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THE SELF-ADJOINT 5-POINT AND 7-POINT DIFFERENCE OPERATORS, THE ASSOCIATED DIRICHLET PROBLEMS, DARBOUX TRANSFORMATIONS AND LELIEUVRE FORMULAE

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Abstract. We present some basic properties of two distinguished discretizations of elliptic operators: the self-adjoint 5-point and 7-point schemes on a two dimensional lattice. We first show that they allow us to solve Dirichlet boundary value problems; then we present their Moutard transformations (distinguished examples of transformation of Darboux type in two dimensions). Finally we construct their Lelieuvre formulae and we show that, at the level of the normal vector and in full analogy with their continuous counterparts, the self-adjoint 5-point scheme characterizes a two dimensional quadrilateral lattice (a lattice whose elementary quadrilaterals are planar), while the self-adjoint 7-point scheme characterizes a generic 2D lattice.

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1. Introduction. In the last two decades of the 19th century and in the beginning of the 20th century many great mathematicians (Bianchi, Darboux and others) developed a differential geometry studying transformations of certain geometric structures and proved theorems of permutability of such transformations, obtaining in turn nonlinear superposition principles for the nonlinear differential equations characterizing the above geometries. These results can be viewed as the pre-history [**1**, **2**] of the modern theory of integrable nonlinear systems, which is based on the existence of linear differential operators possessing symmetry transformations of Darboux type (the Darboux Transformations (DTs)) and nonlinear isospectral symmetries (the integrable nonlinear systems).

After more than one century, in studying discrete integrable systems (which are, in a sense, richer and more fundamental than their continuous counterparts and, for these reasons, worth studying), one often makes use of the mutual interplay between geometry and the theory of integrable systems in the discrete case too [**3**].

The main goal of this paper is to present two examples of such an interplay; we start with the self-adjoint 7-point operator

$$\begin{aligned}\mathcal{L}_7 := a_{m,n}T_m + a_{m-1,n}T_m^{-1} + b_{m,n}T_n + b_{m,n-1}T_n^{-1} + s_{m+1,n}T_mT_n^{-1} \\ + s_{m,n+1}T_m^{-1}T_n - f_{m,n}\end{aligned} \tag{1}$$

and with the self-adjoint 5-point operator

$$\mathcal{L}_5 := a_{m,n}T_m + a_{m-1,n}T_m^{-1} + b_{m,n}T_n + b_{m,n-1}T_n^{-1} - f_{m,n}, \tag{2}$$

for which the existence of Darboux transformations has been recently established [**4**], and we obtain their geometric interpretation through the construction of their Lelieuvre formulae. It is worth mentioning that, very often, one follows the opposite direction: a "geometric insight" allows one to construct an integrable system.

In the equations above T_m and T_n are the translation operators with respect to the discrete variables $(m, n) \in \mathbb{Z}^2$

$$T_mf_{m,n} = f_{m+1,n}, \quad T_nf_{m,n} = f_{m,n+1}$$

and $f_{m,n} = f(m, n)$ is a function of (m, n).

The classical Lelieuvre formulae [**5**]

$$\vec{R},_u = \vec{N},_u \times \vec{N}, \quad \vec{R},_v = \vec{N} \times \vec{N},_v \tag{3}$$

allow one to construct a two-dimensional surface $\vec{R}(u, v)$ in E^3 from its normal (not necessarily the unit one) image $\vec{N}(u, v)$. The coordinate net (u, v) of the surface obtained in this way is the asymptotic one and the normal field satisfies the Moutard equation [**6**]

$$\vec{N},_{uv} = F(u, v)\vec{N}. \tag{4}$$

The Moutard equation is covariant under the Moutard transformation – the second transformation of Darboux type appeared in the literature [**6**] (the first was the transformation of Ribaucour [**7**]). In the modern theory of integrable systems all these transformations are indicated generically as transformations of Darboux type, or DTs, although the famous Darboux transformation for the 1-D Schrodinger equation has been derived by Darboux as a reduction of Moutard's result. This is the reason why we use the terminology "Moutard transformations" (MTs) rather than "Darboux transformations" from now on. The Moutard transformation gave Guichard [**8**] the possibility to describe Weingarten rectilinear congruences in a very elegant way and, in turn, it allowed Bianchi and other geometers to construct many systems of nonlinear differential equations which now are called *soliton systems*.

Another example of a Lelieuvre type formulae was obtained by Bianchi [**9**, p. 253]:

$$\vec{R},_x = \vec{N},_y \times \vec{N}, \quad \vec{R},_y = \vec{N} \times \vec{N},_x . \tag{5}$$

Now the normal field satisfies the 2D Schrödinger equation

$$\vec{N},_{xx} + \vec{N},_{yy} = F\vec{N}, \tag{6}$$

which is also covariant under a Moutard transformation, and the coordinate net (x, y) is, in this case, the isothermally-conjugate one.

An extension of the Lelieuvre formulae to an arbitrary coordinate system (or, better to say, to a coordinate free language) and to hypersurfaces in the equiaffine space of arbitrary dimension has been obtained in [**10**] and [**11**, p. 57]. Another extension of the Lelieuvre formulae can be found in [**12**]. A discretization of the Lelieuvre formulae (3) was first given in the paper [**13**] and a discretization of the notion of Weingarten congruences was proposed in [**14**].

The main result of this paper consists in the construction of the Lelieuvre formulae for a 2D quadrilateral lattice (a lattice whose elementary quadrilaterals are planar) [**15**], [**16**] and for an essentially arbitrary 2D lattice in eA^3 space. This result allows one to establish that the operators (2) and (1) characterize, at the level of the normal vectors, respectively, the above 2D lattices.

For our purposes it is not necessary to deal with the Euclidean space, but it is enough to enrich the affine space with the volume form Vol (by Vol^* we denote the dual form of Vol); i.e., it is enough to deal with the equiaffine space eA^3. This enables one to construct the cross product from an ordered pair of linearly independent vector fields (say $(\vec{a}, \vec{b})$); i.e., to construct the element $\hat{N} \in T^*eA^3$ such that $\langle \hat{N} \mid \vec{a} \rangle = 0 = \langle \hat{N} \mid \vec{b} \rangle$ and $\langle \hat{N} \mid \vec{c} \rangle = Vol\{\vec{a}; \vec{b}; \vec{c}\}$ for every $\vec{c} \in TeA^3$.

The second goal of this paper consists of the illustration of some of the basic criteria for constructing the proper discretizations of partial differential operators. Again we use, as illustrative examples, the operators (1) and (2).

The paper is organized as follows. In Section 2 we present some of the basic criteria which we use as a guide for constructing the proper discretizations of partial differential operators. These criteria are systematically applied, in the remaining sections, to the illustrative examples given by the operators $\mathcal{L}_5$ and $\mathcal{L}_7$. In Section 3 we show that the operators $\mathcal{L}_5$ and $\mathcal{L}_7$ preserve the elliptic character of their differential counterpart, being applicable to solve the Dirichlet problem on a 2D lattice. In Section 4 we show that the operators $\mathcal{L}_5$ and $\mathcal{L}_7$ possess, like their differential counterparts, MTs. In Sections 5, 6 and 7 we derive the Lelieuvre formulae for, respectively, the continuous counterpart of $\mathcal{L}_7$, for $\mathcal{L}_7$, for the continuous counterpart of $\mathcal{L}_5$ and for $\mathcal{L}_5$, verifying that the geometric meaning of the operators $\mathcal{L}_5$ and $\mathcal{L}_7$ is the proper discretization of the geometric meaning of their differential counterparts.

We conclude this introductory section with some general remarks on the operators $\mathcal{L}_5$ and $\mathcal{L}_7$. The operator $\mathcal{L}_7$ can be interpreted as the most general self-adjoint operator on the star of a regular triangular lattice [**17**, **18**]; it possesses a class of Laplace transformations [**17**, **18**] and plays a relevant role in a recently developed discrete complex function theory [**19**]. Its natural continuous limit [**4**]

$$A\partial_x^2 + B\partial_y^2 + 2S\partial_x\partial_y + (A,_x + S,_y)\partial_x + (B,_y + S,_x)\partial_y - F \tag{7}$$

is the most general, second order, linear, self-adjoint operator. The operator $\mathcal{L}_5$ is instead the most general self-adjoint operator on the star of a square lattice [**4**] and its natural continuous limit is the following self-adjoint elliptic (if $AB > 0$) operator

$$A\partial_x^2 + A,_x \partial_x + B\partial_y^2 + B,_y \partial_y - F. \tag{8}$$

It is interesting to remark that the following distinguished gauge equivalent form of the operator $\mathcal{L}_5$ [**4**]

$$\mathcal{L}_{SchInt} := \frac{\Gamma_{m,n}}{\Gamma_{m+1,n}} T_m + \frac{\Gamma_{m-1,n}}{\Gamma_{m,n}} T_m^{-1} + \frac{\Gamma_{m,n}}{\Gamma_{m,n+1}} T_n + \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}} T_n^{-1} - q_{m,n} \tag{9}$$

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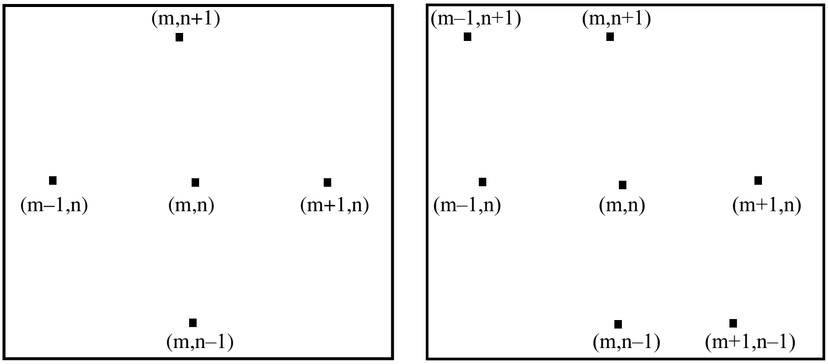

Figure 1. The 5 - and 7 - point schemes on the square lattice

admits MTs and reduces, in the continuous limit, to the celebrated Schrödinger operator in the plane

$$\partial_x^2 + \partial_y^2 - Q. \tag{10}$$

Therefore it can be considered as a distinguished integrable discretization of the Schrödinger operator [**4**].

2. Basic criteria for discretizing partial differential operators. In order to construct the proper discretization of a partial differential operator we are guided by the following criteria.

1. It should possess a large class of (discrete, continuous, isospectral, nonisospectral, ...) symmetries, (at least) as large as that of its differential counterpart.
2. Its spectral properties should be similar to those of its differential counterpart.
3. The discretization should preserve the hyperbolic or elliptic character of the partial differential operator; in particular, if the operator is elliptic, the discretization should be applicable to solve a generic Dirichlet boundary value problem on a 2D lattice.
4. If the continuous operator is geometrically significant, the discretization should possess a geometric meaning which generalizes naturally that of the continuous operator.

In this paper we show that the difference operators (1), (2) are discretizations, respectively, of the partial differential operators (7), (8) that satisfy the properties 1, 3 and 4. The spectral properties of the self-adjoint 5-point scheme in the case of periodic and quasi-periodic potentials are discussed in [**20**].

3. The Dirichlet boundary value problem. As we pointed out in Section 2, a proper discretization of a second order elliptic operator should be applicable to solve Dirichlet boundary value problems on a 2D lattice.

Consider, for the sake of concreteness, the following Dirichlet problem on a bounded domain of $\mathbb{R}^2$ for the operator (8):

$$(A\Psi_{,x})_{,x} + (B\Psi_{,y})_{,y} = F\Psi, \quad (x,y) \in \mathcal{D} \subset \mathbb{R}^2, \quad \Psi(x,y) \text{ given on } \partial\mathcal{D}. \tag{11}$$

This appears very frequently in applications.

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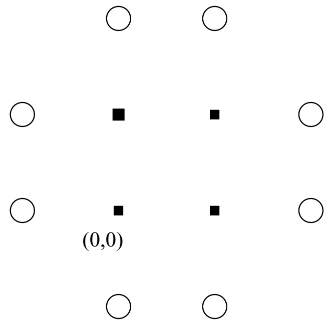

Figure 2. A simple Dirichlet problem for the 5-point scheme

It is easy to convince oneself that the 5-point self-adjoint scheme for the operator $\mathcal{L}_5$ (see equation (2))

$$a_{m,n}\psi_{m+1,n} + a_{m-1,n}\psi_{m-1,n} + b_{m,n}\psi_{m,n+1} + b_{m,n-1}\psi_{m,n-1} = f_{m,n}\psi_{m,n} \tag{12}$$

which, in the natural continuous limit, reduces to the above equation (11), is perfectly adequate to solve a generic Dirichlet boundary value problems on a 2D lattice. The reasoning behind it (wellknown to numerical analysts [**21**]) is clarified by the illustrative example of Figure 2.

Suppose we want to solve the Dirichlet problem associated with the 5-point scheme (12) in the subset of $\mathbb{Z}^2$ consisting of the white and black points in Figure 2. If the field $\psi_{m.n}$ is given at the boundary points (the white points), the unknown values of $\psi_{m.n}$ at the 4 interior points (the black points) are *uniquely* constructed solving the following linear, inhomogeneous, determined system of 4 equations for 4 unknowns:

$$\begin{aligned}
-f_{0,0}\psi_{0,0} + a_{0,0}\psi_{1,0} + b_{0,0}\psi_{0,1} &= -a_{-1,0}\psi_{-1,0} - b_{0,-1}\psi_{0,-1},\\
a_{0,0}\psi_{0,0} - f_{1,0}\psi_{1,0} + b_{1,0}\psi_{1,1} &= -a_{1,0}\psi_{2,0} - b_{1,-1}\psi_{1,-1},\\
b_{0,0}\psi_{0,0} - f_{0,0}\psi_{0,1} + a_{0,1}\psi_{1,1} &= -a_{-1,1}\psi_{-1,1} - b_{0,1}\psi_{0,2},\\
b_{1,0}\psi_{1,0} + a_{0,1}\psi_{0,1} - f_{1,1}\psi_{1,1} &= -a_{1,1}\psi_{2,1} - b_{1,1}\psi_{1,2}
\end{aligned} \tag{13}$$

obtained by applying 4 times the 5-point scheme (12) with center at the interior points.

The same argument holds for more general subsets of $\mathbb{Z}^2$; its only possible failure is associated with the non generic situation in which the relevant matrix determinant of the system (which depends on the coefficients a, b, f) is zero.

The definitions of interior and boundary points used in the illustrative example above are intuitive: the (nearest) neighbourhood of a point (m, n) of the square lattice consists of the four points $(m+1, n)$, $(m, n+1)$, $(m-1, n)$, $(m, n-1)$. Given a subset Ω of $\mathbb{Z}^2$, its interior points are the points of Ω for which all neighbouring points belong to Ω; its boundary points $\partial\Omega$ are instead the points of Ω such that some of the neighbouring points do not belong to Ω.

We remark that the 5-point scheme (12) is, among all possible difference equations adequate to solve Dirichlet problems on 2D lattices, the *simplest* possible scheme.

Using similar considerations, one can show that the 7-point scheme

$$\begin{aligned}
&a_{m,n}\psi_{m+1,n} + a_{m-1,n}\psi_{m-1,n} + b_{m,n}\psi_{m,n+1} + b_{m,n-1}\psi_{m,n-1}\\
&\quad + s_{m+1,n}\psi_{m+1,n-1} + s_{m,n+1}\psi_{m-1,n+1} = f_{m,n}\psi_{m,n},
\end{aligned} \tag{14}$$

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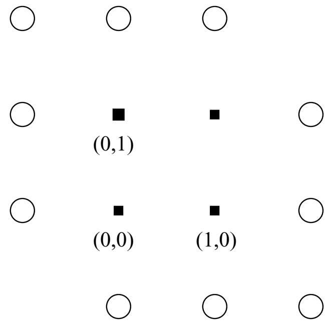

Figure 3. A Dirichlet problem for the 7-point scheme on the square lattice

is applicable to solve Dirichlet problems on a 2D lattice. Notice that, on a square lattice, two white points should be added to the boundary with respect to the 5-point scheme.

4. Moutard transformations. Non isospectral symmetries of Darboux type for linear differential operators play an important role in the theory of nonlinear integrable systems. They allow us, for instance, to construct solutions of these nonlinear systems from simpler solutions through an iterative procedure. As we mentioned in Section 2, a good discretization of a partial differential operator should preserve this type of symmetry.

In this section we present the MTs for the operators $\mathcal{L}_5$ and $\mathcal{L}_7$. These results are extracted from **[4]**.

4.1. MTs for $\mathcal{L}_5$. Consider the operator $\mathcal{L}_5$ together with the associated difference equation

$$a_{m,n}\psi_{m+1,n} + a_{m-1,n}\psi_{m-1,n} + b_{m,n}\psi_{m,n+1} + b_{m,n-1}\psi_{m,n-1} = f_{m,n}\psi_{m,n}, \tag{15}$$

where $a_{m,n}$, $b_{m,n}$ and $f_{m,n}$ are given functions.

The operator $\mathcal{L}_5$ exhibits the following covariance property (gauge invariance):

$$\begin{gathered} \mathcal{L}_5 \to \tilde{\mathcal{L}}_5 = g_{m,n}\mathcal{L}_5 g_{m,n} \\ a_{m,n} \to \tilde{a}_{m,n} = a_{m,n}g_{m,n}g_{m+1,n}, \qquad b_{m,n} \to \tilde{b}_{m,n} = b_{m,n}g_{m,n}g_{m,n+1}, \\ f_{m,n} \to \tilde{f}_{m,n} = f_{m,n}g_{m,n}^2 \end{gathered} \tag{16}$$

and possesses the following MTs.

Let θ be another solution of (15); i.e.

$$a_{m,n}\theta_{m+1,n} + a_{m-1,n}\theta_{m-1,n} + b_{m,n}\theta_{m,n+1} + b_{m,n-1}\theta_{m,n-1} = f_{m,n}\theta_{m,n}; \tag{17}$$

then

$$f_{m,n} = \frac{1}{\theta_{m,n}}\left(a_{m,n}\theta_{m+1,n} + a_{m-1,n}\theta_{m-1,n} + b_{m,n}\theta_{m,n+1} + b_{m,n-1}\theta_{m,n-1}\right). \tag{18}$$

Eliminating $f_{m,n}$ from (15) and (17) we get

$$\begin{aligned} &\Delta_m(a_{m-1,n}\psi_{m,n}\theta_{m-1,n} - a_{m-1,n}\theta_{m,n}\psi_{m-1,n}) \\ &\quad + \Delta_n(b_{m,n-1}\psi_{m,n}\theta_{m,n-1} - b_{m,n-1}\theta_{m,n}\psi_{m,n-1}) = 0, \end{aligned} \tag{19}$$

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where

$$\Delta_m f_{m,n} := f_{m+1,n} - f_{m,n}, \qquad \Delta_n f_{m,n} := f_{m,n+1} - f_{m,n},$$
$$\Delta_{-m} f_{m,n} := f_{m-1,n} - f_{m,n}, \qquad \Delta_{-n} f_{m,n} := f_{m,n-1} - f_{m,n}.$$

It means that there exists a function α such that

$$\begin{aligned} \Delta_n \alpha &= a_{m-1,n}\theta_{m,n}\theta_{m-1,n}\Delta_{-m}\frac{\psi_{m,n}}{\theta_{m,n}}, \\ \Delta_m \alpha &= -b_{m,n-1}\theta_{m,n}\theta_{m,n-1}\Delta_{-n}\frac{\psi_{m,n}}{\theta_{m,n}}, \end{aligned} \tag{20}$$

where

$$\Delta_m f_{m,n} = f_{m+1,n} - f_{m,n}, \qquad \Delta_n f_{m,n} = f_{m,n+1} - f_{m,n},$$
$$\Delta_{-m} f_{m,n} = f_{m-1,n} - f_{m,n}, \qquad \Delta_{-n} f_{m,n} = f_{m,n-1} - f_{m,n}.$$

Setting

$$\psi'_{m,n} = \frac{\alpha_{m,n}}{\theta_{m,n}},$$

we find that $\psi'_{m,n}$ satisfies the following equation

$$a'_{m,n}\psi'_{m+1,n} + a'_{m-1,n}\psi'_{m-1,n} + b'_{m,n}\psi'_{m,n+1} + b'_{m,n-1}\psi'_{m,n-1} = f'_{m,n}\psi'_{m,n}, \tag{21}$$

where

$$a'_{m-1,n} = \frac{\theta_{m,n}}{b_{m-1,n-1}\theta_{m-1,n-1}}, \quad b'_{m,n-1} = \frac{\theta_{m,n}}{a_{m-1,n-1}\theta_{m-1,n-1}} \tag{22}$$

and

$$f'_{m,n} = \theta_{m,n}\left(a'_{m,n}\frac{1}{\theta_{m+1,n}} + a'_{m-1,n}\frac{1}{\theta_{m-1,n}} + b'_{m,n}\frac{1}{\theta_{m,n+1}} + b'_{m,n-1}\frac{1}{\theta_{m,n-1}}\right). \tag{23}$$

Comparing equations (18) and (23), we also infer that $\theta' = 1/\theta$ is a solution of (21). We end this subsection remarking that the superposition principle for the above Moutard transformation for $\mathcal{L}_5$ can be found in [**22**, **20**].

4.2. MTs for $\mathcal{L}_7$. The construction of MTs presented in the previous sub-section applies to the self-adjoint 7-point scheme associated with $\mathcal{L}_7$:

$$\begin{aligned} &a_{m,n}\psi_{m+1,n} + a_{m-1,n}\psi_{m-1,n} + b_{m,n}\psi_{m,n+1} + b_{m,n-1}\psi_{m,n-1} \\ &\quad + s_{m+1,n}\psi_{m+1,n-1} + s_{m,n+1}\psi_{m-1,n+1} = f_{m,n}\psi_{m,n}, \end{aligned} \tag{24}$$

which is a discretization of the most general second order, self-adjoint, linear, differential equation in two independent variables.

Let $\theta_{m,n}$ be another solution of equation (24):

$$\begin{aligned} &a_{m,n}\theta_{m+1,n} + a_{m-1,n}\theta_{m-1,n} + b_{m,n}\theta_{m,n+1} + b_{m,n-1}\theta_{m,n-1} \\ &\quad + s_{m+1,n}\theta_{m+1,n-1} + s_{m,n+1}\theta_{m-1,n+1} = f_{m,n}\theta_{m,n}. \end{aligned} \tag{25}$$

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Eliminating $f_{m,n}$ from (24) and (25) we get

$$\Delta_m \left[a_{m-1,n}\theta_{m,n}\theta_{m-1,n} \left(\frac{\psi_{m,n}}{\theta_{m,n}} - \frac{\psi_{m-1,n}}{\theta_{m-1,n}} \right) + s_{m,n}\theta_{m-1,n}\theta_{m,n-1} \left(\frac{\psi_{m,n-1}}{\theta_{m,n-1}} - \frac{\psi_{m-1,n}}{\theta_{m-1,n}} \right) \right]$$
$$+ \Delta_n \left[b_{m,n-1}\theta_{m,n}\theta_{m,n-1} \left(\frac{\psi_{m,n}}{\theta_{m,n}} - \frac{\psi_{m,n-1}}{\theta_{m,n-1}} \right) \right.$$
$$\left. + s_{m,n}\theta_{m-1,n}\theta_{m,n-1} \left(\frac{\psi_{m-1,n}}{\theta_{m-1,n}} - \frac{\psi_{m,n-1}}{\theta_{m,n-1}} \right) \right] = 0. \tag{26}$$

It means that there exists a function ψ' such that

$$\begin{aligned} \Delta_n(\psi'_{m,n}\theta_{m,n}) &= (a_{m-1,n}\theta_{m,n}\theta_{m-1,n} + s_{m,n}\theta_{m-1,n}\theta_{m,n-1})\, \Delta_{-m}\frac{\psi_{m,n}}{\theta_{m,n}} \\ &\quad - s_{m,n}\theta_{m-1,n}\theta_{m,n-1}\, \Delta_{-n}\frac{\psi_{m,n}}{\theta_{m,n}}, \\ \Delta_m(\psi'_{m,n}\theta_{m,n}) &= -(b_{m,n-1}\theta_{m,n}\theta_{m,n-1} + s_{m,n}\theta_{m-1,n}\theta_{m,n-1})\Delta_{-n}\frac{\psi_{m,n}}{\theta_{m,n}} \\ &\quad + s_{m,n}\theta_{m-1,n}\theta_{m,n-1}\, \Delta_{-m}\frac{\psi_{m,n}}{\theta_{m,n}}. \end{aligned} \tag{27}$$

The function $\psi'_{m,n}$ satisfies the following equation

$$\begin{aligned} a'_{m,n}\psi'_{m+1,n} + a'_{m-1,n}\psi'_{m-1,n} + b'_{m,n}\psi'_{m,n+1} + b'_{m,n-1}\psi'_{m,n-1} \\ + s'_{m+1,n}\psi'_{m+1,n-1} + s'_{m,n+1}\psi'_{m-1,n+1} = f'_{m,n}\, \psi'_{m,n}, \end{aligned} \tag{28}$$

where the new fields are given by

$$\begin{aligned} a'_{m,n} &= \frac{\theta_{m,n}\theta_{m+1,n}a_{m-1,n}}{\theta_{m,n-1}p_{m,n}}, \quad b'_{m,n} = \frac{\theta_{m,n}\theta_{m,n+1}b_{m,n-1}}{\theta_{m-1,n}p_{m,n}}, \quad s'_{m,n} = \frac{s_{m-1,n-1}\theta_{m-1,n}\theta_{m,n-1}}{\theta_{m-1,n-1}p_{m-1,n-1}}, \\ f'_{m,n} &= \theta_{m,n}\left(a'_{m,n}\frac{1}{\theta_{m+1,n}} + a'_{m-1,n}\frac{1}{\theta_{m-1,n}} + b'_{m,n}\frac{1}{\theta_{m,n+1}} + b'_{m,n-1}\frac{1}{\theta_{m,n-1}} \right. \\ &\quad \left. + s'_{m+1,n}\frac{1}{\theta_{m+1,n-1}} + s'_{m,n+1}\frac{1}{\theta_{m-1,n+1}} \right) \end{aligned} \tag{29}$$

and where $p_{m,n} = \theta_{m,n}a_{m-1,n}b_{m,n-1} + \theta_{m-1,n}s_{m,n}a_{m-1,n} + s_{m,n}\theta_{m,n-1}b_{m,n-1}$. Again $\theta'_{m,n} = 1/\theta_{m,n}$ is a solution of (28).

5. Lelieuvre formulae. Lelieuvre's idea of describing the surface parametrized with asymptotic coordinates via its co-normal image [**5**, **23**] plays an important role in the theory of surfaces in equiaffine spaces. Due to the generalizations of Lelieuvre formulae to a coordinate free language [**10**, **11**], one can describe the hyper-surface via its co-normal image in an arbitrary coordinate system.

In this section we show that the co-normal image of a general 2D surface in eA^3 is a vector solution of the partial differential equation

$$(A\Psi,_x)_x + (S\Psi,_y),_x + (B\Psi,_y),_y + (S\Psi,_x),_y = F\Psi \tag{30}$$

associated with the operator (7).

We recall some basic facts. We denote by $\vec{R}$ the position vector $\vec{R} : \mathbb{R}^2 \to eA^3$ of a parametrized surface in eA^3 and we assume that the surface

(i) is regular, i.e.

$$\hat{N} \propto \vec{R},_x \times \vec{R},_y \neq 0; \tag{31}$$

(ii) is twice differentiable; i.e., in particular

$$\vec{R},_{xy} = \vec{R},_{yx}; \tag{32}$$

(iii) is locally strongly convex, so that

$$Vol^*(\hat{N};\hat{N},_x;\hat{N},_y) \neq 0. \tag{33}$$

Then, from (33), we infer that the vector fields $\hat{N} \times \hat{N},_x$ and $\hat{N} \times \hat{N},_y$ are linearly independent and are tangent fields to the surface. Therefore one can decompose the fields $\vec{R},_x$ and $\vec{R},_y$ as follows:

$$\begin{aligned} \vec{R},_y &= A\hat{N} \times \hat{N},_x + P\hat{N} \times \hat{N},_y\,, \\ \vec{R},_x &= -Q\hat{N} \times \hat{N},_x - B\hat{N} \times \hat{N},_y\,, \end{aligned} \tag{34}$$

where, since $\vec{R},_x \times \vec{R},_y = (AB - PQ)Vol^*(\hat{N};\hat{N},_x;\hat{N},_y)\hat{N}$, due to assumption (31), we have $AB - PQ \neq 0$. The equality $\langle \hat{N} \mid \vec{R},_{xy} - \vec{R},_{yx} \rangle = 0$ gives $(P - Q)Vol^*(\hat{N};\hat{N},_x;\hat{N},_y) = 0$, so that we have $P = Q =: S$ and, finally,

$$\begin{aligned} \vec{R},_y &= A\hat{N} \times \hat{N},_x + S\hat{N} \times \hat{N},_y\,, \\ \vec{R},_x &= -S\hat{N} \times \hat{N},_x - B\hat{N} \times \hat{N},_y\,, \end{aligned} \tag{35}$$

$$AB - S^2 \neq 0. \tag{36}$$

The compatibility condition $\vec{R},_{xy} = \vec{R},_{yx}$ of equations (35) leads to the partial differential equation

$$(A\hat{N},_x)_x + (S\hat{N},_y),_x + (B\hat{N},_y),_y + (S\hat{N},_x),_y = F\hat{N} \tag{37}$$

associated with the operator (7), which is nothing but the most general self-adjoint equation of second order in two independent variables.

Conversely, let N_1, N_2 and N_3 be three linearly independent solutions of the self-adjoint equation (30), which we assume to be non parabolic; i.e.

$$AB - S^2 \neq 0. \tag{38}$$

Select any frame in eA^3 and the vector field $\hat{N} = [N_1, N_2, N_3]$ with respect to the co-frame. Since N_1, N_2 and N_3 are linearly independent, we have that $Vol^*(\hat{N};\hat{N},_x;\hat{N},_y) \neq 0$. In addition $\hat{N}$ satisfies equation (37); the vector multiplication of both sides of this equation for $\hat{N}$ by $\hat{N}$ itself yields, after manipulation, the equation

$$(A\hat{N} \times \hat{N},_x + S\hat{N} \times \hat{N},_y),_x + (S\hat{N} \times \hat{N},_x + B\hat{N} \times \hat{N},_y),_y = 0,$$

from which we infer that there exists a vector field $\vec{R}$ such that equations (35) hold. Interpreting $\vec{R}$ as the position vector of a surface, we infer that $\hat{N}$ is a co-normal field to this surface and that this surface is regular, since

$$\vec{R},_x \times \vec{R},_y = (AB - S^2)Vol^*(\hat{N};\hat{N},_x;\hat{N},_y)\hat{N}.$$

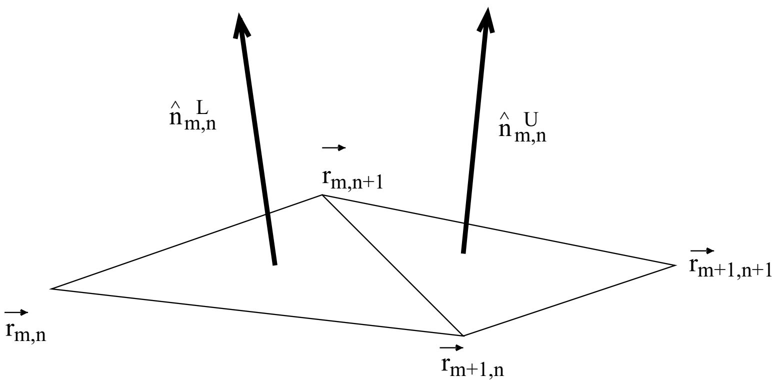

Figure 4. Upper and lower triangles of the 2D lattice

6. Lelieuvre formulae associated with the 7-point scheme. In the previous section we have shown that the operator (7) characterizes the co-normal image of a generic surface in eA^3. According to the last criterion of Section 2, a good discretization of (7) should possess an analogous geometric meaning. Indeed in this section we shall show that the difference operator $\mathcal{L}_7$ describes the co-normal image of a generic 2D lattice in eA^3.

Consider a lattice $\mathbb{Z}^2 \supset \Omega \cup \partial\Omega \to eA^3$ and denote by $\vec{r}_{m,n}$ the position vector with respect to a frame. By "lower" triangles we mean the triangles with vertices $(\vec{r}_{m,n}, \vec{r}_{m+1,n}, \vec{r}_{m,n+1})$ and by "upper" triangles we mean the triangles with vertices $(\vec{r}_{m+1,n+1}, \vec{r}_{m+1,n}, \vec{r}_{m,n+1})$.

We make the following assumptions.

(A) The upper and lower triangles are not degenerate; i.e., the three points of each triangle are not collinear. For the lower triangles of the 2D lattice this condition means that

$$\Delta_m \vec{r}_{m,n} \times \Delta_n \vec{r}_{m,n} \neq 0. \tag{39}$$

Then we denote by $\hat{n}^{L}_{m,n}$ any co-normal non-vanishing field to the lower triangles:

$$\hat{n}^{L}_{m,n} := \lambda^{L}_{m,n} \Delta_m \vec{r}_{m,n} \times \Delta_n \vec{r}_{m,n},$$

where $\lambda^{L}_{m,n}$ is a non-vanishing scalar field. Analogously, the non degeneracy condition

$$\Delta_m \vec{r}_{m,n+1} \times \Delta_n \vec{r}_{m+1,n} \neq 0 \tag{40}$$

for the upper triangles of the 2D lattice allows one to define any co-normal non-vanishing field $\hat{n}^{U}_{m,n}$ to the upper triangles by

$$\hat{n}^{U}_{m,n} := \lambda^{U}_{m,n} \Delta_m \vec{r}_{m,n+1} \times \Delta_n \vec{r}_{m+1,n},$$

where $\lambda^{U}_{m,n}$ is a non-vanishing scalar field.

(B) The fields $\hat{n}^{L}_{m,n}$ and $\hat{n}^{U}_{m,n}$ satisfy the following conditions:

$$V^{L}_{m,n} := Vol^{*}\left(\hat{n}^{L}_{m,n}, \hat{n}^{L}_{m+1,n}, \hat{n}^{L}_{m,n+1}\right) \neq 0, \tag{41}$$

$$V^{U}_{m,n} := Vol^{*}\left(\hat{n}^{U}_{m,n}, \hat{n}^{U}_{m-1,n}, \hat{n}^{U}_{m,n-1}\right) \neq 0. \tag{42}$$

From assumption (41) it follows that the discrete vector fields $\hat{n}^L_{m,n} \times \hat{n}^L_{m-1,n}$ and $\hat{n}^L_{m,n} \times \hat{n}^L_{m-1,n+1}$ are linearly independent and therefore they span the tangent space to the lower triangle (the same is true for fields $\hat{n}^L_{m,n} \times \hat{n}^L_{m,n-1}$ and $\hat{n}^L_{m,n} \times \hat{n}^L_{m+1,n-1}$). Thus we can write

$$\begin{aligned} \Delta_n \vec{r}_{m,n} &= \hat{n}^L_{m,n} \times \left(a_{m-1,n}\hat{n}^L_{m-1,n} + p_{m-1,n}\hat{n}^L_{m-1,n+1}\right), \\ \Delta_m \vec{r}_{m,n} &= -\hat{n}^L_{m,n} \times \left(b_{m,n-1}\hat{n}^L_{m,n-1} + q_{m,n-1}\hat{n}^L_{m+1,n-1}\right). \end{aligned} \tag{43}$$

The equality $\langle \hat{n}^L_{m,n} \mid \Delta_m \Delta_n \vec{r}_{m,n}\rangle = \langle \hat{n}^L_{m,n} \mid \Delta_n \Delta_m \vec{r}_{m,n}\rangle$ is equivalent to $(p-q)V^L_{m,n} = 0$ and so we have (taking into account (41)) $p = q =: s$ and, as a result of this,

$$\begin{aligned} \Delta_n \vec{r}_{m,n} &= \hat{n}^L_{m,n} \times \left(a_{m-1,n}\hat{n}^L_{m-1,n} + s_{m-1,n}\hat{n}^L_{m-1,n+1}\right), \\ \Delta_m \vec{r}_{m,n} &= -\hat{n}^L_{m,n} \times \left(b_{m,n-1}\hat{n}^L_{m,n-1} + s_{m,n-1}\hat{n}^L_{m+1,n-1}\right), \end{aligned} \tag{44}$$

where the coefficients a, b, s are defined by:

$$\begin{aligned} a_{m,n} &= -\frac{\left\langle \hat{n}^L_{m,n+1} \,\middle|\, \Delta_n \vec{r}_{m+1,n}\right\rangle}{V^L_{m,n}}, \qquad b_{m,n} = -\frac{\left\langle \hat{n}^L_{m+1,n} \,\middle|\, \Delta_n \vec{r}_{m,n+1}\right\rangle}{V^L_{m,n}}, \\ s_{m,n} &= \frac{\left\langle \hat{n}^L_{m,n} \,\middle|\, \Delta_n \vec{r}_{m+1,n}\right\rangle}{V^L_{m,n}} = \frac{\left\langle \hat{n}^L_{m,n} \,\middle|\, \Delta_m \vec{r}_{m,n+1}\right\rangle}{V^L_{m,n}}. \end{aligned} \tag{45}$$

From $\Delta_m \Delta_n \vec{r}_{m,n} = \Delta_n \Delta_m \vec{r}_{m,n}$ we finally get that the lower co-normal vector satisfies the self-adjoint 7-point scheme

$$\begin{aligned} &a_{m,n}\hat{n}^L_{m+1,n} + a_{m-1,n}\hat{n}^L_{m-1,n} + b_{m,n}\hat{n}^L_{m,n+1} + b_{m,n-1}\hat{n}^L_{m,n-1} \\ &\quad + s_{m-1,n}\hat{n}^L_{m-1,n+1} + s_{m,n-1}\hat{n}^L_{m+1,n-1} = f_{m,n}\hat{n}^L_{m,n}. \end{aligned} \tag{46}$$

Consider now the co-vector fields:

$$\begin{aligned} X_{m,n} &:= a_{m,n}\hat{n}^L_{m+1,n} + b_{m,n}\hat{n}^L_{m,n+1}, \\ Y_{m,n} &:= a_{m-1,n}\hat{n}^L_{m-1,n} + s_{m-1,n}\hat{n}^L_{m-1,n+1}, \\ Z_{m,n} &:= b_{m,n-1}\hat{n}^L_{m,n-1} + s_{m,n-1}\hat{n}^L_{m+1,n-1}. \end{aligned}$$

Then equation (46) can be re-written in these terms:

$$X_{m,n} + Y_{m,n} + Z_{m,n} = f_{m,n}\hat{n}^L_{m,n}. \tag{47}$$

A direct calculation shows that

$$\Delta_m \vec{r}_{m,n} \times \Delta_n \vec{r}_{m,n} = -V^L_{m,n}\, Vol^*\left(\hat{n}^L_{m,n}; Y_{m,n}; Z_{m,n}\right)\hat{n}^L_{m,n}, \tag{48}$$

from which we infer that

$$Vol^*\left(\hat{n}^L_{m,n}; Y_{m,n}; Z_{m,n}\right) \neq 0. \tag{49}$$

For the normal to the upper triangle, we have

$$\begin{aligned} \hat{n}^U_{m,n} &= \lambda^U_{m,n} \Delta_n \vec{r}_{m+1,n} \times \Delta_m \vec{r}_{m,n+1} \\ &= \lambda^U_{m,n} V^L_{m,n}\left(a_{m,n}b_{m,n}\hat{n}^L_{m,n} + a_{m,n}s_{m,n}\hat{n}^L_{m+1,n} + b_{m,n}s_{m,n}\hat{n}^L_{m,n+1}\right), \end{aligned} \tag{50}$$

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so that

$$a_{m,n}b_{m,n} \neq 0 \qquad \text{or} \qquad a_{m,n}s_{m,n} \neq 0 \qquad \text{or} \qquad b_{m,n}s_{m,n} \neq 0, \tag{51}$$

$$\begin{aligned} V^U_{m,n} &= \lambda^U_{m,n} V^L_{m,n} \lambda^U_{m-1,n} V^L_{m-1,n} \lambda^U_{m,n-1} V^L_{m,n-1} Vol^*\left(\hat{n}^L_{m,n}; Y_{m,n}; Z_{m,n}\right) \\ &\quad * \left(a_{m,n}a_{m-1,n}b_{m,n}b_{m,n-1} + a_{m,n-1}a_{m-1,n}s_{m,n}b_{m-1,n} + b_{m,n-1}b_{m-1,n}s_{m,n}s_{m,n-1}\right). \end{aligned} \tag{52}$$

Therefore

$$a_{m,n}a_{m-1,n}b_{m,n}b_{m,n-1} + a_{m,n-1}a_{m-1,n}s_{m,n}s_{m-1,n} + b_{m,n-1}b_{m-1,n}s_{m,n}s_{m,n-1} \neq 0. \tag{53}$$

Summarizing, we have the following result.

THEOREM 1. *Consider a two-dimensional lattice $\mathbb{Z} \ni \Omega \to eA^3$ such that its position vector $\vec{r}_{m,n}$ and its lower $\hat{n}^L_{m,n}$ and upper $\hat{n}^U_{m,n}$ co-normals obey the conditions* (39), (40), (41) *and* (42)*. Then there exist functions $a_{m,n}$, $b_{m,n}$ and $s_{m,n}$ obeying conditions* (51), *and* (53) *such that the following Lelieuvre type relations hold*

$$\begin{aligned} \Delta_n \vec{r}_{m,n} &= \hat{n}^L_{m,n} \times \left(a_{m-1,n}\hat{n}^L_{m-1,n} + s_{m-1,n}\hat{n}^L_{m-1,n+1}\right), \\ \Delta_m \vec{r}_{m,n} &= -\hat{n}^L_{m,n} \times \left(b_{m,n-1}\hat{n}^L_{m,n-1} + s_{m,n-1}\hat{n}^L_{m+1,n-1}\right), \end{aligned} \tag{54}$$

and such that the lower co-normal field satisfies the 7-point self-adjoint scheme

$$\begin{aligned} & a_{m,n}\hat{n}^L_{m+1,n} + a_{m-1,n}\hat{n}^L_{m-1,n} + b_{m,n}\hat{n}^L_{m,n+1} + b_{m,n-1}\hat{n}^L_{m,n-1} \\ &\quad + s_{m-1,n}\hat{n}^L_{m-1,n+1} + s_{m,n-1}\hat{n}^L_{m+1,n-1} = f_{m,n}\hat{n}^L_{m,n}. \end{aligned} \tag{55}$$

Conversely, consider the field $\hat{n}^L_{m,n}$ satisfying: (*i*) *equation* (55) *with the coefficients obeying conditions* (51) *and* (53)*;* (*ii*) *the conditions* (41) *and* (49)*. Then the Lelieuvre type formulae* (54) *define the position vector $\vec{r}_{m,n}$ of a 2D lattice in eA^3, having $\hat{n}^L_{m,n}$ as a lower co-normal. The position vector, the lower co-normal and the upper co-normal $\hat{n}^U_{m,n}$ given by $\hat{n}^U_{m,n} := \lambda^U_{m,n}\Delta_m\vec{r}_{m,n+1} \times \Delta_n\vec{r}_{m+1,n}$ satisfy the conditions* (39), (40) *and* (42).

7. Lelieuvre formulae associated with the self-adjoint 5-point scheme. In the previous two sections we have shown that, on the level of the Lelieuvre type description, the operator (7) and its discretization $\mathcal{L}_7$ characterize respectively a generic 2D coordinate net and a generic 2D lattice in eA^3. In this section we introduce distinguished reductions on the above generic nets (lattices), showing that (i) the reduction from a generic net to a conjugate net (a surface parametrized by conjugate coordinates) is characterized, on the level of the Lelieuvre type description, by the reduction from the general self-adjoint partial differential operator (7) to the self-adjoint operator (8); (ii) the reduction from a generic 2D lattice to a 2D quadrilateral lattice (a lattice whose elementary quadrilaterals are planar) is characterized, on the level of the Lelieuvre type description, by the reduction from the 7-point scheme (14) to the 5-point scheme (12).

Let $\vec{R}(x, y)$ be the position vector of a conjugate net and let $\hat{N}(x, y)$ be any co-normal vector field; then we have

$$\langle \hat{N} \mid \vec{R}_x \rangle = 0 = \langle \hat{N} \mid \vec{R}_y \rangle \quad \text{(by definition of } \hat{N}\text{)},$$
$$\langle \hat{N} \mid \vec{R}_{xy} \rangle = 0 \quad \text{(by conjugacy)}.$$

Therefore, assuming that the surface is locally strongly convex (condition (33)), one infers that the following Lelieuvre type formulae hold:

$$\vec{R}_x = B\hat{N}_y \times \hat{N}, \quad \vec{R}_y = A\hat{N} \times \hat{N}_x, \tag{56}$$

where

$$B(x, y) = \frac{\langle \hat{N}_x \mid \vec{R}_x \rangle}{Vol^*(\hat{N}, \hat{N}_x, \hat{N}_y)}, \quad A(x, y) = \frac{\langle \hat{N}_y \mid \vec{R}_y \rangle}{Vol^*(\hat{N}, \hat{N}_x, \hat{N}_y)}. \tag{57}$$

The integrability condition $\vec{R}_{xy} = \vec{R}_{yx}$ implies the equation

$$(A\hat{N}_{xx} + A_x\hat{N}_x + B\hat{N}_{yy} + B_y\hat{N}_y) \times \hat{N} = 0, \tag{58}$$

which is equivalent to the desired self-adjoint equation

$$(A\hat{N}_x)_x + (B\hat{N}_y)_y = F\hat{N}, \quad F = F(x, y). \tag{59}$$

Conversely, it is straightforward to prove that, if $\hat{N}$ satisfies equation (59) for some coefficients A, B, F, then the vector $\vec{R}$, defined by (56), is the position vector of a conjugate net having $\hat{N}$ as normal vector.

For the discrete case we follow the same reasoning. Any co-normal vector field of a quadrilateral lattice satisfies the equations

$$\begin{gathered}\langle \hat{N}_{m,n} \mid \Delta_m \vec{r}_{m,n} \rangle = 0 = \langle \hat{N}_{m,n} \mid \Delta_n \vec{r}_{m,n} \rangle \quad \text{(by definition of } \hat{N}\text{)},\\ \langle \hat{N}_{m,n} \mid \Delta_m \Delta_n \vec{r}_{m,n} \rangle = 0 \quad \text{(by quadrilaterality)}.\end{gathered}$$

Therefore, assuming that

$$Vol^*(\hat{N}_{m,n}, \hat{N}_{m+1,n}, \hat{N}_{m,n+1}) \neq 0, \tag{60}$$

the following Lelieuvre type relations exist between the tangent vectors and the co-normal to the lattice

$$\Delta_m \vec{r}_{m,n} = -b_{m,n-1}\hat{N}_{m,n} \times \hat{N}_{m,n-1}, \quad \Delta_n \vec{r}_{m,n} = a_{m-1,n}\hat{N}_{m,n} \times \hat{N}_{m-1,n}, \tag{61}$$

where the scalar fields $a_{m,n}$ and $b_{m,n}$ are defined by:

$$a_{m,n} = -\frac{\langle \hat{N}_{m,n+1} \mid \Delta_n \vec{r}_{m+1,n} \rangle}{Vol^*(\hat{N}_{m,n}, \hat{N}_{m+1,n}, \hat{N}_{m,n+1})}, \quad b_{m,n} = \frac{\langle \hat{N}_{m+1,n} \mid \Delta_m \vec{r}_{m,n+1} \rangle}{Vol^*(\hat{N}_{m,n}, \hat{N}_{m+1,n}, \hat{N}_{m,n+1})}. \tag{62}$$

The integrability condition for equations (61) implies the equation

$$(a_{m,n}\hat{N}_{m+1,n} + a_{m-1,n}\hat{N}_{m-1,n} + b_{m,n}\hat{N}_{m,n+1} + b_{m,n-1}\hat{N}_{m,n-1}) \times \hat{N}_{m,n} = 0, \tag{63}$$

which is equivalent to equation $\mathcal{L}_5\hat{N} = 0$.

Conversely, it is also easy to show that, if $\hat{N}_{m,n}$ satisfies equation (63), then the Lelieuvre formulae (61) define a proper embedding of a 2D quadrilateral lattice having $\hat{N}_{m,n}$ as co-normal.

We remark that this result could have been deduced in a faster way; i.e. from (45c), observing that the reduction from $\mathcal{L}_7$ to $\mathcal{L}_5$, expressed by the equation $s_{m,n} = 0$, is

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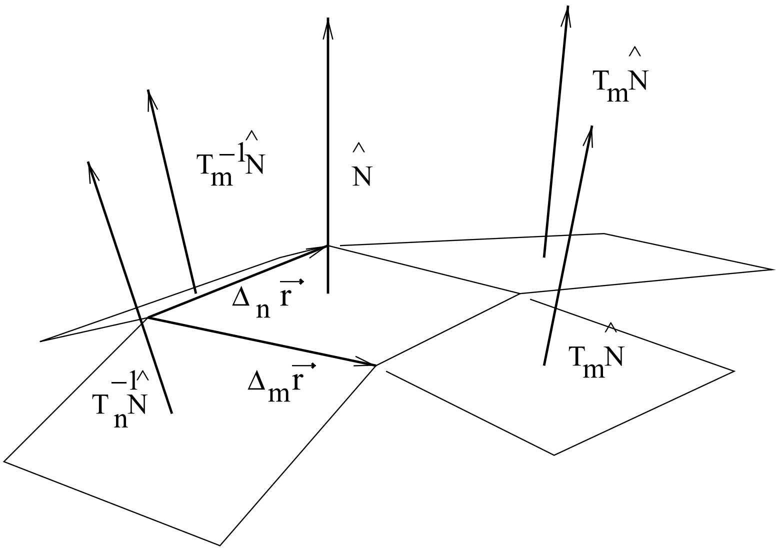

Figure 5. The 5-point scheme for the normal vector

equivalent to the condition that the tangent vectors of the upper triangles of the 2D lattice are perpendicular to $\hat{n}^L_{m,n}$; i.e., it is equivalent to the condition that the 2D lattice is quadrilateral.

We conclude this section by remarking that, due to the gauge covariance properties of the operators (8) and $\mathcal{L}_5$ (see (16) and its continuous limit), the normal vectors appearing in the characterizing equations (59) and $\mathcal{L}_5\hat{N} = 0$ have an arbitrary normalization. The characterizations (see, e.g., [**24**])

$$\hat{N}_{xy} + \alpha\hat{N}_x + \beta\hat{N}_y = 0,$$
$$\Delta_m\Delta_n\hat{N}_{m,n} + \alpha\Delta_m\hat{N}_{m,n} + \beta\Delta_n\hat{N}_{m,n} = 0$$

of respectively a 2D conjugate net and of a 2D quadrilateral lattice in terms of their co-normal vectors are instead gauge dependent.

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