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ON ITERATED POWERS OF POSITIVE DEFINITE FUNCTIONS

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Abstract

We prove that if ρ is an irreducible positive definite function in the Fourier–Stieltjes algebra $B(G)$ of a locally compact group G with $\|\rho\|_{B(G)} = 1$, then the iterated powers (ρ^n) as a sequence of unital completely positive maps on the group C^*-algebra converge to zero in the strong operator topology.

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In this paper we prove the dual version of the following fundamental limit theorem concerning convolution powers of probability measures on locally compact groups.

Theorem 1 ([5, Corollary 4], [7, Theorem 1.8]). *Let μ be a probability measure on a locally compact group G. If G is not compact and μ is irreducible (that is, the semigroup generated by the support of μ is dense in G), then the sequence $\|\mu^n * f\|_\infty$ converges to zero for every $f \in C_0(G)$.*

This result has a number of important consequences in the study of harmonic functions and boundaries of random walks on locally compact groups (see also [10]).

Theorem 1, also known as the *concentration function problem* for locally compact groups, was first considered by Hofmann and Mukherjea in [5], where they proved the above limit theorem for a large class of locally compact groups. Then, in [7], Jaworski *et al.* used the developments in the theory of totally disconnected groups to settle the problem for all locally compact groups.

A dual version of the theory of random walks on groups, harmonic functions and measure–theoretic boundaries has been developed by Biane [1] and Chu and Lau [2] (see also [6, 9, 11]).

The proof of Theorem 1 is based on some deep results on the structure theory of locally compact groups that are not available in the noncommutative setting, and therefore one has to find new arguments to generalise the theorem to the latter case.

In this note we give a proof in the dual setting, that is, for positive definite functions in the Fourier–Stieltjes algebra $B(G)$ of G.

The author has proved a discrete quantum group version of Theorem 1 in [8]. Of course, the dual of a locally compact group is a locally compact quantum group, but the proof in [8], which uses Banach limits and a noncommutative 0–2 law, cannot be applied to general (quantum) probability measures on nondiscrete locally compact quantum groups.

Here we present a different proof for the case of co-commutative quantum groups, that is, duals of locally compact groups.

Recall that for a locally compact group G the set $B(G)$ of all matrix coefficient functions of continuous unitary representations of G forms a subalgebra of bounded continuous functions on G. The algebra $B(G)$ admits a norm $\|\cdot\|_{B(G)}$ with which it becomes a Banach algebra, called the Fourier–Stieltjes algebra of G. It is known that $B(G)$ is isomorphic to the dual Banach space of $C^*(G)$ (the universal group C^*-algebra of G).

For an abelian group G with (Pontryagin) dual group $\hat{G}$, the Fourier–Stieltjes transform $\mathcal{F}_s$ yields an isomorphism of the dual Banach algebras $B(G) \cong M(\hat{G})$, where $(M(\hat{G}), *)$ is the measure algebra (with convolution product) of the dual group $\hat{G}$.

We denote by $P_1(G)$ the set of positive definite functions of norm one. So, $\rho \in P_1(G)$ means that there exist a continuous unitary representation π of G on a Hilbert space H_π and a unit vector $\xi_\rho \in H_\pi$ such that

$$\rho(r) = \langle \pi(r)\xi_\rho, \xi_\rho \rangle \quad (r \in G).$$

A positive definite function $\rho \in P_1(G)$ is *irreducible* if for every nonzero positive $x \in C(G)^*$ there exists $n \in \mathbb{N}$ such that $\langle x, \mu^n \rangle \neq 0$. In the case of an abelian group G, irreducible positive definite functions correspond to the class of probability measures on the dual group $\hat{G}$ with the property that the smallest closed semigroup containing their support is $\hat{G}$.

We need the following general lemma (cf. [3]).

Lemma 2. *Suppose that $\mathfrak{A}$ is a C^*-algebra and T is a positive contraction on $\mathfrak{A}$. Then, for any $0 \leq x \in \mathfrak{A}$,*

$$\lim_n \|T^n x\| = \sup\{|\langle x, \nu \rangle| : \nu \in \mathcal{D}_1\},$$

where $\mathcal{D}_1$ denotes the set of all $0 \leq \nu \in \mathfrak{A}^*$ such that there exists a sequence $\{\nu_n\}_{n=0}^\infty$ of positive elements of $\mathfrak{A}^*$ with $\nu_0 = \nu$, $\|\nu_n\| \leq 1$ for all $n \geq 0$ and

$$T^* \nu_{n+1} = \nu_n, \quad n \geq 0. \tag{0.1}$$

Proof. The '$\geq$' part of the equation follows from $(T^*)^n \nu_n = \nu_0$. For the converse, let $0 \leq x \in \mathfrak{A}$ and $\varepsilon > 0$. Choose $0 \leq \omega_n \in \mathfrak{A}^*$ with $\|\omega_n\| \leq 1$ such that $\|T^n x\| < |\langle T^n x, \omega_n \rangle| + \varepsilon$ for all n. Then let ν_0 be a weak* cluster point of $\{(T^*)^n \omega_n : n \geq 1\}$ in $\mathfrak{A}^*$. We may find a subnet $(T^*)^{n_i} \omega_{n_i}$ that converges weak* to ν_0 and therefore

$$\lim_n \|T^n x\| = \lim_{n_i} \|T^{n_i} x\| \leq \lim_{n_i} |\langle x, (T^*)^{n_i} \omega_{n_i} \rangle| + \varepsilon = |\langle x, \nu_0 \rangle| + \varepsilon.$$

Now let $0 \leq \eta_1$ be a weak* cluster point of $\{(T^*)^{n_i - 1} \omega_{n_i}\}$ in $\mathfrak{A}^*$. Then $\|\eta_1\| \leq 1$ and $T^* \eta_1 = \nu_0$. Continuing in this way, by induction we can construct a sequence $\{\eta_n\}$

of positive elements of $\mathfrak{A}^*$ such that $\|\eta_n\| \le 1$ and $T^*\eta_{n+1} = \eta_n$ for all n. Thus, in particular, $(T^*)^n\eta_n = \nu_0$. Now let $\nu_1 = \lim_{n_j}(T^*)^{n_j-1}\eta_{n_j}$ be a weak* cluster point of $\{(T^*)^{n-1}\eta_n : n \ge 2\}$ in $\mathfrak{A}^*$. Then $\|\nu_1\| \le 1$ and $T^*\nu_1 = \nu_0$. If we choose a weak* cluster point ν_2 of $\{(T^*)^{n_j-2}\eta_{n_j}\}$, then $0 \le \nu_2$, $\|\nu_2\| \le 1$ and $T^*\nu_2 = \nu_1$. Similarly, we can construct a sequence $\{\nu_n\}$ of positive elements of $\mathfrak{A}^*$ with $\|\nu_n\| \le 1$ for all $n \ge 0$ that satisfies (0.1). □

In the following, we denote by $\rho \cdot x$ the canonical action of elements ρ in $B(G)$ on elements x in $C^*(G) \cong B(G)_*$.

Theorem 3. *Let G be a nondiscrete locally compact group and let $\rho \in P_1(G)$ be irreducible. Then, for every $x \in C^*(G)$,*

$$\lim_n \|\rho^n \cdot x\| = 0.$$

Proof. Let $\mathcal{D}_1$ be the set of all $\nu \in B(G)$ such that there exists a sequence $\{\nu_k\}_{k=0}^{\infty}$ in $B(G)^+$ with $\nu_0 = \nu$, $\|\nu_k\| \le 1$ and $\rho\nu_{k+1} = \nu_k$ for all $k \ge 0$. Then $0 < \|\nu_0\|_{B(G)} = \|\nu_k\|_{B(G)} \le 1$ and $\mathrm{support}(\nu_k) \subseteq \mathrm{support}(\rho) \cap \mathrm{support}(\nu_{k+1})$ for all $n \in \mathbb{N}$.

Let $G_{\bar\rho} = \rho^{-1}(\mathbb{T})$. Since $\|\rho\|_\infty = 1$, it follows that

$$|\nu_k(r)| = |\rho(r)|^m|\nu_{k+m}(r)| \le |\rho(r)|^m \longrightarrow 0$$

for all $r \notin G_{\bar\rho}$. Hence, $\mathrm{support}(\nu_k) \subseteq G_{\bar\rho}$ for all $k \ge 0$. Moreover, from the continuity of the ν_k, we conclude that $G_{\bar\rho}$ is open in G.

Thus, we obtain a canonical identification

$$B(G_{\bar\rho}) \cong \{\mathbb{1}_{G_{\bar\rho}} \cdot \omega : \omega \in B(G)\},$$

where $\mathbb{1}_{G_{\bar\rho}} \in B(G)$ is the characteristic function of $G_{\bar\rho}$ (cf. [4]). Now, since $G_{\bar\rho}$ is abelian, the Fourier–Stieltjes transform $\mathcal{F}_s$ induces the identification $B(G_{\bar\rho}) \cong M(\widehat{G_{\bar\rho}})$. In particular, we obtain the sequence $\{\mathcal{F}_s(\mathbb{1}_{G_{\bar\rho}}\nu_k)\}$ in the measure algebra $(M(G_{\bar\rho}), *)$ with $\|\mathcal{F}_s(\mathbb{1}_{G_{\bar\rho}}\nu_k)\| \le 1$ and

$$\mathcal{F}_s(\mathbb{1}_{G_{\bar\rho}}\rho) * \mathcal{F}_s(\mathbb{1}_{G_{\bar\rho}}\nu_{k+1}) = \mathcal{F}_s(\mathbb{1}_{G_{\bar\rho}}\rho\nu_{k+1}) = \mathcal{F}_s(\mathbb{1}_{G_{\bar\rho}}\nu_k)$$

for all $k \ge 0$. Therefore,

$$\begin{aligned}
|\langle x, \nu_0\rangle| &= |\langle \mathbb{1}_{G_{\bar\rho}}x, \mathbb{1}_{G_{\bar\rho}}\nu_0\rangle| \\
&= |\langle \mathcal{F}^{-1}(\mathbb{1}_{G_{\bar\rho}}x), \mathcal{F}^{-1}(\mathbb{1}_{G_{\bar\rho}}\nu_0)\rangle| \\
&\le \lim_n \|\mathcal{F}^{-1}(\mathbb{1}_{G_{\bar\rho}}\rho)^{*n} * \mathcal{F}^{-1}(\mathbb{1}_{G_{\bar\rho}}x)\| \quad &&\text{(by Lemma 2)} \\
&= 0 &&\text{(by Theorem 1).}
\end{aligned}$$

By once again applying Lemma 2, this time to the action of ρ as a contraction on $C^*(G)$, we conclude that

$$\lim_n \|\rho^n \cdot x\| = \sup\{|\langle x, \nu_0\rangle| : \nu_0 \in \mathcal{D}_1\} = 0.$$ □

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Since in the abelian case the reduced C^*-algebra $C_r^*(G)$ coincides with $C^*(G)$, one may consider $B_r(G) = C_r^*(G)^*$ as the dual object to the measure algebra $M(G)$. Using the fact that $C_r^*(G)$ is a quotient C^*-algebra of $C^*(G)$ and that $B_r(G)$ is a subalgebra of $B(G)$, we derive the following result.

COROLLARY 4. *Let G be a nondiscrete locally compact group and let $\rho \in B_r(G)$ be an irreducible positive definite function on G. Then*

$$\lim_n \|\rho^n \cdot x\| = 0$$

for all $x \in C^*(G)$.

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