Conditions for a plane projective metric to be a norm

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Let R be a metrization with distance xy of an open convex
set D in the 2-dimensional real affine plane such that
$xy + yz = xz$ whenever x, y, z lie on an affine line with y
between x and z and such that all the balls $px \leq \rho$ are
compact. The study of such metrics, called open plane
projective metrics falls under the topic of Hilbert's Problem IV
of his famous mathematical problems. In this paper it is proved
that if in R the sets of points equidistant from lines lie
again on lines then D must be the entire affine plane and the
distance must in fact be a norm. The paper contributes to and
gives extensions of similar results proved earlier. The novel
features of the present result are that in the space
collinearity of points x, y, z is taken only as a sufficient
condition for the equality $xy + yz = xz$. Consequently the
solution encompasses *all* normed linear planes, that is, norms
whose unit circles are not necessarily strictly convex are also
admitted.

1. The background

The idea of using a flatness condition on the equidistant loci is due
to Funk [4]. However, he considered only the case when D is either the
entire affine plane or the interior of a strictly convex closed curve.
Also his metric was finslerian. He showed that we obtain a minkowskian
(normed linear) geometry or his "Geometrie der spezifischen

Massbestimmung", see [3, p. 232], when equidistant loci are lines locally,
that is an affine segment S in the space has a convex neighbourhood U
such that the points x in U at a constant distance from the line
containing S lie on lines. Phadke [5] considered the problem in n
dimensions in the context of a straight desarguesian G-space (that is,
the space R as above except that $xy + yz = xz$ if and only if x, y, z
lie on an affine segment in that order). This formulation excluded all
normed linear spaces whose unit balls are not strictly convex because in a
straight G-space spheres are strictly convex whenever they are convex.
Recently Busemann [2] generalized his G-spaces to include spaces in which
segments are not unique even locally. These spaces are called "chord
spaces". In a straight chord space convexity of balls does not always
imply strict convexity. The space R of the present paper is a straight
desarguesian chord space. Thus we show that the property of flatness of
equidistant loci can be used to characterize the class of *all* normed linear
planes among the wider class of all straight desarguesian chord planes.

2.   Statement of the theorem

Let R be a space as defined at the beginning of the paper. To
formulate our result, for a set M and a point p define
$Mp = \inf\{mp \mid m \in M\}$ and for $\alpha \geq 0$ the equidistant locus
$E(M, \alpha) = \{x \mid Mx = \alpha\}$. Denoting the two sides of a line L by $\sigma^{\pm}(L)$,
we define $E^{\pm}(L, \alpha) = E(L, \alpha) \cap \sigma^{\pm}(L)$. We prove the following.

THEOREM. *The space R is a normed linear space if and only if the
equidistants of affine lines are again affine lines.*

A word about notation. Whenever $Mp = qp$ we say that q is a foot
of p on M. The notation (pqr) means that p, q, r satisfy
$pq + qr = pr$. When p, q, r are points of an affine segment with q
lying between p and r we will write $[pqr]$. The symbol $F(p, L)$
stands for the set of feet of p on L. $Q(a, b)$ denotes the affine
segment joining a and b.

3.   Convexity of circles and the perpendicularity

To prove the theorem stated above the major step is to show that D
must be the entire affine plane. We first show that the circles must be

convex, that is, whenever u and v lie in the disc $px \leq \rho$ the affine segment joining u and v also lies in that disc. Note first that $L_1 = E(L_2, \alpha)$ implies $L_2 = E(L_1, \alpha)$ because choosing $x \in L_1$, $y \in L_2$ with $xy = \alpha$ we draw L_3 through y such that $L_3 = E(L_1, \alpha)$. If L_3 did not coincide with L_2 then L_3 would contain points z on the side of L_2 not containing L_1 . All these points are at a distance greater than α which is a contradiction to $z \in E(L_1, \alpha)$.

We use this fact to prove that the set of feet of a point on a line is a connected set. Let $x, y \in F(p, L)$, $xp = yp = \alpha$. If $L' = E(L, \alpha)$ with $p \in L$ and $[xmy]$ then m has a foot n on L' . $Q(m, n)$ meets either $Q(p, y)$ or $Q(p, x)$, say, $Q(p, y)$ in l . If $lm > ly$ we would have $ny \leq nl+ly < nl+lm = nm = \alpha$. But $ny < \alpha$ contradicts the symmetry of "being equidistant" which was proved above. Hence $lm \leq ly$. Consequently $\alpha \leq pm \leq pl+lm \leq pl+ly = py = \alpha$. Thus m is also a foot of p on L . This proves that $F(p, L)$ is a connected set for all p and all L . A result of Busemann [2, p. 115] now implies that all the circles $px \leq \rho$ are convex.

A line H is said to be perpendicular to the line L if H intersects L in a point f and each point of H has f as its foot on L . Busemann's results [2, p. 115] again imply that when all the circles are convex each $p \notin L$ is the origin of perpendiculars whose union intersects L in a point or an affine segment. We note that although perpendiculars exist, they are not unique (which was the case in Phadke [5]).

4.   Classification of the domains D

Although the perpendiculars are not unique the following still holds. Given two equidistants L_1 and L_2 and a point x between them there exists a common perpendicular to L_i through x . In fact let H be a perpendicular through x to L_1 . Let H meet L_1 in p and suppose $L_2 = E(L_1, \alpha)$. Take $q \in H$ with $qp = \alpha$. Since H is perpendicular to L_1 , p is a foot of q on L_1 . Hence q is at a distance α from

L_1 . Therefore q lies on L_2 . Now by the symmetry of the relation of "being equidistant" q is also a foot of p on L_2 . Hence the line joining q to p , that is H , is perpendicular to L_2 also. (See Busemann [2, p. 115].)

In virtue of the above observation no extreme point of the affine closure $\overline{D}$ of D can lie between two equidistants. Hence we can prove in the same manner as in Phadke [5, p. 315] that D must either be a triangle or a strip enclosed by two affine parallel lines or the entire affine plane.

5. D is the entire affine plane

In this section we prove that our geometry cannot be defined inside a triangle or a strip. The proof of the impossibility of the strip as given in Phadke [5, p. 315, 316] does not need any change except one writes "a perpendicular" in place of "*the* perpendicular". Hence we consider only the case of a triangle. Even in this case we can prove as in [5, p. 315] that all equidistants of a line through a vertex must again pass through the same vertex and that if a line does not pass through a vertex then all the perpendiculars to that line pass through a vertex. However, the rest of the proof given there cannot be carried through without modifications because that proof used the continuous change of feet and perpendiculars which is not available to us now because the feet and the perpendiculars are no longer unique.

Nevertheless we prove that lines through one vertex are perpendicular to lines through another vertex. Let a, b, c be the vertices of the triangle D . Let L be a line passing through the vertex a and let $b \in \sigma^-(L)$ and $c \in \sigma^+(L)$. Let $x \in \sigma^+(L)$ have $p \in L$ as its foot. If $[cxp]$ does not hold let the line joining c and x meet L in q . We prove $xq = xp$. For if $xp < xq$ we can find a point u with $[xuq]$ such that $xp < xu$ and the line L' joining u and p does not pass through a vertex. But then the perpendicular to L' through x must pass through c which is a contradiction to $xu > xp$. Thus lines passing through a vertex are always perpendiculars. This implies that the line joining c and p is perpendicular to all other lines through p . This

is clearly impossible because otherwise a circle through p would shrink to a segment on $Q(c, p)$. Thus the geometry cannot be defined inside a triangle. As noted at the beginning of this section this implies that the geometry can only be defined in the entire affine plane.

6. The metric is a norm

Since the geometry is defined in the entire affine plane, the equidistant of a line is its affine parallel. We will show that every metric circle $px = \rho$ possesses parallel supporting lines at the endpoints of every chord (diameter) through the centre. In fact if L is a supporting line to the circle to an endpoint a of a diameter H then H is a perpendicular to L . Consequently the equidistant to L through the other end of H is also a supporting line of the circle. Since equidistant lines are affine parallels we have proved our assertion. By a result of Busemann [1, p. 91] we know that a closed convex curve C has a point z inside it as its affine centre if C possesses parallel supporting lines at the endpoints of every chord through z . Hence in our geometry the metric centre p of a circle $px = \rho$ is also its affine centre. Therefore for any segment the metric midpoint is the same as its affine midpoint. It is well known, see Busemann [1, p. 94], that this implies that the metric is a norm. This completes the proof of our theore theorem.

References

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