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POSITIONING MATRICES WITH RESPECT TO THE BOUNDARY OF THE MAXIMAL GROUP

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Let Δ denote the Banach algebra of all conservative triangular matrics, M the maximal group of invertible elements of Δ, B the boundary of M and $N = \Delta \setminus \overline{M}$. In this note little Nörlund means are located with respect to the disjoint decomposition $M \cup B \cup N$ of Δ in terms of the zeros of the generating power series. Further, corridor matrices of finite type, that is, conservative methods with finitely many convergent diagonals, are located with respect to $M \cup B \cup N$.

1. Introduction

In [10] B.E. Rhoades considered the problem of locating classes of summability methods with respect to the boundary of the maximal group of invertible elements in the Banach algebra Δ of all conservative triangular methods. In particular, he determined the location of most of the Nörlund polynomial methods in terms of where the roots of the generating polynomial sit with respect to the unit disk. The one case yet to be settled is the location of those methods whose polynomials admit

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roots inside and on the unit disk. In the special case of polynomials of degree two these were shown by I.D. Berg [1] to be in the complement of the closure of the maximal group.

In [13] Sharma considered the collection of little Nörlund means which form a closed subalgebra of Δ containing its inverses. In this smaller algebra those methods whose series admit roots inside and on the unit disk are shown to be located in the complement of the closure of the maximal group of this smaller algebra. In section 3 we improve several of the results of [13] using arguments quite distinct from those of [10] or [13].

In [4] the authors considered the class of bi-diagonal methods. That is, methods with convergent sequences $\{a_n\}$ and $\{b_n\}$ on the main and lower diagonal respectively and zeros elsewhere. These were shown to locate according as the Nörlund polynomial method generated by $p_0 + p_1 z$ where $p_0 = \lim_{n\to\infty} a_n$ and $p_1 = \lim_{n\to\infty} b_n$. In section 4 we extend this result to include the class of corridor matrices. That is, conservative methods with finitely many convergent diagonals. This allows one to locate many methods which do not fit into various classes determined by Rhoades [10] and Sharma [13].

It should be pointed out that [13, Theorem 6 and comment 2, p.288] are stated incorrectly. The author claims, that if a little Nörlund mean is on the boundary of the maximal group of invertible elements of the Banach algebra of little Nörlund means, then the generating function is identically zero on the closed unit disk or has zeros only on the boundary of the closed unit disk. However, the product of a little Nörlund mean on the boundary of the maximal group with one in the maximal group is back in the boundary of the maximal group. Therefore, the correct statement in this case is either the generating function is identically zero on the closed unit disk or has no zeros in the interior of the closed unit disk. This in fact is what is proven in [13].

2. Preliminaries

If $A = (a_{nk})$ is an infinite matrix, A defines a sequence to sequence transformation, mapping the sequence x to Ax where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k \qquad n = 0,1,2,\ldots$$

Let $c_A = \{x | Ax \in c\}$ where c denotes the space of convergent sequences. Well known necessary and sufficient conditions for A to be conservative are:

(i) $\quad ||A|| \equiv \sup_n \sum_k |a_{nk}| < \infty$

(ii) $\quad t \equiv \lim_n \sum_k a_{nk}$ exists, and

(iii) $\quad a_k \equiv \lim_n a_{nk}$ exist for $k = 0,1,2\ldots$.

If $\lim_n (Ax)_n = \lim_n x_n$ for all $x \in c$, then A is called regular. A conservative matrix A is regular if and only if $t = 1$ and $a_k = 0$ for all k. A conservative matrix A is said to be of type M provided $b \in \ell_1 = \{x | \sum_k |x_k| < \infty\}$ and $bA = 0$ implies $b = 0$ where $(bA)_k = \sum_n b_n a_{nk}$.

Let $B(c)$ denote the Banach algebra of bounded linear operators on c, Γ the subalgebra of conservative matrices and Δ the subalgebra of triangular (that is $a_{nk} = 0$ for $k > n$) conservative matrices. An element $A \in \Delta$ is a triangle provided $a_{nn} \neq 0$ for all n. With the norm given in (i) above Δ and Γ are closed Banach subalgebras of $B(c)$.

If X is any Banach algebra we denote the maximal group of invertible elements by $M(X)$, $\mathcal{B}(X)$ the boundary of the maximal group, and $N(X) = X \setminus \overline{M(X)}$. Since $M(X)$ is open, $\mathcal{B}(X) = \overline{M(X)} \setminus M(X)$ and $M(X) \cup \mathcal{B}(X) \cup N(X)$ is a disjoint decomposition of X. For simplicity we write $M = M(\Delta)$, $\mathcal{B} = \mathcal{B}(\Delta)$ and $N = N(\Delta)$.

3. Little Nörlund Means

Let $p(z) = \sum_{k=0}^{\infty} p_k z^k$ be a complex power series. The little Nörlund mean $N_p = (a_{nk})$ associated with $p(z)$ is defined by [see for example 6]

$$a_{nk} = \begin{cases} p_{n-k} & k \le n, \\ 0 & k > n. \end{cases}$$

It follows easily that a little Nörlund mean N_p is conservative if and only if the generating sequence $p = (p_k) \in \ell_1$. In this case the radius of convergence R_p of the power series $p(z)$ satisfies $R_p \geq 1$. In this section we restrict our attention to little Nörlund means N_p for which $R_p > 1$.

Given little Nörlund means N_p and N_q, the matrix product $N_h = N_p N_q$ is a little Nörlund mean generated by the function $h(z) = p(z)q(z)$. Also the matrix product $N_p N_q$ and operator composition $N_p(N_q^{\cdot})$ agree for little Nörlund means. For any little Norlund mean N_p we define three sets

$$\nu_1 = \{z | 1 < |z| < R_p, \ p(z) = 0\},$$

$$\nu_2 = \{z | 1 = |z|, \ p(z) = 0\} \text{ and}$$

$$\nu_3 = \{z | \ |z| < 1, \ p(z) = 0\} .$$

Finally let

$$A = \{N_p \mid p(z) = \sum_{n=0}^{\infty} p_n z^n , \ p \in \ell_1 \ .$$

In [10] Norlund polynomial methods are positioned with respect to the decomposition $M(\Delta) \cup B(\Delta) \cup N(\Delta)$ and in [13] little Nörlund means are positioned with respect to the decomposition $M(A) \cup B(A) \ N(A)$. In particular it is shown in [13] that for a little Nörlund mean N_p, with no restriction on the radius of convergence of the generating series, $N_p \in M(A)$ if and only if $\nu_2 = \emptyset$ and $\nu_3 = \emptyset$ and $N_p \in B(A)$ if and only if $\nu_2 \neq \emptyset$ and $\nu_3 = \emptyset$. The question first posed in [10] and examined in [4] and [5], which remains of interest, is whether it is true that $N_p \in B(\Delta)$ if and only if $\nu_2 \neq \emptyset$ and $\nu_3 = \emptyset$. The sufficiency follows immediately from Sharma's result since $B(A) \subset B(\Delta)$. For little Nörlund means with $R_p > 1$ the following theorem improves a result given in [13].

THEOREM 3.1. *If N_p is a little Nörlund mean with $R_p > 1$ and such that $\nu_2 = \emptyset$ and $\nu_3 \neq \emptyset$, then $N_p \in N(\Delta)$.*

Proof. Since $p(e^{i\theta}) \neq 0 \ \forall$ real θ, by Wiener's Theorem [11, 14] there exists a g defined on $T \equiv \{z \mid |z| = 1\}$ such that

$$g(z) = \sum_{n=-\infty}^{\infty} b_n z^n , \quad \sum_{n=-\infty}^{\infty} |b_n| < \infty$$

with $p.g = g.p = I$ the identity on T. Moreover

$$p(e^{i\theta})g(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} a_{n-k} b_k \Big) e^{in\theta}$$

for all real θ. Since $p(z) \not\equiv 0$, $\nu_3 \cap \{z \mid |z| \leq R'\}$ is finite for any R' with $0 < R' < R_p$. Hence there exists an annulus $R_1 < |z| < R_2 < R_p$, $R_1 < 1$, $R_2 > 1$ so that the Laurent expansion of $1/p(z)$ is defined and hence equals $g(z)$. Since $\nu_3 \neq \emptyset$, $1/p(z) = g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$ is such that b_n is non-zero for some negative indices. Define $N_g = (b_{nk})$ by $b_{nk} = b_{n-k}$ $k = 0,1,2,\ldots$, Then N_g is not triangular and moreover

$$N_g . N_p = I \neq N_p N_g .$$

Thus N_p has a left inverse in Γ which is not a right inverse and it follows that N_p does not have a two-sided inverse in Γ and hence $N_p \notin M(\Gamma)$. Indeed, if B is a two-sided inverse for N_p in Γ, then it is the unique two-sided inverse of N_p in Δ (equalling the unique right inverse in Δ). Then

$$N_g = N_g I = N_g(N_p B) = (N_g N_p)B = B ,$$

a contradiction. Associativity holds since N_g is row finite and N_p and B are triangles. Moreover, since N_p has a left inverse in Γ, it follows that N_p is not a left topological divisor of zero in Γ. Thus $N_p \notin B(\Gamma)$ which implies $N_p \in N(\Gamma)$ and hence $N_p \in N$.

THEOREM 3.2. *If* N_p *is a little Nörlund mean with* $R_p > 1$ *and such that* $\nu_2 \neq \emptyset$ *and* $\nu_3 \neq \emptyset$, *then* N_p *is in the closure of* $N(\Delta)$.

Proof. Since $p(z) \not\equiv 0$ on $|z| \leq 1$, $p(z)$ has a finite number of zeros with modulus less than 1. Let (r_m) be a sequence of real numbers $0 < r_m < 1$ for $m \geq 1$ such that $\lim_{m\to\infty} r_m = 1$. Define $g_m(z) = p(r_m z)$ for each m. Then for r_m sufficiently close to 1, $g_m(z)$ will satisfy $\nu_2 = \emptyset$ and $\nu_3 \neq \emptyset$. By Theorem 3.1, $N_{g_m} \in N$ for m sufficiently large. We claim that $(N_{g_m})_m$ converges to N_p in norm. Indeed, let $\epsilon > 0$ be given. Consider

$$\|N_p - N_{g_m}\| = \sup_n \sum_{k=0}^{\infty} |p_{n-k}|\,|1 - r_m^{n-k}|$$

$$= \sup_n \sum_{k=0}^{n} |p_k|\;|1 - r_m^{k}|$$

for any m. Since $|1 - r_m^k| < 1\ \forall m,\ \forall k$

$$\sum_{k=0}^{\infty} |p_k|\;|1 - r_m^k| \leq \sum_{k=0}^{\infty} |p_k| < \infty\ \forall m.$$

Choose M so that $\sum_{k=M+1}^{\infty} |p_k| < \epsilon/2$. Then choose L such that $m \geq L$ implies for $0 \leq k \leq M$

$$|1 - r_m^k| < \epsilon/(2\sum_{j=0}^{M} |p_j|)\ .$$

It then follows that $\|N_p - N_{g_m}\| \to 0$ as $m \to \infty$ and hence $N_p \in \overline{N(\Delta)}$.

In summary we have

THEOREM 3.3. *Let* N_p *be a little Nörlund mean.*

a) $\nu_2 = \nu_3 = \emptyset \Leftrightarrow N_p \in M(\Delta)$ *([13, Theorem 4]);*

b) $\nu_2 \neq \emptyset,\ \nu_3 = \emptyset \Rightarrow N_p \in B(\Delta)$ *([13, Theorem 6]);*

c) *If* $R_p > 1$ *and* $\nu_2 = \emptyset$, $\nu_3 \neq \emptyset \Rightarrow N_p \in N(\Delta)$

d) *If* $R_p > 1$ *and* $\nu_2 \neq \emptyset$, $\nu_3 \neq \emptyset \Rightarrow N_p \quad \overline{N(\Delta)}$.

Question 3.3. Can the hypothesis that $R_p > 1$ be dropped in the two preceeding theorems?

Question 3.4. Is $B(\Delta) \cap A = B(A)$?

If the answer to this question is in the affirmative, then little Nörlund means would position in Δ as they do in A and thus would settle the remaining positioning question. As in the case of Nörlund polynomial methods [10] the question of whether or not a little Nörlund mean with $\nu_2 \neq \emptyset$ and $\nu_3 \neq \emptyset$ is in $N(\Delta)$ remains unresolved. In [1] it was shown that Nörlund polynomial methods, generated by polynomials of degree two, with roots in and on the unit disk are in $N(\Delta)$. From [5] it follows that a little Nörlund mean is of type M if and only if $\nu_3 = \emptyset$. Consequently, to show that a little Nörlund mean with roots in and on the unit disk is in $N(\Delta)$ is equivalent to showing that any little Norlund mean in $B(\Delta)$ must be of type M.

4. Corridor Summabilty Methods.

In this section we use Nörlund polynomial methods to determine the position in $M \cup B \cup N$ of a large class of matrices which occur frequently in summability theory.

DEFINITION 4.1. A conservative triangular matrix $A = (a_{nk})$ is called a corridor matrix provided

(i) $p_n \equiv \lim\limits_{k\to\infty} a_{n+k,k}$ exists for each $n = 0,1,2,\ldots$ and

(ii) $p \in \ell_1$.

If A is not the zero matrix and if there exists a least j such that for each $n \geq j$ and for all $k \geq 1$, $a_{n+k,k} = 0$ the corridor matrix is said to be of finite type j.

We remark here that Nörlund polynomial methods are corridor matrices of finite type. Given a corridor matrix A of finite type M let

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$p(z) = \sum_{k=0}^{M} p_k z^k$ and associate with A the Nörlund polynomial method N_p. If a corridor matrix A of finite type satisfies $p_n = \lim_{k\to\infty} a_{n+k,k} = 0$ for all n, then A is coercive (that is, A maps each bounded sequence into c) and hence is in the boundary of the maximal group [12]. The next result positions a class of corridor matrices.

THEOREM 4.2. *Let $A = (a_{nk})$ be a corridor matrix of finite type M and suppose the associated Nörlund polynomial method N_p satisfies $p_0 \neq 0$. Then*

(i) $A \in M \Leftrightarrow N_p \in M$

(ii) $A \in B \Leftrightarrow N_p \in B$ *and*

(iii) $A \in N \Leftrightarrow N_p \in N$

Proof. We first verify part (ii) of the theorem.

Since $A = (a_{nk})$ is of finite type M we have $a_{nk} = 0$ for all $k > n$ and $a_{n+k,k} = 0$ for each $n > M$ and for all k. For $m = 1,2,...$ define $A^{(m)} = (a^{(m)}_{nk})$ as follows:

$$a^{(m)}_{n+k,k} = \begin{cases} a_{n+k,k} & \text{for each } n \geq 0 \text{ and for all } k \leq m, \\ p_n & \text{for } 0 \leq n \leq M,\ k \geq m+1 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since $(A^{(m)}x)_n = (N_p x)_n$ for all $n \geq m + M$ it is clear that $c_{N_p} = c_{A^{(m)}}$. We claim that $\|A^{(m)} - A\| \to 0$. But $\|A^{(m)} - A\|$ is computed from

$$\max \{|a_{m+1,m+1} - p_0|,\ |a_{m+2,m+1} - p_1| + |a_{m+2,m+2} - p_0|, \ldots,$$

$$|a_{m+M,m+1}\ p_M| + \ldots + |a_{m+M,m+M} - p_0|\} \text{ and}$$

$$\sup_{j \geq m+M+1} \sum_{k=0}^{M} |a_{j,j-M+k} - p_{M-k}| .$$

Since $\lim_{k} a_{n+k,k} = p_n \; \forall \; 0 \le n \le M$ we can choose M_1 such that $|a_{n+k,k} - p_n| < \epsilon/M \; \forall \; 0 \le n \le M$ and $\forall \; k > M_1$. It then follows that $\| A^{(m)} - A \| < \epsilon \; \forall \; m > M_1$ and hence $\|A^{(m)} - A\| \to 0$ as $m \to \infty$. Now suppose that $N_p \in \mathcal{B}$. Since $c_{N_p} = c_{A^{(m)}}$ for all m by [4, Lemma 2.4] $A \in \mathcal{B}$ and since $\mathcal{B}$ is closed this gives the sufficiency of part (ii). To prove the necessity for $m = 1, 2, \ldots$ define $B^{(m)} = (b_{nk}^{(m)})$ as as follows:

$$b_{n+k,k}^{(m)} = \begin{cases} p_n, & 0 \le n \le M, \quad k \le m, \\ a_{n+k,k}, & \text{for each } n, \; k > m, \\ 0 & \text{otherwise.} \end{cases}$$

It then follows that $c_{B^{(m)}} = c_A$ and as in the previous argument $\|N_p - B^{(m)}\| \to 0$. Now suppose $A \in \mathcal{B}$. Then by [4, Lemma 2.4] $B^{(m)} \in \mathcal{B}$ for all m and hence $N_p \in \mathcal{B}$.

Suppose $N_p \in M$ and define $A^{(m)}$ as above. It then follows that $c_{A^{(m)}} = c_A = c$ and hence $A^{(m)} \in M$ for all m. Moreover as $\| A^{(m)} - A\| \to 0$ as $m \to \infty$ and hence $A \in \overline{M}$. But from the first part of the proof $A \notin \mathcal{B}$ and hence $A \in M$. Similarly $A \in M$ implies that $N_p \in M$. This gives part (i). Part (iii) now follows immediately.

It would be of interest to know if Theorem 4.2 remains valid for general corridor methods. By considering the Cesàro matrix one observes that the techniques used in the proof of 4.2 will not carry over to this more general setting.

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Added in proof. Professor W. Beekmann has pointed out that the hypothesis $R_p > 1$ can be dropped from Theorem 3.2 by choosing the sequence (r_m) so that $\lim r_m = 1$ and for each m, $0 < r_m < 1$ with $p(z) \neq 0$ for all z satisfying $|z| = r_m$.