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# ON INTEGRAL ABEL-TYPE AND LOGARITHMIC METHODS OF SUMMABILITY

BY

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1. **Introduction.** In this paper, we define an integral logarithmic method of summability, extending the integral Abel-type methods defined by Jakimovski [6]. We examine the behaviour of the product of this method with integral Hausdorff methods. A full scale of strict inclusions for integral Abel-type methods is obtained and the integral logarithmic method is placed in this scale. For the analogous theorems for sequence-to-sequence Abel-type and logarithmic methods, see Borwein [1], [2].

We use the notation $M_1 \supseteq M_2$ to mean that any sequence or function summable by the method $M_2$ is also summable by the method $M_1$ to the same limit. If $M_1 \supseteq M_2$ and $M_2 \supseteq M_1$, then we write $M_1 \simeq M_2$. The notation $M_1 = M_2$ indicates that the methods are equal; that is, their transforms are the same.

Throughout this paper, we require that $f$ be a real function which is bounded and Borel-measurable on every finite interval $[0, X]$. We shall suppose that $\sigma$ is real, that $\alpha > 0$, that $\gamma > 0$, and that $\beta > -1$.

2. **Integral Hausdorff methods of summability.** Integral (or continuous) Hausdorff methods of summability have been defined by Rogosinski [8] (see also [6]).

Let $\chi$ be a function of bounded variation on $[0, 1]$. We extend $\chi$ to the entire real line by defining $\chi(t) = \chi(0)$ for $t < 0$, and $\chi(t) = \chi(1)$ for $t > 1$. For $y > 0$, let $\mathbf{H}_\chi(y) = \mathbf{H}_{f;\chi}(y) = \int_0^1 f(yt)\, d\chi(t)$, where the integral is the Lebesgue–Stieltjes integral over $[0, 1]$. Under our assumptions, $\mathbf{H}_\chi(y)$ exists for all $y > 0$. If $\mathbf{H}_\chi(y) \to \sigma$ as $y \to \infty$, then we say that $f$ is $\mathbf{H}_\chi$-summable to $\sigma$, and write $f(x) \to \sigma(\mathbf{H}_\chi)$.

We may assume that for $0 < t < 1$, $\chi(t) = \frac{1}{2}\{\chi(t-) + \chi(t+)\}$, and that $\chi(0) = 0$. Hence each integral Hausdorff method corresponds to a unique $\chi$. Such a $\chi$ also generates a sequence-to-sequence Hausdorff method which we denote by $H_\chi$ (see, for example, [5, Ch. XI]). The conditions for the regularity of the integral and of the sequence-to-sequence Hausdorff methods are identical ([7], [8]; see also [5, Ch. XI]): $\mathbf{H}_\chi$ (or $H_\chi$) *is regular if and only if* $\chi(0+) = \chi(0)$ *and* $\chi(1) - \chi(0) = 1$.

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Certain other properties of the familiar sequence-to-sequence Hausdorff methods are also found in their functional analogues. For example, we have ([8]; cf. [7] or [5, Thm. 197]) $\mathbf{H}_\chi\mathbf{H}_\psi=\mathbf{H}_\psi\mathbf{H}_\chi$.

For $n=1,2,\ldots$, the *moments of order n* of $\mathbf{H}_\chi$ (or of $H_\chi$) are given by $\mu_n=\int_0^1 t^n\,d\chi(t)$. Rogosinski [8] has proved the following result.

THEOREM A. *If $H_\chi\supseteq H_\psi$ and at most finitely many of the moments of $H_\psi$ vanish, then $\mathbf{H}_\chi\supseteq\mathbf{H}_\psi$.*

3. **Integral Cesàro-type methods.** A specific class of integral Hausdorff methods is the integral Cesàro-type methods (see, for example, [2]). For $y>0$, let

$$C_{\alpha,\beta}(y)=C_{f;\alpha,\beta}(y)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^1(1-t)^{\alpha-1}t^\beta f(yt)\,dt.$$

If $C_{\alpha,\beta}(y)\to\sigma$ as $y\to\infty$, then we write $f(x)\to\sigma$ $(C,\alpha,\beta)$. This is a regular summability method. The integral Cesàro method $(C,\alpha)$ is defined to be $(C,\alpha,0)$ (see, for example, [5, p. 110] or [8]). The sequence-to-sequence analogues are well known [4], [5, Ch. V]. It is easy to show (using results in [4] and Theorem A) that $(C,\alpha,\beta)\simeq(C,\alpha)$, that $(C,\alpha,\gamma)(C,\gamma)=(C,\alpha+\gamma)$, and that $(C,\alpha+\gamma)\supseteq(C,\gamma)$, all analogues of results in [4].

4. **Integral Abel-type summability.** Jakimovski [6] has defined an integral Abel-type method of summability: for $\lambda>-1$ and $y>0$, let

$$A_\lambda(y)=A_{f;\lambda}(y)=\frac{1}{\Gamma(\lambda+1)}\int_0^\infty e^{-x}x^\lambda f(xy)\,dx.$$

If $A_\lambda(y)$ exists as a Cauchy–Lebesgue integral for all $y>0$ and if $A_\lambda(y)\to\sigma$ as $y\to\infty$, then we say that $f$ is $A_\lambda$-summable to $\sigma$ and we write $f(x)\to\sigma(A_\lambda)$.

We note that the case $\lambda=0$ gives the Laplace transform. The method $A_\lambda$ is regular. It is the integral analogue of the sequence-to-sequence Abel-type method defined by Borwein [2], which generalizes the well-known Abel method.

The following three results are all due to Jakimovski [6].

THEOREM B. *Let $\mathbf{H}_\chi$ be a regular integral Hausdorff method and let $\lambda>-1$. Then $A_\lambda\mathbf{H}_\chi\supseteq A_\lambda$.*

THEOREM C. *Let $\lambda>\mu>-1$ and $y>0$. If $A_{f;\lambda}(y)$ exists, then $A_{f;\mu}(y)=C_{\lambda-\mu,\mu}A_{f;\lambda}(y)$.*

THEOREM D. *For $\lambda>\mu>-1$, $A_\mu\supseteq A_\lambda$.*

We now show that the inclusion in Theorem D is strict.

THEOREM 1. (cf. [2]) *Let $\lambda>-1$. Then there exists a function which is*

*$A_\mu$-summable to* 0 *for all $\mu$ satisfying $\lambda>\mu>-1$, but which is not $A_\lambda$-summable.*

**Proof.** For all real $x$, we define $f(x)=\sum_{n=0}^{\infty} a_n x^{2n}$, where $a_n=(-1)^n\Gamma(\lambda+1)/(2n+1)!\,\Gamma(\lambda+2n+2)$. It is easy to show that $A_{f;\lambda}(y)=\sin y$. Thus $f$ is not $A_\lambda$-summable.

Suppose now that $\lambda>\mu>-1$. Using theorem C and the Riemann–Lebesgue theorem, it now follows that

$$A_{f;\mu}(y)=C_{\lambda-\mu,\mu}A_{f;\lambda}(y)=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu)\Gamma(\mu+1)}\int_0^1(1-t)^{\lambda-\mu-1}t^\mu\sin(yt)\,dt$$

$$\to 0 \quad \text{as} \quad y\to\infty;$$

that is, $f(x)\to 0(A_\mu)$.

5. **The method** $A_{-1}$. We now define an integral logarithmic method of summability which may be regarded as an extension of the integral Abel-type method $A_\lambda$ to the case $\lambda=-1$. Accordingly, we use the notation $A_{-1}$ to denote this method. For $y>0$, let

$$A_{-1}(y)=A_{f;-1}(y)=\frac{1}{\log(1+y)}\int_0^\infty\frac{e^{-(x+1)/y}}{x+1}f(x)\,dx.$$

If $A_{-1}(y)$ exists as a Lebesgue integral for all $y>0$ and if $A_{-1}(y)\to\sigma$ as $y\to\infty$ then we write $f(x)\to\sigma(A_{-1})$. This is a regular integral method, analogous to a sequence-to-function method given by Hardy [5, p. 81] (see also [1]).

We remark that changing the function $f$ on any finite interval does not affect its $A_{-1}$-summability.

LEMMA. (cf. [1]) *Let $\delta$ be real and let $g(x)=f(x)/(x+\delta)$ for $x\ge|\delta|+1$ and zero otherwise. If $f(x)\to\sigma(A_{-1})$, then $g(x)\to 0(A_{-1})$.*

**Proof.** Let $M>|\delta|+1$ be constant. Let $\phi(t)=\int_M^\infty f(x)e^{-t(x+\delta)}/(x+1)\,dx$. Then we have that $\{\log(1+1/t)\}^{-1}\phi(t)\to\sigma$ as $t\to 0+$. Hence there exists a constant $K$ such that for all $t$ in $(0,1)$, $|\phi(t)|\le K\,|\log(1+1/t)|$. We also obtain that, as $t\to 0+$,

$$A_{g;-1}(1/t)\sim\frac{e^{-t(1-\delta)}}{\log\left(1+\frac{1}{t}\right)}\int_M^\infty\frac{e^{-t(x+\delta)}}{(x+\delta)(x+1)}f(x)\,dx$$

$$=\frac{e^{-t(1-\delta)}}{\log\left(1+\frac{1}{t}\right)}\int_t^\infty dz\int_M^\infty\frac{e^{-z(x+\delta)}}{(x+1)}f(x)\,dx\sim\frac{1}{\log\left(1+\frac{1}{t}\right)}\int_t^\infty\phi(z)\,dz.$$

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Let $\varepsilon \in (0,1)$ be given. Then for $0<t<\varepsilon$ we have that

$$\frac{1}{\log\left(1+\frac{1}{t}\right)} \int_{\varepsilon}^{\infty} \phi(z)\, dz \to 0 \quad \text{as} \quad t \to 0+,$$

and

$$\left|\frac{1}{\log\left(1+\frac{1}{t}\right)} \int_{t}^{\varepsilon} \phi(z)\, dz\right| \le \frac{K}{\log\left(1+\frac{1}{t}\right)} \int_{t}^{\varepsilon} \log\left(1+\frac{1}{t}\right) dz \le K\varepsilon.$$

Thus, for any $\varepsilon>0$, we have that $\limsup_{t\to 0+} |A_{g;-1}(1/t)| \le K\varepsilon$, where $K$ is independent of $\varepsilon$. Thus it follows that $g(x) \to 0(A_{-1})$.

LEMMA. (cf. [1]) *Let $\delta$ be real. Then $f(x) \to \sigma(A_{-1})$ if and only if $f(x+\delta) \to \sigma(A_{-1})$.*

**Proof.** Let $g(x)=f(x+\delta)$. Suppose that $f(x) \to \sigma(A_{-1})$. From the previous lemma, it follows that

$$A_{g;-1}(y) \sim \frac{e^{\delta/y}}{\log(1+y)} \int_{|\delta|+1}^{\infty} \frac{e^{-(x+1)/y}}{(x+1)} \left(1+\frac{\delta}{x+1-\delta}\right) f(x)\, dx \to \sigma \quad \text{as} \quad y \to \infty;$$

that is, $f(x+\delta) \to \sigma(A_{-1})$. Since $\delta$ is arbitrary, the result follows.

To investigate the product of integral Hausdorff methods and the method $A_{-1}$, we use a result due to Borwein [1].

THEOREM E. *Let $\mathbf{H}_\chi$ be a regular integral Hausdorff method. For $x \ge 0$, let $g(x)$ be a continuous function. If $g(x) \to \sigma$ as $x \to \infty$, then*

$$\{\log(1+y)\}^{-1} \int_0^1 \log(1+yt) g(yt)\, d\chi(t) \to \sigma \quad \text{as} \quad y \to \infty.$$

THEOREM 2. (cf. [1]) *Let $\mathbf{H}_\chi$ be a regular integral Hausdorff method. Then $A_{-1}\mathbf{H}_\chi \supseteq A_{-1}$.*

**Proof.** Suppose that $f(x) \to \sigma(A_{-1})$. We may further suppose that $f(x)=0$ for $x \le 1$. Hence we have that

$$A_{-1}\mathbf{H}_\chi(y) \sim \{\log(1+y)\}^{-1} \int_1^{\infty} \frac{e^{-u/y}}{u}\, du \int_0^1 f(ut)\, d\chi(t)$$

$$= \{\log(1+y)\}^{-1} \int_0^1 \log(1+yt) g(yt)\, d\chi(t),$$

where

$$g(x) = \{\log(1+x)\}^{-1} \int_1^{\infty} \frac{e^{-u/x}}{u} f(u)\, du.$$

Using an argument as in [10, p. 181], we can show that $g$ is continuous. The desired result now follows from Theorem E.

**6. A scale of inclusions for integral Abel-type methods.** We shall now place the $A_{-1}$ method in our scale of inclusions for integral Abel-type methods. From here on, we assume that $A_\lambda(y)$ exists as a Lebesgue integral.

THEOREM 3. (cf. [1]) *For $\lambda > -1$, $A_{-1} \supseteq A_\lambda$, and the inclusion is strict.*

Before proving this theorem, we must develop some machinery.

THEOREM F. ([7], [3]) *Let $c_0$ be a real constant. If $F(s)$ is an analytic function of $s = \rho + ir$ in the region $\rho > c_0$, and if there is a constant $K$ such that $\int_{-\infty}^{\infty} |F(c+it)|^2\, dt < K$ for all $c > c_0$, then $F(s) = \int_0^1 t^s \phi(t)\, dt$ for $\rho > c_0$, where $t^c \phi(t)$ is Lebesgue integrable on $[0, 1]$ for all $c > c_0$.*

LEMMA. (cf. [3]) *For $\lambda \geq 1$, $(x/(x+1))^\lambda - 1 = \int_0^1 t^x \phi(t)\, dt$, where $t^c \phi(t)$ is Lebesgue integrable on $[0, 1]$ for all $c > 0$.*

**Proof.** Setting $F(s) = (s/(s+1))^\lambda - 1$, it is easy to show that, for $c > 0$, $|F(c+it)| \leq \lambda/\{(c+1)^2 + t^2\}^{1/2}$, and further that $\int_{-\infty}^{\infty} |F(c+it)|^2\, dt \leq \lambda^2 \int_{-\infty}^{\infty} dt/1 + t^2$. The result now follows from Theorem F.

By adapting a method defined by Watson [9, p. 41], we define a method $J_\lambda$ as follows. For $y > 0$ and $\lambda > -1$, let

$$J_\lambda(y) = J_{f;\lambda}(y) = \frac{1}{y^\lambda \log(1+y)} \int_0^y (y-x)^\lambda e^{-1/x} x^{-1} f(x)\, dx.$$

If $J_\lambda(y)$ exists for all $y > 0$ and if $J_\lambda(y) \to \sigma$ as $y \to \infty$, then we say that $f$ is $J_\lambda$-summable to $\sigma$, and write $f(x) \to \sigma(J_\lambda)$.

LEMMA. (cf. [9, §4.6]) *Suppose that $\lambda > -1$. For $x > 0$, let $g(x) = (x/(x+1))^\lambda f(x)$. If $A_\lambda$ can be applied to $f$, then $J_\lambda A_{f;\lambda} = A_{g;-1}$.*

**Proof.** For $y > 0$, we have that

$$\begin{aligned} J_\lambda A_{f;\lambda}(y) &= \frac{1}{y^\lambda \log(1+y)} \int_0^y \frac{(y-x)^\lambda e^{-1/x}}{\Gamma(\lambda+1) x^{\lambda+2}}\, dx \int_0^\infty e^{-u/x} u^\lambda f(u)\, du \\ &= \frac{1}{\log(1+y)} \int_0^\infty \frac{e^{-(u+1)/y}}{u+1} \left(\frac{u}{u+1}\right)^\lambda f(u)\, du = A_{g;-1}(y). \end{aligned}$$

By letting $f(x) = 1$ in the above, we see that $J_\lambda$ is regular, and further that the following result is true.

LEMMA. *Let $\lambda > -1$. If $f(x) \to \sigma(A_\lambda)$, then $(x/(x+1))^\lambda f(x) \to \sigma(A_{-1})$.*

The next two lemmas extend this result.

LEMMA. (cf. [3]) *Let $\lambda \geq 1$. If $f(x) \to \sigma(A_{-1})$, then $(x/(x+1))^\lambda f(x) \to \sigma(A_{-1})$.*

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**Proof.** Write $g(x)=(x/(x+1))^\lambda f(x)$. Without loss of generality, assume that $f(x)=0$ for $x<1$ and that $\sigma=0$. By an earlier lemma, we have that $(x/(x+1))^\lambda - 1=\int_0^1 t^x\phi(t)\,dt$, where $t^c\phi(t)$ is Lebesgue integrable on $[0,1]$ for all $c>0$. Thus $(z/(x+1))^\lambda-1=\int_0^\infty e^{-x/z}\psi(z)\,dz$, where $\psi(z)=z^{-2}\phi(e^{-1/z})$. We note that $\psi(z)$ is Lebesgue integrable on $[\varepsilon,\infty)$ for any $\varepsilon>0$. In particular we have that $\int_1^\infty|\psi(z)|\,dz<\infty$. It now follows that

$$\begin{aligned}A_{g;-1}(y)&\sim\frac{1}{\log(1+y)}\int_1^\infty u^{-1}e^{-u/y}\left(\frac{u}{u+1}\right)^\lambda f(u)\,du\\&=\frac{1}{\log(1+y)}\int_1^\infty u^{-1}e^{-u/y}f(u)\,du\int_0^\infty e^{-u/z}\psi(z)\,dz\\&\quad+\frac{1}{\log(1+y)}\int_1^\infty u^{-1}e^{-u/y}f(u)\,du.\end{aligned}$$

The second integral is $A_{f;-1}(y)$, which, by assumption, tends to zero as $y\to\infty$.

Let $Y$ be a constant greater than one, and which will be specified later. Then for $y>Y$, we have that

$$\begin{aligned}A_{g;-1}(y)&\sim\frac{1}{\log(1+y)}\left(\int_0^Y+\int_Y^\infty\right)\psi(z)\,dz\int_1^\infty u^{-1}e^{-u(y^{-1}+z^{-1})}f(u)\,du\\&=I_1+I_2,\quad\text{say.}\end{aligned}$$

It is easy to show that $I_1\to 0$ as $y\to\infty$. Further, since $y>yz/(y+z)\geq Y/2$, we obtain

$$\begin{aligned}|I_2|&\leq\int_Y^\infty\frac{|\psi(z)|}{\log(1+yz/(y+z))}\,dz\left|\int_1^\infty u^{-1}e^{-u(y^{-1}+z^{-1})}f(u)\,du\right|\\&\leq\sup_{x\geq Y/2}\left|\frac{1}{\log(1+x)}\int_1^\infty u^{-1}e^{-u/x}f(u)\,du\right|\cdot\int_Y^\infty|\psi(z)|\,dz\\&\leq\sup_{x\geq Y/2}|A_{f;-1}(x)|\cdot\int_1^\infty|\psi(z)|\,dz.\end{aligned}$$

Since $Y$ can be chosen sufficiently large, the result now follows.

LEMMA. *Suppose that* $\lambda>-1$. *Then* $f(x)\to\sigma(A_{-1})$ *if and only if* $(x/(x+1))^\lambda f(x)\to\sigma(A_{-1})$.

**Proof.**

(i) Necessity. Suppose that $f(x)\to\sigma(A_{-1})$. For $\lambda\geq 1$, the preceding lemma gives the result. For $0\leq\lambda<1$, we observe that

$$\begin{aligned}\left(\frac{x}{x+1}\right)^\lambda f(x)&=\left(\frac{x+1}{x}\right)\left(\frac{x}{x+1}\right)^{\lambda+1}f(x)\\&=\left(\frac{x}{x+1}\right)^{\lambda+1}f(x)+\left(\frac{x}{x+1}\right)^{\lambda+1}\frac{f(x)}{x}\to\sigma(A_{-1}),\end{aligned}$$

since $\lambda+1\geq 1$. The result for $-1<\lambda<0$ follows in a similar manner.

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(ii) Sufficiency. Suppose that $(x/(x+1))^{\lambda}f(x)\to\sigma(A_{-1})$. For $-1<\lambda<1$, we note that $f(x)=(x/(x+1))^{-\lambda}\cdot(x/(x+1))^{\lambda}f(x)\to\sigma(A_{-1})$, by the necessity part of this lemma, since $-\lambda>-1$. Suppose now that $\lambda\geq 1$. Then we have that $\lambda=\lambda_0+\lambda_1$, where $\lambda_0$ is a positive integer and $-1<\lambda_1\leq 0$. It now follows that

$$\begin{aligned}\left(\frac{x}{x+1}\right)^{\lambda_1}f(x)&=\left(\frac{x+1}{x}\right)^{\lambda_0}\left(\frac{x}{x+1}\right)^{\lambda}f(x)\\&=\left(1+\frac{\lambda_0}{x}+\cdots+\frac{1}{x^{\lambda_0}}\right)\cdot\left(\frac{x}{x+1}\right)^{\lambda}f(x)\to\sigma(A_{-1}).\end{aligned}$$

By the first part of the sufficiency, we obtain that $f(x)\to\sigma(A_{-1})$, since $-1<\lambda_1\leq 0$.

By combining the results of this section with Theorem D and Theorem 1, we can now obtain a full scale of strict inclusions for integral Abel-type methods of summability.

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