \ldots, h_\nu \in H$ and $M_1, \ldots, M_\nu \trianglelefteq_0 X$ such that

$$L \geqq \{f: H \to X \mid (h_i)f \in M_i \quad \text{for} \quad 1 \leqq i \leqq \nu\}.$$

Put $M = \bigcap \{M_i \mid 1 \leqq i \leqq \nu\} \trianglelefteq_0 X$. Then $M\tau \leqq L \cap X\tau$. □

3. General results

Let us begin with some elementary applications of Corollary 2.5. We will denote by π the set of all primes which divide the order of some $\mathfrak{X}$-group.

Theorem 3.1. *The following assertions hold for every e.c. Lc$\mathfrak{X}$-group G.*

(a) *G is verbally complete.*

(b) *If $1 \neq N \trianglelefteq G$, then $C_G(N) = 1$.*

(c) *If U and V are f.g. subgroups of G, then $[U, V^g] = 1$ for some $g \in G$.*

(d) *Every non-trivial normal subgroup of G contains for every π-number μ an element of order μ.*

(e) *For each π-number μ, every element of G is in the normal closure of some element of order μ.*

Proof. Let E be any countable e.c. $L\mathfrak{X}$-group. Then parts (a)–(c) hold for E by [**16**, Theorem 2.1 and p. 212], and by the argument of [**15**, Lemma 2.2]. Note that (b) can be encoded by the $\forall_2$-sentence $\forall g, h \exists x\ (g \neq 1 \neq h \to [g, h^x] \neq 1)$. For the proof of (d) and (e), regard the $\forall_2$-sentences

$$\forall g\,\exists x,y\left[\bigwedge_{k=1}^{\mu-1} x^k\neq 1 \wedge x^\mu=1 \wedge (g\neq 1\rightarrow x=[g,y])\right]$$

and

$$\forall g\,\exists x,y,z\left[\bigwedge_{k=1}^{\mu-1} x^k\neq 1 \wedge x^\mu=1 \wedge g=[x,y,z]\right]$$

These hold for E by the arguments of [**16**, Lemmata 2.5/4.3(b) and 3.3]. Therefore, Corollary 2.5 applies in all cases. □

Because every f.g. $c(\mathfrak{F}_\pi\cap\mathfrak{G})$-group H is hypoabelian, it follows from the argument of [**14**, Proposition 1.B.3] that minimal normal subgroups of $Lc(\mathfrak{F}_\pi\cap\mathfrak{G})$-groups are abelian. Hence Theorem 3.1(b) implies that e.c. $Lc(\mathfrak{F}_\pi\cap\mathfrak{G})$-groups have no minimal normal subgroups.

Next, let us extend the notion of a G-subgroup given in [**21**, p. 114] for $L\mathfrak{F}_p$-groups. Suppose that $\mathscr{L}$ is the local system of all f.g. subgroups of an $Lc\mathfrak{X}$-group G. Then we say that N is a *G-subgroup* of $U\in\mathscr{L}$ ($N\trianglelefteqq_G U$), if every $V\in\mathscr{L}$ with $U\leqq V$ contains some $M\trianglelefteqq_0 V$ such that $M\cap U=N$. Put $U_G^*=\bigcap\{N\mid N\trianglelefteqq_G U\}\trianglelefteqq U$. It is readily verified that $U\leqq V\in\mathscr{L}$ and $N\trianglelefteqq_G V$ implies $N\cap U\trianglelefteqq_G U$, and so we may form

$$G^*=\bigcup\{U_G^*\mid U\in\mathscr{L}\}\trianglelefteqq G.$$

Note that $G^*\leqq\bigcap\{N\trianglelefteqq G\mid N\cap U\trianglelefteqq_0 U \text{ for all } U\in\mathscr{L}\}$. A subgroup $\tilde{N}=\bigcup\{N_X\mid X\in\mathscr{L}\}$ is said to be a *locally closed* normal subgroup in G, if every N_X is a closed normal subgroup in X, and if $N_X\leqq N_Y$ whenever $X\leqq Y$. Every $N\trianglelefteqq G$ is contained in the locally closed normal subgroup $\tilde{N}=\bigcup\{N_X\mid X\in\mathscr{L}\}$, where $N_X=\bigcap\{K\trianglelefteqq_0 X\mid N\cap X\leqq K\}$ is the closure of $N\cap X$ in X. Another kind of locally closed normal subgroup $\tilde{N}$ can be obtained from each $N\trianglelefteqq_G U\in\mathscr{L}$ by choosing $N_X=\bigcap\{K\trianglelefteqq_0 X\mid K\cap U=X\cap N\}$. In the latter case $\tilde{N}\cap U=N$, and we immediately have:

Lemma 3.2. *Let $N\trianglelefteqq_G U$ where U is a f.g. subgroup of the $Lc\mathfrak{X}$-group G. Then $\langle N^G\rangle\cap U=N$.*

Theorem 3.3. *Let $\tilde{N}=\bigcup\{N_X\mid X\in\mathscr{L}\}$ be a locally closed normal subgroup of the e.c. $Lc\mathfrak{X}$-group G. Then the following hold for every $g\in G-\tilde{N}$.*

(a) *Every finite system of equations and inequalities with coefficients from $\tilde{N}$, which is solvable in G, already has a solution in every verbal subgroup of $\langle g^G\rangle$. In particular, $\tilde{N}\leqq\langle g^G\rangle$, the locally closed normal subgroups of G are totally ordered via inclusion, and the G-subgroups in each f.g. subgroup of G form a descending chain of length $\leqq x$.*

(b) *If $0(\tilde{N}g)=n<\infty$, then any two elements in $\tilde{N}g$ of order n are conjugate in every verbal subgroup of $\langle g^G\rangle$.*

Proof. (a) Let $\mathscr{S}$ be a finite system of equations and inequalities with coefficients

$c_1,\ldots,c_r \in \tilde{N}$, unknowns $x_1,\ldots,x_s$, and a solution $g_1,\ldots,g_s \in G$. Fix any reduced word $v(\xi_1,\ldots,\xi_t) \neq 1$. Adjoin to $\mathscr{S}$ the equations

$$x_i = v(\xi_{i1},\ldots,\xi_{it}) \quad \text{and} \quad \xi_{ij} = \prod_{k=1}^{3} [g, y_{ijk}, z_{ijk}] \qquad (1 \leqq i \leqq s, 1 \leqq j \leqq t)$$

with coefficients g. In the case when $0(\tilde{N}g) = n < \infty$, let $h_1, h_2 \in \tilde{N}g$ with $n = 0(h_i)$, and adjoin the additional equations

$$h_1^{x_{s+1}} = h_2, \quad x_{s+1} = v(\xi_{s+1,1},\ldots,\xi_{s+1,t}), \quad \text{and}$$

$$\xi_{s+1,j} = \prod_{k=1}^{3} [g, y_{s+1,j,k}, z_{s+1,j,k}] (1 \leqq j \leqq t)$$

with coefficients g, h_1, h_2. Denote the resulting system by $\mathscr{T}$.

Since $\tilde{N}$ is locally closed, there exists a f.g. $V \leqq G$ with $g_1,\ldots,g_s, g \in V$ and $c_1,\ldots,c_r \in N v$ (and with $h_1, h_2 \in N_V g$ in the case when $n < \infty$). Let $\mathscr{L}_G = \{X \in \mathscr{L} \mid V \leqq X\}$. Because N_X is closed in $X \in \mathscr{L}_G$, there exists $L_X \trianglelefteq_0 X$ with $N_X \leqq L_X$ and $g \notin L_X$ (and with $\langle h_i \rangle \cap L_X = 1$ in the case when $n < \infty$). Regard in X the residual system $\mathscr{R}_X$ of all $M \trianglelefteq_0 X$ satisfying $M \leqq L_X$ and $w(c_1,\ldots,c_r,g_1,\ldots,g_s) \notin M$ for every inequality $w(c_1,\ldots,c_r,x_1,\ldots,x_s) \neq 1$ in $\mathscr{S}$. Fix some $M \in \mathscr{R}_X$. Since every e.c. $L\mathfrak{X}$-group is verbally complete, we can find an $\mathfrak{X}$-group $F_{X,M} \geqq X/M$ such that, for every $x \in X$, there exist elements $f_{x,l} \in F_{X,M}$ with $xM = v([f_{x,1}, f_{x,2}],\ldots,[f_{x,2t-1}, f_{x,2t}])$. Identify X/M with its image in L_X/M Wr $X/L_X \leqq W_{X,M} = F_{X,M}$ Wr X/L_X under some Krasner-Kaloujnine embedding.

Let $m_X = 0(L_X g)$, and choose a left transversal T of $\langle L_X g \rangle$ in X/L_X. Then every $f: X/L_X \to F_{X,M}$ can be decomposed in $W_{X,M}$ into a product $f = f_1 f_2 f_3$, where

$$\operatorname{supp} f_1 \subseteq T,$$

$$\operatorname{supp} f_2 \subseteq \bigcup \{T \cdot (L_X g^i) \mid i \in \{1,\ldots,m_X - 1\} \text{ is odd}\},$$

$$\operatorname{supp} f_3 \subseteq \bigcup \{T \cdot (L_X g^i) \mid i \in \{1,\ldots,m_X - 1\} \text{ is even}\}.$$

With this decomposition, the arguments of [16, Theorem 4.7 and Lemmata 4.2/4.3] actually show that the system $\mathscr{T}/M$ with coefficients $Mc_1,\ldots,Mc_r, Mg$ (and Mh_1, Mh_2) has a solution in $W_{X,M}$. Therefore, Theorem 2.3 yields that $\mathscr{T}$ has a solution in G, whence $\mathscr{S}$ (and the equation $h_1^x = h_2$) have a solution in the verbal subgroup of $\langle g^G \rangle$ generated by $v(\xi_1,\ldots,\xi_t)$.

Finally, if $h \in \tilde{N}$, then an application of the above to the system $\mathscr{S}$ consisting only of the equation $h = x$ with coefficient h shows that $h \in \langle g^G \rangle$. Hence $\tilde{N} \leqq \langle g^G \rangle$. □

Note, that Theorem 3.3 does in fact hold for unions and intersections of locally closed normal subgroups.

Question. *Is every normal subgroup of an e.c. $Lc\mathfrak{X}$-group the union or intersection of a chain of locally closed normal subgroups?*

If $N \trianglelefteq G \in Lc\mathfrak{X}$ is such that $N \cap U$ is closed in U for every f.g. $U \leqq G$, then each $U/U \cap N$ is a $c\mathfrak{X}$-group via the quotient topology. Note however, that this is in general not enough to ensure that $G/N \in Lc\mathfrak{X}$ since, for $U \leqq V$, the canonical monomorphism $U/(U \cap N) \to V/(V \cap N)$ need not be an embedding [**11**, p. 23].

By combining Theorem 2.1 with [**15**, Theorem 2.1], we can apply the technique of [**16**, §2] in order to extend [**16**, Theorems 2.3–2.6] to the normal $L\mathfrak{F}$-subgroups of an e.c. $Lc\mathfrak{X}$-group G. In particular, the normal $L\mathfrak{F}$-subgroups of G form a chain. We will now turn to the $L\mathfrak{F}$-quotients of G. Note that by Theorem 3.3, for every f.g. subgroup U of an e.c. $Lc\mathfrak{X}$-group G, we have $G^* \cap U = U^*_G$ or $G^* \cap U \trianglelefteq_G U$.

Theorem 3.4. *Let G be e.c. in $Lc\mathfrak{X}$.*

(a) *If $N \trianglelefteq G$, then $N \leqq G^*$ or $G^* \leqq N$.*

(b) *If $G^* < N \trianglelefteq G$, then $N \cap U \trianglelefteq_G U$ for every f.g. $U \leqq G$. In particular, $G/N \in L\mathfrak{X}$.*

(c) *If G/G^* has a minimal normal subgroup, then (b) also applies to $N = G^*$.*

(d) *$G^* = \bigcap \{N \trianglelefteq G \mid N \cap U \trianglelefteq_0 U$ for all f.g. $U \leqq G\}$. In particular, G^* contains the $L\mathfrak{X}$-residual of G.*

(e) *If for every infinite f.g. $U \leqq G$ the chain of G-subgroups in U has length ω, then G^* is the $L\mathfrak{F}$-residual of G.*

Proof. (a) Suppose that N is not in G^*. Fix some $g \in N - G^*$. Since G^* is a locally closed normal subgroup in G, Theorem 3.3 yields $G^* \leqq \langle g^G \rangle \leqq N$.

(b) Since $G^* < N$, we may assume that $G^* \cap U < N \cap U$. Choose $M \trianglelefteq_G U$ maximal with respect to $N \cap U \nsubseteq M$. Choose $g \in (N \cap U) - M$. Then Theorem 3.3 yields $M \leqq \langle g^G \rangle \cap U \leqq N \cap U$, whence $N \cap U \trianglelefteq_0 U$. Since this also holds for every f.g. $V \leqq G$ containing U, we even obtain $N \cap U \trianglelefteq_G U$.

(c) Let N/G^* be a minimal normal subgroup in G/G^*. Fix some $g \in N - G^*$. Regard any f.g. $U \leqq G$ containing g. From (b) we have $N \cap U \trianglelefteq_G U$. Choose $M \trianglelefteq_G U$ maximal with respect to $G^* \cap U \leqq M < N \cap U$. Now $\langle M^G \rangle \cap U = M$ by Lemma 3.2. Because N/G^* is minimal normal in G/G^*, we conclude from (a) that $\langle M^G \rangle \leqq G^*$. But then $G^* \cap U = M \trianglelefteq_G U$.

(d) We have already noted that $G^* \leqq S = \bigcap \{N \trianglelefteq G \mid N \cap U \trianglelefteq_0 U$ for all f.g. $U \leqq G\}$. Assume that $G^* < S$. Then (b) implies that S/G^* is a minimal normal subgroup in G/G^*. But now (c) leads to $S \leqq G^*$, a contradiction.

(e) Let $N \trianglelefteq G$ with $G/N \in L\mathfrak{F}$. Then, for every infinite f.g. $U \leqq G$, we have $|U : U \cap N| < \infty$. By our assumption, $U \cap N$ is therefore not contained in the intersection of all G-subgroups of U. Choose $M \trianglelefteq_G U$ maximal with respect to $U \cap N \nsubseteq M$. As in (b), $U \cap N \trianglelefteq_0 U$. The latter also holds for finite U. It thus follows from (d) that $G^* \leqq N$. □

Theorem 3.4 enables us to apply Construction 2.6 to all normal subgroups of an e.c.

$Lc\mathfrak{X}$-group G which contain G^* (see Theorem 2.7). We can therefore extend most of the results from [15] and [18] about normal subgroups and chief factors of e.c. $L\mathfrak{X}$-groups to normal subgroups and chief factors of G above G^*. Here is a list of theorems which can be transposed literally: [**15**, Theorems 2.3, 2.5 and 2.6], [**18**, Theorems 3.1–4.3]. Also, [**15**, §3] and the corresponding results about embeddings of countable supersoluble π-groups into e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{S})$-groups carry over (cf. [**18**, end]). As a slight extension of [**15**, Theorem 2.5] we note that, if $N \trianglelefteq G$ contains an element h of infinite order, then so does each coset of N in G (apply Theorem 3.1(c) with $U = \langle x \rangle$ and $V = \langle h \rangle$; then $h^g x$ is the desired element in the coset Nx). Finally, [**16**, Theorem 5.1] about locally inner automorphisms of countable e.c. $L\mathfrak{X}$-groups holds as well for countable e.c. $Lc\mathfrak{X}$-groups.

4. G-subgroups

Up to now, we have not established the existence of proper G-subgroups in infinite f.g. subgroups of an $Lc\mathfrak{X}$-group G. In fact, they need not exist, since we will show in Theorem 6.1 that e.c. $Lc\mathfrak{F}$-groups are simple. However, the situation is not always so bad.

Theorem 4.1 *Suppose that all simple $\mathfrak{X}$-groups are contained in a variety $\mathscr{V}$ such that every free $\mathscr{V}$-group of finite rank has only finitely many subgroups of finite index. Then every infinite f.g. subgroup U of an $Lc\mathfrak{X}$-group G contains a descending chain of G-subgroups of length ω.*

Proof. Suppose that $M \trianglelefteq_G U$. Let $\mathscr{N}$ be the set of all $N \trianglelefteq_0 M$ containing $\mathscr{V}(M)$. Then $\mathscr{N}$ is finite by hypothesis. Assume that $\mathscr{N}$ contains no G-subgroup of U. Then we can find a f.g. $X \leqq G$ such that $U \leqq X$, and such that $L \cap U \notin \mathscr{N}$ for all $L \trianglelefteq_0 X$. On the other hand, there exists $K \trianglelefteq_0 X$ with $K \cap U = M$, and we can choose $L \trianglelefteq_0 X$ maximal with respect to $L < K$. As a chief factor of $X/L \in \mathfrak{X}$, the factor K/L lies in $\mathscr{V}$. But now $L \cap U \in \mathscr{N}$, a contradiction. This shows that the desired chain can be constructed inductively. □

Theorem 4.1 applies for example in the case when $\mathfrak{X} = \mathfrak{F}_\pi \cap \mathfrak{S}$ for a finite set π of primes. More generally, Theorem 4.1 holds whenever the exponents of the simple $\mathfrak{X}$-groups are uniformly bounded (use Zelmanov's solution of the restricted Burnside problem [**24**], [**25**]).

Because chief factors of $\mathfrak{F}_p$-groups are cyclic of order p, Theorem 4.1 has the following Corollary.

Corollary 4.2. *Every f.g. subgroup U of an $Lc\mathfrak{F}_p$-group G contains G-subgroups of every index $p^\alpha \leqq |U|$ $(\alpha \in \omega)$.*

This of course generalizes the fact, that every chief series of an $L\mathfrak{F}_p$-group induces a chief series on each of its finite subgroups. Conversely, we know from [**15**, Corollary

3.3], that every series of an $\mathfrak{F}_p$-group U is induced by the unique chief series of some countable e.c. $L\mathfrak{F}_p$-group $G \geqq U$. We will also extend this result to the $Lc\mathfrak{F}_p$-case.

Construction 4.3. *Let $(n_\alpha)_{\alpha<\omega}$ be a sequence with $n_\alpha \leqq \omega$. Suppose that U_0 is an infinite f.g. $c\mathfrak{F}_p$-group, and that $\{U_{0\beta} \mid \beta<\omega\}$ is a descending chain of open normal subgroups of index p^β in U_0. Then there exists an ascending chain $\{U_\alpha \mid \alpha<\omega\}$ of f.g. $c\mathfrak{F}_p$-groups, and for every α a descending chain $\{U_{\alpha\beta} \mid \beta<\omega\}$ of open normal subgroups of index p^β in U_α such that*

(1) *$G = \bigcup \{U_\alpha \mid \alpha<\omega\}$ is an e.c. $Lc\mathfrak{F}_p$-group,*

(2) *$\{U_{\alpha\beta} \mid \beta<\omega\}$ is the unique chain of G-subgroups in U_α,*

(3) *for every $\beta \leqq n_\alpha$, there exists $\beta'<\omega$ such that $U_{\alpha\beta} = U_\alpha \cap U_{\alpha+1,\beta'}$, and*

(4) *if $n_\alpha<\omega$, and if m_α is minimal with respect to $U_{\alpha,n_\alpha} = U_\alpha \cap U_{\alpha+1,m_\alpha}$, then $U_{\alpha,n_\alpha} = U_\alpha \cap U_{\alpha+1,\beta'}$ for all $\beta' \geqq m_\alpha$.*

In the course of the construction it suffices to choose $n_{\alpha+1}$ after determining U_α and the chain $\{U_{\alpha\beta} \mid \beta<\omega\}$.

Proof. (a) Using Cantor's diagonal enumeration of $\omega \times \omega$ we can find a bijection $\chi: \omega \times \omega \to \omega$ such that $\alpha \leqq (\alpha,\beta)\chi$ for all $\alpha,\beta<\omega$ (see also [**9**, Satz 3.5]). The construction is now performed inductively in such a way that,

$$\text{for all } \alpha' \leqq \alpha, \beta<\omega, \text{ there exists } N \trianglelefteqq_0 U_\alpha \text{ such that } U_{\alpha'\beta} = N \cap U_{\alpha'}. \tag{4.1}$$

In the step $\alpha \to \alpha+1$, let $\mathscr{S}_{\alpha\gamma}$, $\gamma<\omega$, be an enumeration of all finite systems of equations and inequalities with coefficients from U_α. Fix $i,j<\omega$ with $\alpha=(i,j)\chi$. Then $\alpha \geqq i$, and $\mathscr{S}_{ij}$ does already exist by induction. If $\mathscr{S}_{ij}$ is solvable in a f.g. $c\mathfrak{F}_p$-group $V \geqq U_\alpha$ such that,

$$\text{for all } \alpha' \leqq \alpha, \beta<\omega, \text{ there exists } N \trianglelefteqq_0 V \text{ such that } U_{\alpha'\beta} = N \cap U_{\alpha'}, \tag{4.2}$$

then put $V_\alpha = V$; otherwise, let $V_\alpha = U_\alpha$. It is now easy to find a descending chain $\{V_{\alpha\beta} \mid \beta<\omega\}$ of open normal subgroups of index p^β in V_α such that $\{U_{\alpha\beta} \mid \beta \leqq n_\alpha\} \subseteq \{U_\alpha \cap V_{\alpha\beta} \mid \beta<\omega\}$. If $n_\alpha=\omega$, then we put $U_{\alpha+1}=V_\alpha$ and $U_{\alpha+1,\beta}=V_{\alpha\beta}$ for all $\beta<\omega$. In this case, (4.1) and (3) are satisfied, while (4) is empty. Otherwise, we identify V_α with the first factor of $U_{\alpha+1}=V_\alpha \times V_\alpha$, regard $U_{\alpha+1}$ as a $c\mathfrak{F}_p$-group via the product topology, and define

$$U_{\alpha+1,\beta} = \begin{cases} V_{\alpha\beta} \times V_\alpha & \text{if } \beta \leqq l_\alpha, \\ V_{\alpha,l_\alpha} \times V_{\alpha,\beta-l_\alpha} & \text{if } \beta \geqq l_\alpha, \end{cases}$$

where l_α is minimal with respect to $U_{\alpha,n_\alpha} = U_\alpha \cap V_{\alpha,l_\alpha}$. Then (3), (4) and (4.1) hold, and the induction is completed.

(b) Fix some $\alpha,\beta<\omega$ and $g \in U_\alpha - U_{\alpha\beta}$, $h \in U_{\alpha\beta}$. We will show now that the system $\mathscr{S}$ consisting only of the equation

$$h=\prod_{j=1}^{\nu}[g,x_j,y_j] \quad \text{where} \quad \nu=0(U_{\alpha\beta}g)$$

has a solution in G. Clearly, $\mathscr{S}=\mathscr{S}_{\alpha\gamma}$ for some $\gamma<\omega$. Let $\mu=(\alpha,\gamma)\chi$. Because of the construction given in (a) it suffices to show that $\mathscr{S}$ has a solution in some f.g. $c\mathfrak{F}_p$-group $V\geqq U_\mu$ satisfying (4.2).

Because of (4.1) there exists $L\trianglelefteq_0 U_\mu$ such that $L\cap U_\alpha=U_{\alpha\beta}$. Therefore, $\mathscr{R}_\mu=\{M\trianglelefteq_0 U_\mu \mid M\leqq L\}$ is a residual system in U_μ which gives the topology on U_μ. Since every e.c. $Lc\mathfrak{F}_p$-group is verbally complete, there exists a f.g. $c\mathfrak{F}_p$-group $P\geqq\langle h\rangle$ such that h is a commutator in P. Applying Theorem 2.1 and Remark 2.2. to the amalgam $U_{\alpha\beta}\cup P\mid\langle h\rangle$, we obtain a f.g. $c\mathfrak{F}_p$-group $F\geqq U_{\alpha\beta}$ such that h is a commutator in F, and such that

$$\text{every } N\trianglelefteq_0 U_{\alpha\beta} \text{ is induced in } U_{\alpha\beta} \text{ by some open normal subgroup of } F. \tag{4.3}$$

Let $\sigma: U_\alpha\to U_{\alpha\beta}\operatorname{Wr} U_\alpha/U_{\alpha\beta}\leqq W=F\operatorname{Wr} U_\alpha/U_{\alpha\beta}$ be a Krasner–Kaloujnine embedding. Denote the base group of W by Ω. For $K\trianglelefteq_0 F$, let $\check{K}=\{f\in\Omega \mid \operatorname{Im} f\subseteq K\}$. Note that, for $M\in\mathscr{R}_\mu$, we have

$$\check{K}\cap U_\alpha\sigma=(M\cap U_\alpha)\sigma \quad \text{if and only if} \quad K\cap U_{\alpha\beta}=M\cap U_{\alpha\beta}. \tag{4.4}$$

In particular, if $\mathscr{R}_W=\{\check{K} \mid K\trianglelefteq_{0\,0}F$ with $K\cap U_{\alpha\beta}=M\cap U_{\alpha\beta}$ for some $M\in\mathscr{R}_\mu\}$, then $\mathscr{R}_W\cap U_\alpha\sigma=(\mathscr{R}_\mu\cap U_\alpha)\sigma$. Put $R=\bigcap\mathscr{R}_W$, and regard $H=W/R$ as a $c\mathfrak{F}_p$-group under the topology given by the residual system $\mathscr{R}_H=\mathscr{R}_W/R$. Then the composition of σ andthe canonical epimorphism $W\to H$ embeds U_α into the $c\mathfrak{F}_p$-group H. In the following, we identify U_α with its image in H under this embedding. Now the proof of [**16**, Lemma 4.3(b)] actually shows that $\mathscr{S}$ has a solution in H. Our aim is to obtain the desired group V from an application of Theorem 2.1 to the amalgam $U_\mu\cup H\mid U_\alpha$.

To this end, regard some $K\trianglelefteq_0 F$ and $M\in\mathscr{R}_\mu$ with $K\cap U_{\alpha\beta}=M\cap U_{\alpha\beta}$. Because of (4.3) and (4.4), we can find for any chief series of open normal subgroups in U_μ, which refines $1\leqq M\leqq L<U_\mu$, a chief series of open normal subgroups in H, which refines $1\leqq\check{K}/R\leqq\Omega/R\leqq H$, such that both series induce the same on U_α. It thus follows from [**12**] that the amalgam $U_\mu/M\cup H/N\mid U_\alpha M/M\equiv U_\alpha N/N$, where $N=\check{K}/R$, is contained in an $\mathfrak{F}_p$-group. This shows that we may apply Theorem 2.1 and Remark 2.2 to find a f.g. $c\mathfrak{F}_p$-group V containing the amalgam $U_\mu\cup H\mid U_\alpha$, and satisfying

whenever $M\trianglelefteq_0 U_\mu$ and $N\trianglelefteq_0 H$ are such that $M\cap U_\alpha=N\cap U_\alpha$ and such that amalgam $U_\mu/M\cup H/N\mid U_\alpha M/M\equiv U_\alpha N/N$ is contained in an $\mathfrak{F}_p$-group, then there exists $K\trianglelefteq_0 V$ with $K\cap U_\mu=M$ and $K\cap H=N$. (4.5)

It remains to prove that V satisfies (4.2).

To this end, fix $\alpha\leqq\kappa\leqq a$. From (4.1) there exists a descending chain $\{M_\lambda \mid \lambda<\omega\}$ of open normal subgroups in U_μ which induces $\{U_{\kappa\lambda} \mid \lambda<\omega\}$ in U_κ, and hence also $\{U_{\alpha\beta'} \mid \beta'\leqq n_\alpha\}$ in U_α. Therefore, (4.3) and (4.4) yield a descending chain of open normal

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subgroups in H which induces $\{U_{\alpha\beta'} \mid \beta' \leqq n_\alpha\}$ in U_α. Now (4.5) gives $K_\lambda \trianglelefteqq_0 V$ with $K_\lambda \cap U_\mu = M_\lambda$. Suppose now that $\kappa < \alpha$. By the above argument, there exists a descending chain $\{K_\lambda \mid \lambda \leqq n_\kappa\}$ of open normal subgroups in V which induces $\{U_{\kappa\lambda} \mid \lambda \leqq n_\kappa\}$ in U_κ. If $\lambda > n_\kappa$, then (3) and (4) yield $U_{\kappa\lambda} \leqq U_{\alpha\beta}$, and using (4.1) we can extend $\{K_\lambda \cap U_\mu \mid \lambda \leqq n_\kappa\}$ to a descending chain $\{M_\lambda \mid \lambda < \omega\}$ of open normal subgroups in U_μ, which induces $\{U_{\kappa\lambda} \mid \lambda < \omega\}$ in U_κ, and which satisfies $M_\lambda \in \mathscr{R}_\mu$ for all $\lambda > n_\kappa$. Again, (4.3)–(4.5) apply.

(c) Clearly, (4.1) ensures that every $U_{\alpha\beta}$, $\beta < \omega$, is a G-subgroup of U_α. Conversely, if $N \trianglelefteqq_G U_\alpha$, then N has finite index in U_α. Therefore, N is not contained in $\bigcap \{U_{\alpha\beta} \mid \beta < \omega\}$. Choose β minimal with respect to $N \nleqq U_{\alpha,\beta+1}$. If $g \in N - U_{\alpha,\beta+1}$, then (b) and Lemma 3.2 yield that $U_{\alpha,\beta+1} < \langle g^G \rangle \cap U_\alpha \leqq \langle N^G \rangle \cap U_\alpha = N \leqq U_{\alpha\beta}$. It follows that $N = U_{\alpha\beta}$. This shows that (2) holds.

(d) Finally, let us prove (1). Suppose, that $\mathscr{S}$ is a finite system of equations and inequalities with coefficients $g_1, \ldots, g_r \in G$ and a solution $h_1, \ldots, h_s$ in some $Lc\mathfrak{F}_p$-supergroup H of G. Choose $\alpha < \omega$ such that $g_1, \ldots, g_r \in U_\alpha$. Then $\mathscr{S} = \mathscr{S}_{\alpha\gamma}$ for some $\gamma < \omega$. Let $\mu = (\alpha, \gamma)\chi$, and regard $V = \langle U_\mu, h_1, \ldots, h_s \rangle$. By the construction given in (a), $\mathscr{S}$ will have a solution in $U_{\mu+1} \leqq G$ if, for all $\kappa \leqq \mu$, $\lambda < \omega$, there exists $N \trianglelefteqq_0 V$ such that $U_{\kappa\lambda} = N \cap U_\kappa$. But this is true, since the $U_{\kappa\lambda}$ are the only G- and hence also the only H-subgroups of U_κ (Corollary 4.2). □

Corollary 4.4. *Let U be an infinite f.g. $c\mathfrak{F}_p$-group. If $\{N_\alpha \mid \alpha < \omega\}$ is a descending chain of open normal subgroups of index p^α in U, then U is contained in a countable e.c. $Lc\mathfrak{F}_p$-group G such that the N_α are precisely the G-subgroups in U.*

Construction 4.3 enables us to build various examples of countable e.c. $Lc\mathfrak{F}_p$-groups.

Example 4.5. *Each of the following properties is shared by $2^{\aleph_0}$ (pairwise non-isomorphic) non-periodic countable e.c. $Lc\mathfrak{F}_p$-groups G: $G^* = G$, $1 \neq G/G^* \in L\mathfrak{F}_p$, $G/G^* \notin L\mathfrak{F}_p$.*

Proof. Regard the free group U_0 of rank two as a $c\mathfrak{F}_p$-group under the topology given by the residual system of all $N \trianglelefteqq U_0$ of p-power index. Since the commutator factor group of every subgroup of finite index in U_0 is free abelian of rank $\geqq 2$ [**20**, Proposition I.3.9], there exist $2^{\aleph_0}$ descending chains of subgroups $\{U_{0\beta} \mid \beta < \omega\}$ of index p^β in U_0 with $\bigcap \{U_{0\beta} \mid \beta < \omega\} = 1$. An application of Construction 3.4 to these chains clearly yields $2^{\aleph_0}$ countable e.c. $Lc\mathfrak{F}_p$-groups G, which are not isomorphic as $Lc\mathfrak{F}_p$-groups, since they contain U_0 in too many non-compatible ways. Choose $n_\alpha = 0$ (resp. $n_\alpha = \omega$) for all α to establish $G^* = G$ (resp. $G/G^* \notin L\mathfrak{F}_p$). Moreover, if $0 < n_0 < \omega$, then $n_{\alpha+1} = m_\alpha$ for all α leads to $1 \neq G/G^* \in L\mathfrak{F}_p$, since $U_{0,n_0-1}G^*/G^*$ is a minimal normal subgroup in G/G^* in this case. □

5. E.c. $Lc\mathfrak{F}_p$-groups

In the case when $\mathfrak{X} = \mathfrak{F}_p$, we can make more progress by considering further applications of Corollary 2.5 to e.c. $Lc\mathfrak{F}_p$-groups. To this end we will have to regard specific $\forall_2$-sentences. For any reduced word $w(\xi_1,\ldots,\xi_v)\neq 1$, we define the term $t_w(g,\bar{x})$ as follows.

$$t_w(g,\bar{x})=w\left(\prod_{i=0}^{p-1}[g,x_{1,i},x_{1,p+i}],\ldots,\prod_{i=0}^{p-1}[g,x_{v,i},x_{v,p+i}]\right). \tag{5.1}$$

Then the $\forall_2$-sentence ϕ_w is given by

$$\phi_w=\forall g,h\,\exists\bar{x}\left[\bigvee_{k=0}^{p-1}h=g^k\cdot t_w(g,\bar{x})\vee\bigvee_{k=0}^{p-1}g=h^k\cdot t_w(h,\bar{x})\right]. \tag{5.2}$$

Lemma 5.1. *Every e.c. $Lc\mathfrak{F}_p$-grouup satisfies the above $\forall_2$-sentences ϕ_w.*

Proof. Note that $[x_{1j},x_{2j}]=[(b,f),x_{1j},x_{2j}]$ actually holds for *all* $j\in\{1,\ldots,m\}$ in the proof of [**16**, Lemma 4.3(b)]. Therefore, a detailed analysis of the proof of [**16**, Theorem 4.7] shows that the unique countable e.c. $L\mathfrak{F}_p$-group satisfies each ϕ_w. Now Corollary 2.5 applies. □

Lemma 5.1 provides the key for the proof of the following theorem.

Theorem 5.2. *Every e.c. $Lc\mathfrak{F}_p$-group G has a unique chief series Σ. The chief factors of G are central and cyclic of order p, and the order type of Σ is a dense order without endpoints.*

Proof. Regard any $K,L\trianglelefteq G$. Suppose that there exists $g\in K-L$. From $g\notin L$ we have $g\notin\langle h^G\rangle$ for every $h\in L$. Therefore it follows from Lemma 5.1 that $h\in\langle g^G\rangle$ for every $h\in L$, whence $L\leqq K$. This shows that the normal subgroups of G are totally ordered via inclusion. Equivalently, G has a unique chief series.

Regard a chief factor M/N in G. If $g\in M-N$ then, from Lemma 3.2, Σ must induce the unique chain of G-subgroups in $\langle g\rangle$, when $g^p\in N$. Therefore, $\exp(M/N)=p$. Regard the reduced word $w(\xi)=\xi^p$ and some $g,h\in M-N$. Then Lemma 5.1 yields that $g\in h^kN$ or $h\in g^kN$ for some $k\in\{1,\ldots,p-1\}$. This shows that M/N is cyclic of order p. Since G is verbally complete, G has no finite epimorphic image, and so M/N must be central.

Lemma 5.1 applied to the word $w(\xi_1,\xi_2)=[\xi_1,\xi_2]$ yields that $M'=N$ for every chief factor M/N in G. Since G is also perfect and has trivial centre (Theorem 3.1), we may deduce as in the proof of [**19**, Theorem $B(d)$] that Σ has the desired order type. □

Theorem 5.2 implies that *every* $Lc\mathfrak{F}_p$-group has a chief series with central and cyclic factors of order p. Note also, that the existence of a unique chief series with dense order type and elementary-abelian factors could also be shown for e.c. $Lc(\mathfrak{F}_\pi\cap\mathfrak{G})$-groups, where π is a finite set of primes, if there would exist a pendant to Lemma 5.1, i.e., if it could be shown, for example, that there exists some fixed $m<\omega$ such that, for every chief factor M/N of a countable e.c. $L(\mathfrak{F}_\pi \cap \mathfrak{G})$-group, and for all $g, h \in M - N$, the element $Nh \in G/N$ is a product of at most m conjugates of powers of Ng in G/N (cf. [**16**, Theorem 4.9 and p. 214]).

Theorem 5.3. *Let $K \trianglelefteq G$, where G is e.c. in $Lc\mathfrak{F}_p$, and suppose that $K \neq \langle g^G \rangle$ for all $g \in G$. Then the following hold.*

(a) *K is e.c. in G. In particular, every normal subgroup of K is already normal in G, and conjugation with elements from G induces locally inner automorphisms on K.*

(b) *If $K \in L\mathfrak{F}_p$, then K is e.c. in $L\mathfrak{F}_p$.*

Proof. Let $\mathscr{S}$ be a finite system of equations and inequalities with coefficients $\bar{c}$ and unknowns $\bar{x}$.

(a) Denote by $\phi(\bar{c}, \bar{x})$ the conjunction of all equations and inequalities from $\mathscr{S}$. Consider the $\forall_2$-sentence

$$\forall g, \bar{c}, \bar{d}, \bar{y}_i \exists \bar{h}, \bar{z}_j \left[g \neq 1 \wedge \phi(\bar{c}, \bar{d}) \wedge \bigwedge_i c_i = t_w(g, \bar{y}_i) \rightarrow \phi(\bar{c}, \bar{h}) \wedge \bigwedge_j h_j = t_w(g, \bar{z}_j) \right],$$

where $w(\xi_1, \xi_2) = [\xi_1, \xi_2]$. This sentence expresses that, whenever $\mathscr{S}$ has coefficients $\bar{c}$ in $[g, G, G]'$ and a solution $\bar{d}$ in G, then there exists a solution $\bar{h}$ to $\mathscr{S}$ in $\langle g^G \rangle'$. It is satisfied in the countable e.c. $L\mathfrak{F}_p$-group [**16**, Theorem 4.8]. Thus Corollary 2.5, Lemma 5.1 and Theorem 5.2 yield that the groups N in the chief factors M/N of G are e.c. in G. It now follows from the arguments of [**16**, Theorem 4.8], that K satisfies (a).

(b) Suppose that $\mathscr{S}$ has coefficients in K and a solution in the $L\mathfrak{F}_p$-group $H \geqq K$. Then $U = \langle \bar{c} \rangle$ is finite. Since K is e.c. in G, and since G is e.c. in $Lc\mathfrak{F}_p$, it suffices to embed the amalgam $G \cup H \,|\, U$ into an $Lc\mathfrak{F}_p$-group. To this end, we check the conditions of Theorem 2.1. By (a), K has a unique chief series. So [**21**, Hilfssatz 1] yields the existence of a finite group $V \geqq U$ in K such that every chief series in V induces the K-chief series in U. Let $\mathscr{L}_G$ and $\mathscr{L}_H$ be the local systems in G resp. H of all f.g. subgroups containing V. For $(X, Y) \in \mathscr{L}_G \times \mathscr{L}_H$, put $\mathscr{R}_{X,Y} = \{M \trianglelefteq_0 X \,|\, M \cap V = 1\}$ and $\mathscr{R}_{Y,X} = \{1\}$. Then [**12**] ensures that, for all $M \in \mathscr{R}_{X,Y}$, the amalgam $X/M \cup Y \,|\, UM/M \equiv U$ is contained in an $\mathfrak{F}_p$-group, since $X/M \geqq VM/M \cong V$ and $Y \geqq V$ enforce that all chief series in X/M and Y induce the K-chief series in $UM/M \equiv U$. □

It remains open, whether Theorem 5.3(b) can be extended to arbitrary K. Note that, if K is the unique countable e.c. $L\mathfrak{F}_p$-group E_p, then G is contained canonically in the group $\mathrm{LokInn}(E_p)$ of all locally inner automorphisms of E_p by Theorem 3.1(b).

Theorem 5.4. *Let G be e.c. in $Lc\mathfrak{F}_p$. Then every subnormal subgroup of G is already normal in G.*

Proof. Assume that there exists a subnormal subgroup S of defect 2 in G. Because of Theorem 5.3, there exists a chief factor M/N in G such that $S \lhd M \lhd G$, and such that S is not in N. Regard the $\forall_2$-sentence

$$\forall g, h, \bar{x}\, \exists \bar{y}, \bar{z}_i \left[h = t_w(g, \bar{x}) \rightarrow h = t_v(g, \bar{y}) \wedge \bigwedge_i y_i = t_w(g, \bar{z}_i) \right],$$

where $v(\xi) = \xi$ and $w(\xi_1, \xi_2) = [\xi_1, \xi_2]$. A detailed analysis of the proof of [**16**, Theorem 4.11(f)] shows that this sentence is satisfied by the countable e.c. $L\mathfrak{F}_p$-group. Hence it also holds in G by Corollary 2.5. Because of Lemma 5.1 and Theorem 5.2 we conclude that $N \leqq [g, N, N] \leqq S < M$ for any $g \in S - N$. But this enforces $S = N$, a contradiction. □

We can also extend [**15**, §3] and [**16**, Theorems 4.1/2] literally to results about embeddings of countable $L\mathfrak{F}_p$-groups into e.c. $Lc\mathfrak{F}_p$-groups and to results about partial complements to normal subgroups $\neq \langle g^G \rangle$ in e.c. $Lc\mathfrak{F}_p$-groups G (here, G_0/N resp. G_0/K must be in $L\mathfrak{F}_p$). To this end we just transform the systems of equations and inequalities used in the proofs of these theorems into suitable $\forall_2$-sentences. Example 4.5 shows that the full generalization of the above embedding results (without restriction to $L\mathfrak{F}_p$-groups) does not hold. As far as conjugacy of f.g. subgroups in e.c. $Lc\mathfrak{F}_p$-groups is concerned, we have the following result.

Theorem 5.5. *Let G be an e.c. $Lc\mathfrak{F}_p$-group.*

(a) *An isomorphism $\phi: A \rightarrow B$ between f.g. subgroups of G is induced by conjugation in G, if and only if there exists a local system $\mathscr{L}$ in G of f.g. subgroups which contain $\langle A, B \rangle$ such that, in every $X \in \mathscr{L}$, there is a chief series $\{N_\alpha\}_{\alpha \in \omega}$ of open normal subgroups such that $a^{-1} \cdot a\phi \in N_{\alpha+1}$ for all $a \in (A \cap N_\alpha) - N_{\alpha+1}$.*

(b) *An element $g \in G$ of infinite order is conjugate in G to g^n ($n \in \mathbf{Z}$), if and only if $n \equiv 1$ (mod p).*

Proof. Combine Theorem 2.3 with [**15**, Corollary 3.3(b)] and [**17**, Theorem 6.1]. □

It remains open whether every automorphism of G, which stabilizes the unique chief series in G, is a locally inner automorphism (cf. [**17**, Theorem 6.1]).

6. E.c. $Lc\mathfrak{F}$-groups

The techniques developed in Section 2 yield the following informations about e.c. $Lc\mathfrak{F}$-groups.

Theorem 6.1. *The following assertions hold for every e.c. $Lc\mathfrak{F}$-group G.*

(a) *Every isomorphism between finite subgroups of G is induced by conjugation in G.*

(b) *If $G \cup H \mid U$ is an amalgam of G with a countable $L\mathfrak{F}$-group H over a finite subgroup U, then $id: U \rightarrow G$ can be extended to an embedding $H \rightarrow G$.*

(c) *For all $g, h \in G - 1$ there exist $x, y \in G$* such that *$h = g^x g^y$. In particular, G is simple.*

(d) *On every f.g. abelian subgroup of G, inversion is induced by conjugation in G. In particular, every element in G is conjugate to its inverse.*

(e) *Let* $w_1, \ldots, w_n$ *be words in unknowns* $x_1, \ldots, x_n$ *and elements of G. Denote by* d_{ij} *the exponent sum of* x_j *in* w_i. *If* $\det(d_{ij}) \neq 0$, *then the system of equations* $w_i = 1$ for $1 \leqq i \leqq n$ *has a solution in G.*

Proof. The assertions (a), (c), (d), (e) can be encoded as $\forall_2$-sentences and hold in every e.c. $L\mathfrak{F}$-group by [**14**, Theorem 6.1] and [**4**, Theorem 2]. Moreover, (b) follows from an iterated application of the corresponding statement for finite H, which in turn can be expressed as an $\forall_2$-sentence that holds in the unique countable e.c. $L\mathfrak{F}$-group [**14**, Theorem 6.1]. □

Of course, one is tempted to ask in how far the assertion (a) of Theorem 6.1 extends to isomorphisms between infinite f.g. subgroups of an e.c. $Lc\mathfrak{F}$-group G. We will show now that it is hardly possible to make much progress in this direction.

Theorem 6.2. *Let G be an e.c.* $Lc\mathfrak{F}$-*group.*

(a) *An isomorphism* $\phi: A \to B$ *between f.g. subgroups of G is induced by conjugation in G, if and only if there exists a local system* $\mathscr{L}$ *of f.g. subgroups of G which contain* $\langle A, B \rangle$ *such that, for every* $X \in \mathscr{L}$, *the topology on X is given by a residual system* $\mathscr{R}_X$, *such that* ϕ *induces an isomorphism* $AN/N \to BN/N$ *for every* $N \in \mathscr{R}_X$.

(b) *An element* $g \in G$ *of infinite order is conjugate in G to* g^n $(n \in \mathbb{Z})$, *if and only if* $0(g^n N) = 0(gN)$ *for every* $N \trianglelefteq_0 \langle g \rangle$. *In particular, no element of infinite order in G is conjugate to all of its non-trivial powers.*

Proof. (a) follows from Theorem 2.3 and [**7**, Lemma 1].

(b) If $0(g^n N) = 0(gN)$ for every $N \trianglelefteq_0 U = \langle g \rangle$, then (a) implies that g is conjugate to g^n. Conversely, suppose that $0(g^n N) \neq 0(gN)$ for some $N \trianglelefteq_0 U$. Then $0(g^n M) \neq 0(gM)$ for every $M \trianglelefteq_0 U$ with $M \leqq N$. But every residual system in U contains some $M \trianglelefteq_0 U$ such that $M \leqq N$, whence (a) implies that g is not conjugate to g^n. □

Let $C = \langle c \rangle$ be the infinite cyclic group with the topology given by the residual system of all $N \trianglelefteq C$ of finite index. Then Theorem 6.2(b) shows that, in every e.c. $Lc\mathfrak{F}$-group $G \geqq C$, the element c is conjugate to c^n $(n \in \mathbb{Z})$ if and only if $n = \pm 1$.

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FACHBEREICH 17-MATHEMATIK
JOHANNES GUTENBERG-UNIVERSITÄT
SAARSTR. 21
D-6500 MAINZ
GERMANY