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On knots that divide ribbon knotted surfaces

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Abstract

We define a knot to be *half ribbon* if it is the cross-section of a ribbon 2-knot, and observe that ribbon implies half ribbon implies slice. We introduce the *half ribbon genus* of a knot K, the minimum genus of a ribbon knotted surface of which K is a cross-section. We compute this genus for all prime knots up to 12 crossings, and many 13-crossing knots. The same approach yields new computations of the double slice genus. We also introduce the *half fusion number* of a knot K, that measures the complexity of ribbon 2-knots of which K is a cross-section. We show that it is bounded below by the Levine–Tristram signatures, and differs from the standard fusion number by an arbitrarily large amount.

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1. *Introduction*

Knots in S^3 naturally appear as equatorial cross-sections of knotted surfaces in S^4. In this paper we restrict to *ribbon* knotted surfaces, those that are particularly simple Morse-theoretically (see Definition 3). We define a knot K to be *half ribbon* if it is the cross-section of a ribbon knotted 2-sphere in S^4. Ribbon knots are half ribbon (see Proposition 6), but the converse is an open question. Half ribbon knots are slice, but the converse is also an open question (as not every knotted 2-sphere is ribbon [**25**]).

Answering these questions would resolve the slice-ribbon conjecture, that posits that every slice knot is ribbon [**6**]. Despite much effort, and results in both directions, it remains open (see, for example, [**1, 8, 9, 14, 16**]). The notion of half ribbon arises naturally by splitting the slice-ribbon conjecture into two questions: (i) if K is a cross-section of a 2-knot is it the cross-section of a ribbon 2-knot? and (ii) if K is a cross-section of a ribbon 2-knot does it possess a ribbon disc?

We also introduce the *half ribbon genus*, $g_{hr}(K)$, of a knot K : the minimum genus of a ribbon knotted surface of which K is a cross-section. The half ribbon genus is an intermediate between the slice genus, $g_4(K)$, and double slice genus, $g_{ds}(K)$, in that

$$2g_4(K) \le g_{hr}(K) \le g_{ds}(K). \qquad \text{(Proposition 8)}$$

It follows that a knot of half ribbon genus one would be a counterexample to the slice-ribbon conjecture. More generally, a knot of odd half ribbon genus would have distinct slice and ribbon genera (see Question 3).

We determine the half ribbon genus of every prime knot of up to 12 crossings to be even. In addition, we calculate 8 of the 65 previously unknown double slice genera of such knots. The following result is proved in Section 3·2.

THEOREM 1. *Let K be a prime knot with up to 12 crossings. Then $g_{hr}(K) = 2g_4(K)$. If K is one of the following knots then $g_{ds}(K) = 2g_4(K)$ also*:

$$9_{37}, 10_{74}, 11n148, 12a554, 12a896, 12a921, 12a1050, 12n554.$$

We also calculate the half ribbon genus of 2156 13-crossing knots, and in 247 such cases determine the double slice genus.

Orson and Powell constructed a knot $K_{M,N}$ with $2g_4(K_{M,N}) = M$ and $g_{ds}(K_{M,N}) = N$, for all integers $0 \le M \le N$ with M even [**21**]. In Section 2·3 we observe that $g_{hr}(K_{M,N}) = M$, so that the half ribbon and double slice genera differ by an arbitrarily large amount. The analogous question regarding the slice and half ribbon genera is open. Answering it in full generality is at least as hard as resolving the slice-ribbon conjecture (see Question B).

Suppose that K is a cross-section of a knotted surface S. The calculations of Theorem 1 rely on realising band attachments to K as 3-dimensional 1-handle attachments to S (see Theorem 13). This allows us to prove that if J is obtained from K via a sequence of ℓ band attachments then

$$\begin{aligned} g_{ds}(J) - \ell \le g_{ds}(K) \le g_{ds}(J) + \ell \\ g_{hr}(K) \le 2g_r(J) + \ell, \end{aligned} \qquad \text{(Corollary 14)}$$

where $g_r(J)$ denotes the ribbon genus. We also prove a version of this result for general ribbon cobordisms, Theorem 15, from which results of McDonald [**19**, Theorems 3·1, 3·2] can be recovered.

We consider a finer notion of complexity for half ribbon knots. Ribbon 2-knots are precisely those obtainable from a disjoint union of trivial 2-spheres by attaching 1-handles. The minimum number of 1-handles required to form a 2-knot S in this way is the *fusion number*, denoted $f(S)$. We introduce the *half fusion number*, $f_h(K)$, of a half ribbon knot K : it is the minimum $f(S)$ such that K is a cross-section of S.

As only the trivial 2-knot has fusion number zero it follows that $f_h(K) = 0$ if and only if $g_{ds}(K) = 0$. The Levine–Tristram signatures bound the double slice genus from below [**21**], and we prove that this also holds for the half fusion number:

$$f_h(K) \geq \max_{\omega \in S^1 \setminus \{1\}} \{|\sigma_\omega(K)|\}. \qquad \text{(Theorem 11)}$$

Together with the observation (made in Section 2·4) that $f(K) - g_{ds}(K)$ can be arbitrarily large, this poses the question: what is the precise relationship between the double slice genus and half fusion number?

The *fusion number* of ribbon knot K, $f(K)$, is the minimum number of bands in a ribbon disc for K. As described in Section 2·4 the fusion number is bounded below by the half fusion number. We prove that these quantities can differ by an arbitrarily large amount: for all integers $0 \leq M < N$ there exists a ribbon knot K with

$$f_h(K) = M \text{ and } f(K) \geq N. \qquad \text{(Proposition 12)}$$

For such a knot K arbitrarily more bands are required in a ribbon disc than 1-handles are required to form a ribbon 2-knot (of which K appears as a cross-section). In other words, the structure of the set of ribbon discs for K is in some sense distinct to that of such ribbon 2-knots.

Conventions. All manifolds and embeddings are smooth and orientable. Knots are labelled as per KnotInfo [**18**].

2. *Dividing ribbon knotted surfaces*

We recall necessary background in Section 2·1 before tackling our main objects of study in Sections 2·2 to 2·4.

2·1. *Background*

A *1-link* is a link in S^3, and 1-link of one component is a *1-knot*. A *2-knot* is an embedding $S^2 \hookrightarrow S^4$, and a *surface-knot* is an embedding $F \hookrightarrow S^4$ for F a closed orientable surface. A surface-knot is *trivial* if it bounds an embedded handlebody in S^4. All of the above objects are considered up to ambient isotopy.

Henceforth we denote by S^3_0 an equator in S^4, and by B^4_+, B^4_- the associated hemispheres (so that $S^4 = B^4_+ \cup_{S^3_0} B^4_-$).

Definition 2. Let K be a 1-knot and S a surface-knot. We say that K divides S if there exists an equator S^3_0 such that $S^3_0 \cap S = K$. We also refer to K as a cross-section of S.

Every 1-knot divides a surface-knot, and this relationship has been of great interest to low-dimensional topologists for almost a century. Questions on this relationship are broadly of two kinds. First, given a fixed 1-knot how complex are the surface-knots that it divides? Obversely, for surface-knots of a given complexity what are the 1-knots that divide them?

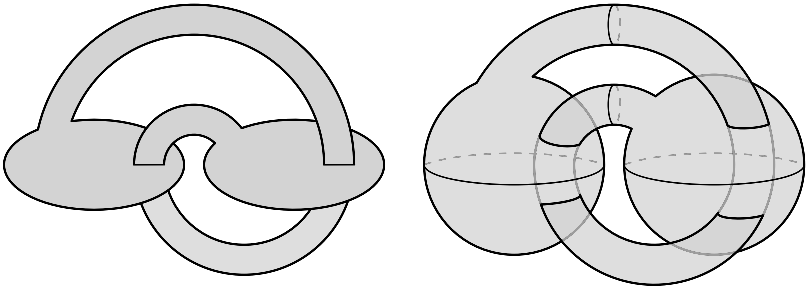

Fig. 1. On the left: a ribbon surface F formed of discs and bands. On the right: the induced sphere-tube presentation of the double of F.

The focus of this paper is a question of the second type: studying the 1-knots that divide *ribbon* surface-knots.

Definition 3 *(Ribbon surface-knot)*. We say that a surface-knot is ribbon if it bounds a properly embedded handlebody in B^5 on which the radial height function restricts to a Morse function without critical points of index 2.

Before presenting formal definitions of our main objects of interest we fix some further terminology.

Definition 4 *(Slice, ribbon surface)*. A slice surface for a 1-link K is a compact orientable surface F properly embedded in B^4 such that $\partial F = K$. A ribbon surface is a slice surface on which the radial height function restricts to a Morse function without critical points of index 2.

This Morse-theoretic definition of a ribbon surface is equivalent to the definition via surfaces immersed in S^3 with ribbon singularities (see, for example, [11, Lemma 11·9]). Note that slice surface and surface-knot are distinct concepts, likewise ribbon surface and ribbon surface-knot.

Given a slice surface F for a 1-knot K we may form a surface-knot divided by K as follows. Regard K as lying in an equator S^3_0 and F in B^4_+. Denote by $\overline{F}$ the surface obtained by reflecting F through S^3_0. The surface-knot $F \cup_K \overline{F}$ is known as the *double of F,* and K divides it by construction.

An embedded 2-sphere in S^4 is *standard* if it is in Morse position with exactly two critical points (of index 0 and 2 necessarily). An isotopy representative of a surface-knot is a *sphere-tube presentation* if it is obtained by attaching 3-dimensional 1-handles to a disjoint union of standard 2-spheres. A surface-knot is ribbon if and only if it possesses a sphere-tube presentation [12, Section 5·6].

We frequently make use of the fact that the double of a ribbon surface for K is a ribbon surface-knot divided by K. This is described in [19, Figure 2] and the related discussion (for full details see, for example, [12, Section 5]). We suffice ourselves by observing, as in Figure 1, that a ribbon surface is made up of discs and bands; upon doubling discs become trivial 2-spheres and bands become 1-handles in a sphere-tube presentation of the double.

2·2. *Dividing spheres*

We say that a 1-knot K is *slice* if it divides a 2-knot [**2, 7**], and that K is *doubly slice* if it divides the trivial 2-knot [**6, 24**].

Using Definition 3 we introduce a notion that lies between slice and doubly slice.

Definition 5 *(Half ribbon)*. We say that a 1-knot K is half ribbon if it divides a ribbon 2-knot.

Notice that as the trivial 2-knot is ribbon it follows that a doubly slice 1-knot is half ribbon (the converse fails, for example, on the 1-knot 6_1). Half ribbon knots are of course slice, but the status of the converse is an open question (as outlined in Question 1).

Recall that a 1-knot K is *ribbon* if it bounds a ribbon surface of genus 0; such a surface is known as a *ribbon disc* for K.

PROPOSITION 6. *Ribbon 1-knots are half ribbon.*

Proof. Suppose that D is a ribbon disc for a 1-knot K. The double of D is a ribbon 2-knot that K divides by construction.

Thus doubly slice implies slice but the converse is false, and ribbon implies half ribbon but the converse may be false. In other words, ribbon is to the property of dividing a ribbon 2-knot as doubly slice is to slice, whence the name half ribbon.

Proposition 6 shows that the slice-ribbon conjecture splits into the following questions.

Question A. Let K be a 1-knot.

1. If K divides a 2-knot must it divide a ribbon 2-knot?
2. If K divides a ribbon 2-knot must it possess a ribbon disc?

A negative answer to (i) or (ii) would yield a counterexample to the slice-ribbon conjecture.

2·3. *Dividing surfaces*

The *slice genus* of a 1-knot K, $g_4(K)$, is the minimum genus of a slice surface for K. The *ribbon genus* of K, $g_r(K)$, is the minimum genus of a ribbon surface for K.

Notice that taking the double of a slice surface for K yields a surface-knot divided by K. Generically this surface-knot will be nontrivial. Restricting to trivial surface-knots (not necessarily doubles of slice surfaces for 1-knots) yields the *double slice genus* of K, $g_{ds}(K)$, the minimum genus of a trivial surface-knot divided by K [**17**].

Just as in Section 2·2 we can use Definition 3 to define a quantity intermediate to the slice and double slice genera.

Definition 7 *(Half ribbon genus)*. Let K be a 1-knot. The half ribbon genus of K, $g_{hr}(K)$, is the minimum genus of a ribbon surface-knot divided by K.

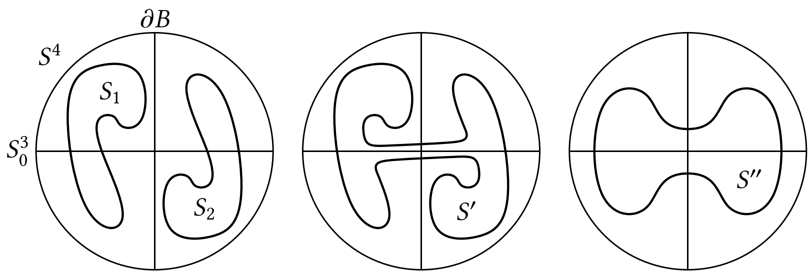

Fig. 2. Schematic diagrams of the split union of S_1 and S_2, and the surface-knots S' and S''.

Of course, K is of half ribbon genus zero if and only if it is half ribbon. The half ribbon genus is finite for all 1-knots as it is bounded above by the double slice genus and twice the ribbon genus.

PROPOSITION 8. *Let K be a 1-knot, then*

$$2g_4(K) \leq g_{hr}(K) \leq 2g_r(K) \leq g_{ds}(K).$$

Proof. Let F be a ribbon surface for K with $g(F) = g_r(K)$. The double of F is a ribbon surface-knot of genus $2g_r(K)$ and is divided by K, so that $g_{hr}(K) \leq 2g_r(K)$. As the double of F is not necessarily trivial we have $2g_r(K) \leq g_{ds}(K)$.

PROPOSITION 9. *The half ribbon genus is subadditive with respect to the connected sum of 1-knots.*

Proof. Let K_1, K_2 be 1-knots and S_1, S_2 ribbon surface-knots such that K_i divides S_i. Denote by S the split union of S_1 and S_2. That is, S is a disjoint union of S_1 and S_2, and there exists a 4-ball B such that $S_1 \cap B = S_1$, $S_2 \cap B = \varnothing$.

Let S' be the result of adding a 1-handle between the components of S, chosen so that its core intersects ∂B in exactly one point and the connected sum of K_1 and K_2 appears as the equatorial cross-section.

As S_1 and S_2 are ribbon they possess sphere-tube presentations. Let S'' be the surface-knot obtained by first isotoping S_1 and S_2 into such presentations and then adding a 1-handle between them, the core of which intersects ∂B in exactly one point. (Notice that the isotopy taking S_1 and S_2 to their sphere-tube presentations may be chosen to fix ∂B pointwise.) A schematic for the split union of S_1 and S_2, together with S' and S'', is given in Figure 2.

It follows that S'' is a ribbon surface-knot as it possesses a sphere-tube presentation. By [12, Proposition 1·2·11] the surface-knots S' and S'' are identical. Thus S' is a ribbon surface-knot divided by $K_1\#K_2$ of genus $g(S_1) + g(S_2)$, so that $g_{hr}(K_1\#K_2) \leq g_{hr}(K_1) + g_{hr}(K_2)$.

W. Chen constructed the first examples of 1-knots of arbitrarily large double slice genus [4]. These 1-knots are ribbon and are therefore of half ribbon genus zero by Proposition 6.

Let $\overline{K}$ denote the mirror image of a 1-knot K. Orson and Powell showed that the knot $J = \left(\#^{\frac{M}{2}}\overline{5_2}\right) \# \left(\#^{N-M} 8_{20}\right)$ satisfies $2g_4(J) = M$ and $g_{ds}(J) = N$, for all integers $0 \leq M \leq N$ with M even [21]. As $2g_4\left(\overline{5_2}\right) = g_{ds}\left(\overline{5_2}\right) = 2$ and 8_{20} is ribbon we have that $g_{hr}\left(\overline{5_2}\right) = 2$ and $g_{hr}(8_{20}) = 0$ by Proposition 8, and $g_{hr}(J) = M$ by Proposition 9. It follows that the half ribbon and doubly slice genera differ by an arbitrarily large amount. The analogous question regarding the slice genus remains open.

Question B. Given integers $0 \leq M \leq N \leq P$ does there exist a 1-knot K such that

$$2g_4(K) = M,\ g_{hr}(K) = N,\ g_{ds}(K) = P?$$

Does there exist a prime 1-knot with this property?

Answering Question 2 for all M, N, P is at least as hard as finding a 1-knot with distinct slice and ribbon genera.

Question C. Does there exist a 1-knot J of odd half ribbon genus?
Such a J must have distinct slice and ribbon genera by Proposition 8.

In Section 3 we determine the half ribbon genus of all 1-knots up to 12 crossings to be even.

Specialising further, establishing the existence of 1-knots of half ribbon genus one would resolve the slice-ribbon conjecture in the negative.

Question D. Does there exist a 1-knot K of half ribbon genus one?
Such a K would be a counterexample to the slice-ribbon conjecture: $g_4(K) = 0$ by Proposition 8, but K is not ribbon as $g_{hr}(K) \neq 0$.

Satoh defined a surjective map from the category of welded knots to that of ribbon tori [**23**]. Can this map be used to address Question 4?

2·4. *Fusion numbers*

In Sections 2·2 and 2·3 we consider the problem of minimising the genus of a surface-knot divided by a given 1-knot. In this section we consider minimising the following alternative measure of complexity.

Let K be a ribbon 1-knot. The *fusion number of K*, $f(K)$, is the minimum number of bands in a ribbon disc for K. Let S be a ribbon 2-knot. The *fusion number of S*, $f(S)$, is the minimum number of 1-handles in a sphere-tube presentation for S.

For half ribbon 1-knots we define a new quantity in terms of the fusion number of the ribbon 2-knots they divide.

Definition 10 *(Half fusion number)*. Let K be a half ribbon 1-knot. The half fusion number of K, $f_h(K)$, is the minimum fusion number of a ribbon 2-knot divided by K.

Note that $f(S) = 0$ if and only if S is a trivial 2-knot, so that $f_h(K) = 0$ if and only if $g_{ds}(K) = 0$. The proof of Proposition 9 also establishes the subadditivity of the half fusion number with respect to the connected sum of 1-knots.

Orson and Powell showed that the Levine–Tristram signatures bound the double slice genus from below [**21**]. The same is true of the half fusion number.

THEOREM 11. *Let K be a half ribbon 1-knot. Then*

$$f_h(K) \geq \max_{\omega \in S^1 \setminus \{1\}} \{|\sigma_\omega(K)|\}.$$

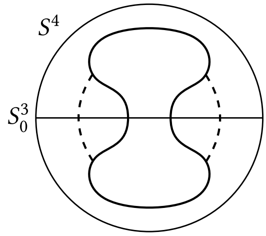

Fig. 3. Attaching 1-handles with cores given by the dashed arcs introduces an unlink to the equatorial cross-section.

Proof. Let S be a ribbon 2-knot divided by K such that $f(S) = f_h(K)$. By adding $f\ (S)$ 1-handles to S it can be converted into a trivial surface-knot S' [**20**]. Moreover, as the result of adding a 1-handle depends only on its core [**12**, Proposition 5·1·6] these handles may be chosen so that $K \cup U$ divides S', for $K \cup U$ the split union of K and an unlink U. Generically the handles may intersect the equatorial S^3 so that we cannot avoid the appearance of this unlink, as per the schematic given in Figure 3.

The genus of S' is equal to $f\ (S)$ and bounds from above the weak double slice genus of $K \cup U$, denoted $g^1_{ds}(K \cup U)$ [**5**, Equation 1]. Conway and Orson showed that the Levine–Tristram signatures bound this quantity from below [**5**, Corollary 1·3], so that

$$|\sigma_\omega(K \cup U)| \leq g^1_{ds}(K \cup U) \leq g(S') = f_h(K).$$

The proposition follows by the additivity of the signature under disjoint union.

In the proof above the surface-knot divided by K is trivialised by attaching $f_h(K)$ 1-handles. However, this does not allow us to conclude that $g_{ds}(K)$ bounds $f_h(K)$ from below, as we can guarantee only that $K \cup U$ divides this trivial surface-knot.

Let D be a ribbon disc for a 1-knot K. The bands of D yield 1-handles in the double of D, that is a ribbon 2-knot divided by K. It follows that $f_h(K) \leq f(K)$. There exist ribbon knots whose fusion and half fusion numbers are arbitrarily far apart.

PROPOSITION 12. *For all integers $0 \leq M < N$ there exists a ribbon 1-knot K such that $f_h(K) = M$ and $f(K) \geq N$.*

Proof. Juhaśz, Miller and Zemke defined an invariant of 1-knots using knot Floer homology, denoted $\mathrm{Ord}_v(K)$, and proved that it bounds the fusion number of ribbon 1-knots from below [**10**, Corollary 1·7]. Denote by $T_{p,q}$ the positive (p,q)-torus knot and let $C_{p,q} = T_{p,q}\#\overline{T_{p,q}}$. It is established in [**10**, Equation 1·7] that

$$\mathrm{Ord}_v(C_{p,q}) = f(C_{p,q}) = \min\{p, q\} - 1.$$

Notice that $f_h(C_{p,q}) = 0$ as $C_{p,q}$ is doubly slice. Let $K_M = C_{p,q}\#\left(\#^M 8_{20}\right)$. The 1-knot 8_{20} is chosen as for $\omega = e^{\pi i/3}$ we have $\sigma_\omega(8_{20}) = f_h(8_{20}) = f(8_{20}) = 1$. Observe that $M = \sigma_\omega(K_M) \leq f_h(K_M)$ by Theorem 11 and the additivity of the signature with respect to connected sum. That the half fusion number is subadditive with respect to connected sum implies that $f_h(K_M) = M$, in fact.

Further, applying [10, Equation 1·4] yields

$$\mathrm{Ord}_v(K_M) = \max\left\{\mathrm{Ord}_v(C_{p,q}), \mathrm{Ord}_v(8_{20})\right\}$$
$$= \min\{p, q\} - 1$$

so that $\min\{p, q\} - 1 \leq f(K_M)$. A suitably large choice of p and q completes the proof.

Proposition 12 establishes that the set of ribbon 2-knots divided by a 1-knot K is in some sense distinct to the set of ribbon discs for K.

Note that as $g_{ds}(8_{20}) = 1$ and the double slice genus is subadditive the proof of Proposition 12 shows that the difference $f(K) - g_{ds}(K)$ can also be made arbitrarily large.

3. *Calculations*

The calculation of the slice genus of prime 1-knots of up to 12 crossings has recently been completed, with input from a large number of authors [3, 13, 15, 22]. Karageorghis and Swenton also calculated the double slice genus of all but 68 of these 1-knots [13], three of which were later determined by Brittenham and Hermiller [3].

In this section we calculate the half ribbon genus of every prime 1-knot up to 12 crossings. Additionally, we compute the double slice genus in 8 of the 65 previously undetermined cases. We apply our methods to 13-crossing 1-knots, calculating the half ribbon genus of 2156 of them. In 247 such cases we are also able to determine the double slice genus.

Section 3·1 describes how ribbon cobordisms between 1-knots can be realised on surface-knots they divide, and Section 3·2 gives the results of our calculations.

3·1. *Handle attachments defined by ribbon cobordisms*

Recent calculations of the slice and double slice genera have employed upper bounds obtained from various sequences of band attachments. Our calculations rely on the observation that if K divides a surface-knot S, attaching bands to K may be realised by adding 1-handles to S.

THEOREM 13. *Let K, J be 1-links, C a connected cobordism from K to J defined by attaching ℓ bands to K, and $\overline{C}$ its reverse. Suppose that J divides a surface-knot S with $S = S_+ \cup_J S_-$. Then*:

(i) *the surface-knot $S' = S_+ \cup C \cup_K \overline{C} \cup S_-$ is obtained from S by attaching ℓ 1-handles*;

(ii) *if S_+ is a ribbon surface for J and $\overline{S_-} = S_+$ then S' is ribbon*;

(iii) *if S is trivial then S' is trivial.*

Proof.

(i) The bands of C become 1-handles in S', that may be thought of as being attached to the cylinder $J \times [0, 1]$ in $S_+ \cup (J \times [0, 1]) \cup S_-$.

(ii) If S_+ is a ribbon surface for J then $S_+ \cup C$ is a ribbon surface for K, the double of which is a ribbon surface-knot.

(iii) Attaching 1-handles to a trivial surface-knot yields a trivial surface-knot [11, Proposition 11·2].

The *superslice genus*, $g^s(K)$, of a 1-knot K is the minimum genus of a slice surface for K the double of which is a trivial surface-knot [4].

COROLLARY 14. *Let K, J be 1-knots. Suppose that J is obtained from K via a sequence of ℓ band attachments. Then*:

(i) $g_{ds}(J) - \ell \leq g_{ds}(K) \leq g_{ds}(J) + \ell$;
(ii) $2g^s(J) - \ell \leq 2g^s(K) \leq 2g^s(J) + \ell$;
(iii) $g_{hr}(K) \leq 2g_r(J) + \ell$.

Proof.

(i) Let J divide a trivial surface-knot S with $g(S) = g_{ds}(J)$. By Theorem 13 K divides a trivial surface-knot of genus $g_{ds}(J) + \ell$, whence the rightmost inequality. Reversing the roles of K and J gives the leftmost.

(ii) Let F be a slice surface for J with $g(F) = g^s(J)$. Denote by C the cobordism defined by the sequence of ℓ bands. Then $F \cup C$ is a slice surface for K of genus $g^s(J) + (\ell/2)$ (the number of bands is even as C is orientable with two boundary components). The double of $F \cup C$ is a surface-knot of genus $2g^s(J) + \ell$, and is trivial by Theorem 13. The rightmost inequality follows from the fact that K divides this surface-knot by construction. Reversing the roles of K and J gives the leftmost.

(iii) Let F be a ribbon surface for J with $g(F) = g_r(J)$. A doubling process similar to that given above shows that K divides a ribbon surface of genus $2g_r(K) + \ell$.

Picking J as the unknot in Corollary 14 (i), (ii) recovers [19, Theorem 3·1]. It is unknown if attaching a 1-handle to a ribbon surface-knot preserves the ribbon property. This causes Corollary 14 (iii) to be of a different form to Corollary 14 (i), (ii). The result of attaching a 1-handle h to a surface-knot S depends only on the homotopy class of the core, γ, of h in the complement of S [12, Proposition 5·1·6]. If S is ribbon it may be isotoped into a sphere-tube presentation. The trace of this isotopy is of codimension 1 so that it and γ generically intersect in points. It is therefore unclear if the isotopy can be completed in the presence of h.

Let K and J be 1-knots and C a cobordism between them. We say that C is a *ribbon cobordism from K to J* if we do not encounter a birth of an unknotted and unlinked component when traversing C from K to J. A *ribbon concordance* is a ribbon cobordism of genus 0.

Theorem 13 is similar to the following description of the union of a ribbon concordance with its reverse given by Zemke [26][1]. If C is a ribbon concordance from K to J then $C \cup_K \overline{C}$ is obtained from $J \times [0, 1]$ by taking a disjoint union with trivial 2-knots and attaching them to $J \times [0, 1]$ via 1-handles. This operation is known as taking a *tube sum with trivial 2-knots*.

This description can be combined in a straightforward manner with Theorem 13 to obtain the following results. We expect these more general results to be useful in further study of the double slice and half ribbon genera.

[1] Our definition is consistent with the *reverse* of what Zemke refers to as a ribbon concordance.

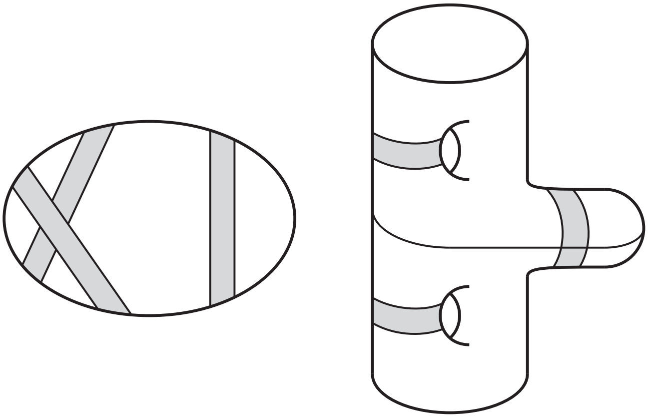

Fig. 4. On the left: bands defining a ribbon cobordism C. On the right: a schematic of $C \cup \overline{C}$ (some handles have been isotoped away from the equator for aesthetic purposes).

THEOREM 15. *Let K, J be 1-links, C a connected ribbon cobordism from K to J with s saddles and d deaths. If J divides a surface-knot S then K divides a surface-knot, S', obtained from S by attaching $(s-d)$ 1-handles to S and taking the tube sum with d trivial 2-knots.*

Proof. The saddles of C split into two types. Up to isotopy we may assume that as we move in reverse through C (from J to K) we first see the creation of a d-component unlink, the components of which are then joined together by bands to produce a 1-knot. The saddles of C associated to these bands yield the tube sums that contribute to S'. The remaining $(s-d)$ saddles of C yield the 1-handles attached to S. A schematic example is given in Figure 4.

COROLLARY 16. *Suppose that there exists a ribbon cobordism from K to J with s saddles and d deaths. Then*:

(i) $g_{ds}(J) - s \leq g_{ds}(K) \leq g_{ds}(J) + s$;

(ii) $g_{hr}(K) \leq 2g_r(J) + s - d$.

Proof. (i)Suppose that J divides a trivial surface-knot S with $g(S) = g_{ds}(J)$. By Theorem 15 K divides a surface-knot, S', obtained from S by attaching $(s-d)$ 1-handles to S and taking d tube sums with trivial 2-knots. Denote by S_1 the result of attaching the $(s-d)$ 1-handles to S. As S is trivial S_1 is trivial, and $g(S_1) = g(S) + s - d$.

Denote by T the set of trivial 2-knots that will be tube summed to S_1 to produce S'. A 1-handle is *trivial* if it bounds an embedded $D^1 \times D^2$. As the d components of T are formed by doubling a ribbon cobordism they may be connected with $(d-1)$ trivial 1-handles to produce a single trivial 2-knot, T', without altering the equatorial cross-section. We may add an additional trivial 1-handle between S_1 and T' to produce a trivial surface-knot, S_2, also without altering the cross-section. Notice that $g(S_2) = g(S) + s - d$.

Finally, consider the set of d 1-handles defined by the tube sums that produce S'. We may add these 1-handles to S_2 (as it includes both S and T) to produce a trivial surface-knot, S_3, of genus $g(S_2) + d = g_{ds}(J) + s$. That the trivial 1-handles above are attached without altering the cross-section ensures that K divides S_3.

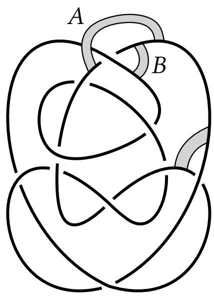

Fig. 5. A diagram of the 1-knot 10_{74}, together with three bands. Attaching the bands labelled A and B realises a crossing change that yields a diagram of the 1-knot 9_{46}. Subsequently attaching the band labelled C defines a ribbon disc for 9_{46}.

(ii) Let F be a ribbon surface for J with $g(F) = g_r(J)$. Then J divides the genus $2g_r(J)$ surface-knot S with sphere-tube presentation given by the double of F. By Theorem 15 K divides a surface-knot S' obtained from S by attaching 1-handles and taking tube sums with trivial 2-knots. Thus S' is ribbon as it also has a sphere-tube presentation, and $g(S') = 2g_r(J) + s - d$ (only the $(s - d)$ 1-handles attached to S affect the genus of S').

Picking J as the unknot in Corollary 16 (i) recovers [**19**, Theorem 3·2].

3·1. *Determining genera*

As employed by Lewark–McCoy and Brittenham–Hermiller the operations of switching a crossing, switching a pair of crossings of zero writhe, and taking the oriented resolution at two crossings are all realizable by attaching two bands [**15**, Lemma 5].

To calculate the half ribbon genus we combine specific examples of these operations found by Lewark–McCoy and Brittenham–Hermiller, calculations of the double slice genus by Karageorghis–Swenton, and Corollary 14.

For the 1-knots that Lewark–McCoy and Brittenham–Hermiller do not provide a suitable operation we made an independent computer search for crossing changes to ribbon or doubly slice 1-knots.

Proof of Theorem 1. The slice-ribbon conjecture has been verified up to 12 crossings. Therefore $2g_4(K) = g_{hr}(K) = 0$ for all slice 1-knots.

If K is not slice and $g_{ds}(K)$ was determined prior to this work then $2g_4(K) = g_{ds}(K)$ [**18**], so that $2g_4(K) = g_{hr}(K)$ also by Theorem 8.

This leaves 58 cases of undetermined half ribbon genus, as described in Table I (7 of the 65 cases undetermined by Karageorghis and Swenton are slice). These 1-knots are obtainable from a ribbon 1-knot by attaching two bands as shown by Lewark–McCoy [**15**, Appendix A], Brittenham–Hermiller [**3**, Section 4], or our crossing change search. Therefore these 1-knots have half ribbon genus equal to 2 by Corollary 14 and Proposition 8 (recall that they are not slice). An example of this process in the case of 10_{74} is given in Figure 5.

Finally, as depicted in Table 1 the 1-knots 9_{37}, 10_{74}, $11n148$, $12a554$, $12a896$, $12a921$, $12a1050$, $11n148$, $12n554$ are obtainable from a doubly slice 1-knot by attaching two bands, so that they have double slice genus 2 by Corollary 14 (they were shown to have double slice genus 2 or 3 by Karageorghis and Swenton).

Table 1. *The second and third columns list the result of attaching two bands, as given by [15, Appendix A] and [3, Section 4], respectively. The fourth column lists the result of a crossing change, found by our computer search. If we were able to calculate a previously unknown value of g_{ds} it is listed in the fifth column.*

Knot	L-M	B-H	C.c.	g_{ds}
9_{37}			9_{46}	2
9_{48}			8_{20}	
10_{74}			9_{46}	2
10_{103}			8_8	
11a135	6_1			
11a155	8_{20}			
11a173	8_{20}			
11a327	8_{20}			
11a352	6_1			
11n71			8_{20}	
11n75			8_{20}	
11n148	$4_1\#4_1$			2
11n167	6_1			
12a164	8_{20}			
12a166	8_{20}			
12a177	6_1			
12a247	8_8			
12a265	6_1			
12a298	8_{20}			
12a327	8_8			
12a396	8_{20}			
12a413	8_{20}			
12a449	6_1			
12a493	6_1			
12a503	10_{75}			
12a554			9_{46}	2
12a735	6_1			
12a750	6_1			
12a769	6_1			
12a873	8_{20}			
12a895	10_{87}			
12a896	$3_1\#\overline{3_1}$			2
12a905		10_{87}		
12a921	$4_1\#4_1$			2
12a971	6_1			
12a1050			9_{46}	2
12a1085	6_1			
12a1194	8_8			
12a1200	6_1			
12a1226	8_{20}			

Table 1. Continued.

Knot	L-M	B-H	C.c.	g_{ds}
12n147	8_8			
12n334	6_1			
12n379	8_{20}			
12n388	6_1			
12n396	6_1			
12n460	6_1			
12n480	6_1			
12n495	8_{20}			
12n524	6_1			
12n537	6_1			
12n554			9_{46}	2
12n555		6_1		
12n577	10_{140}			
12n583	6_1			
12n737	6_1			
12n813	6_1			
12n846	6_1			
12n869	8_{20}			

This leaves 57 prime 1-knots of up to 12 crossings with unknown double slice genus.

Our methods extend fruitfully into 13-crossing 1-knots. Specifically, we restricted to 13-crossing 1-knots of signature 2 and searched for crossing changes to ribbon or doubly slice 1-knots. This allows us to show that 2156 such 1-knots have half ribbon genus 2, of which 247 have double slice genus 2. The full results of these calculations are provided on the webpage https://www.mas.ncl.ac.uk/william.rushworth/ccdata.html and on the arXiv listing https://arxiv.org/abs/2209.15577.

Just as this paper studies 1-knots that appear as cross-sections of ribbon 2-knots, one could study those that appear as cross-sections of *homotopy ribbon* 2-knots. In addition to being possibly distinct to half ribbon in the smooth category, such a definition extends to the topological category. The most basic question one might ask in this setting is as follows.

Question E. Is every topologically slice 1-knot the cross-section of a homotopy-ribbon 2-knot?

Finally, although we do not pursue it here, the notion of half ribbon genus can readily be extended to allow for knotted surfaces of more than one component, as is done for the double slice genus [**5**, Equation 1]. There are natural generalisations of Theorems 13 and 15 to this setting.

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