ASSOCIATED BASIC HYPERGEOMETRIC SERIES

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(Received 22nd May, 1953)

1. *Introduction.* The purpose of the present note is to give some interesting and simple identities connected with basic hypergeometric series of the types ${}_2\Phi_1$ and ${}_3\Phi_2$.

The difference operator

$$Df(x) \equiv \frac{f(x) - f(qx)}{x}, \quad (q = 1 - \epsilon, \epsilon > 0)$$

is of much importance in the theory of basic hypergeometric functions and has been used by many authors : e.g., Heine (**1**), Rogers (**2**), Jackson (**3**) and Hahn (**4**), etc., in developing the theory of basic functions. The operator D in the theory of basic functions replaces the ordinary differential operator d/dx.

In § 3, I use this operator to obtain some identities involving the function ${}_2\Phi_1$. In § 4, a basic generalisation of Gauss's theorem (extended by Riemann), that any three series of the ordinary hypergeometric type $F(a+l, b+m ; c+n ; x)$, where l, m, n are integers (positive or negative) are connected by a linear homogeneous relation with polynomial coefficients, is given.

2. *Notation.* Let

$$(a)_n = (1 - q^a)(1 - q^{a+1}) \dots (1 - q^{a+n-1}), \quad |q| < 1,$$
$$(a)_0 = 1,$$

and

$${}_{s+1}\Phi_s\begin{pmatrix} a_1, \dots, a_{s+1} ; x \\ b_1, \dots, b_s \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_{s+1})_n}{(1)_n (b_1)_n \dots (b_s)_n} x^n.$$

Also, for the sake of brevity, we will use the notation

$$\alpha \equiv (q^{-a} - 1), \quad \beta \equiv (q^{-b} - 1), \quad \gamma \equiv (q^{-c} - 1),$$
$$\delta \equiv (q^{-d} - 1) \text{ and } \epsilon \equiv (q^{-e} - 1).$$

3. We now prove the following identities :

(i) $(a)_n x^{a-1} {}_2\Phi_1(a+n, b ; c ; x) = D^n [x^{a+n-1} {}_2\Phi_1(a, b ; c ; x)]$,

(ii) $(c-n)_n x^{c-1-n} {}_2\Phi_1(a, b ; c-n ; x) = D^n [x^{c-1} {}_2\Phi_1(a, b ; c ; x)]$,

(iii) $(a)_n (b)_n {}_2\Phi_1(a+n, b+n ; c+n ; x) = (c)_n D^n [{}_2\Phi_1(a, b ; c ; x)]$,

(iv) $$(c-a)_n x^{c-a-1} \prod_{m=0}^{\infty} \frac{(1 - xq^{c-a-b+n+m})}{(1 - xq^m)} {}_2\Phi_1(a-n, b ; c ; xq^{c-a-b+n})$$
$$= D^n \left[x^{c-a+n-1} \prod_{m=0}^{\infty} \frac{(1 - xq^{c-a-b+m})}{(1 - xq^m)} {}_2\Phi_1(a, b ; c ; xq^{c-a-b}) \right],$$

(v) $$(c-n)_n x^{c-1-n} \prod_{m=0}^{\infty} \frac{(1 - xq^{c-a-b+n+m})}{(1 - xq^m)} {}_2\Phi_1(a-n, b-n ; c-n ; xq^{c-a-b+n})$$
$$= D^n \left[x^{c-1} \prod_{m=0}^{\infty} \frac{(1 - xq^{c-a-b+m})}{(1 - xq^m)} {}_2\Phi_1(a, b ; c ; xq^{c-a-b}) \right],$$

(vi) $$(c-a)_n(c-b)_n \prod_{m=0}^{\infty} \frac{(1-xq^{c-a-b+n+m})}{(1-xq^m)}\, {}_2\Phi_1(a, b\,;\; c+n\,;\; xq^{c-a-b+n})$$
$$=(c)_n D^n\left[\prod_{m=0}^{\infty} \frac{(1-xq^{c-a-b+m})}{(1-xq^m)}\, {}_2\Phi_1(a, b\,;\; c\,;\; xq^{c-a-b})\right].$$

To prove the first three we expand the right-hand ${}_2\Phi_1$ in powers of x and use the relation

$$Dx^a=(1-q^a)x^{a-1}$$

term by term.

The last three are variants of the first three in order. They are obtained from the first three by using the well-known transformation

$${}_2\Phi_1(a, b\,;\; c\,;\; x)=\prod_{n=0}^{\infty} \frac{(1-xq^{a+b-c+n})}{(1-xq^n)}\, {}_2\Phi_1(c-a, c-b\,;\; c\,;\; xq^{a+b-c})$$

on both sides of (i), (ii) and (iii) respectively to transform the ${}_2\Phi_1$. The identity (v) is the basic analogue of the well-known result due to Jacobi (**5**) for the ordinary hypergeometric function.

4. In this section I will generalise Gauss's* theorem for ordinary hypergeometric associated series by showing that between any four series of the type

$${}_3\Phi_2\left[\begin{matrix} a+l, b+m, c+n\,;\; x \\ d+p, e+s \end{matrix}\right],$$

where l, m, n, p and s are integers (positive or negative), there always exists a linear homogeneous relation with polynomial coefficients.

To prove this we can easily verify that the difference equation satisfied by

$${}_3\Phi_2(a, b, c\,;\; d, e\,;\; x)$$

is

$$\{\vartheta(\vartheta+q^{1-d}-1)(\vartheta+q^{1-e}-1)-xq^{a+b+c-d-e+2}(\vartheta+\alpha)(\vartheta+\beta)(\vartheta+\gamma)\}\Phi=0, \quad \text{.........(4.1)}$$

where $\vartheta\equiv xD$.

Also, it is easily verified that

$$(\vartheta+\alpha)\,\Phi=\alpha\Phi_{a+}\,, \quad \text{...(4.2)}$$

and

$$(q^{1-e}-1)\;\Phi_{e-}=(\vartheta+q^{1-e}-1)\,\Phi, \quad \text{...............................(4.3)}$$

where Φ denotes the function ${}_3\Phi_2$ and

$$\Phi_{a+}={}_3\Phi_2\left[\begin{matrix} a+1, b, c\,;\; x \\ d, e\,; \end{matrix}\right];$$

with similar abbreviated notations for other associated series.

Now (4.1), with $a-1$ in place of a, can be written as

$$\{\vartheta^2+\vartheta(q^{1-d}+q^{1-e}-q^{1-a}-1)+q^2(\epsilon-\alpha)(\delta-\alpha)-xq^{a+b+c-d-e+1}(\vartheta+\beta)(\vartheta+\gamma)\}(\vartheta+q^{1-a}-1)\Phi_{a-}$$
$$=q^2(\epsilon-\alpha)(\delta-\alpha)(q^{1-a}-1)\;\Phi_{a-}. \quad \text{...(4.4)}$$

Using (4.2) with $a-1$ in place of a we get

$$q^2(\epsilon-\alpha)(\delta-\alpha)\,\Phi_{a-}=\{\vartheta^2+\vartheta(q^{1-d}+q^{1-e}-q^{1-a}-1)+q^2(\epsilon-\alpha)(\delta-\alpha)-xq^{a+b+c-d-e+1}$$
$$\times(\vartheta+\beta)(\vartheta+\gamma)\}\,\Phi. \quad \text{.................(4.5)}$$

* For similar results for ordinary hypergeometric series see Bailey, *Quart. J. of Math.*, Oxford, 8 (1937), pp. 115–118.

Next, replacing e by $e+1$ in (4.1) and proceeding as above, we get, on using (4.3)

$$xq^{1-d-e+a+b+c}(\alpha-\epsilon)(\beta-\epsilon)(\gamma-\epsilon)\ \Phi_{e+}$$
$$=\{\vartheta(\vartheta+q^{1-d}-1)-xq^{a+b+c-d-e+1}(\vartheta^2+\vartheta(\alpha+\beta+\gamma-\epsilon)+\alpha(\beta-\epsilon)+\beta(\gamma-\epsilon)+\gamma(\alpha-\epsilon)+\epsilon^2)\}\,\epsilon\Phi. \quad \text{......(4.6)}$$

Now, by repeated applications of the relations (4.2), (4.3), (4.5) and (4.6) and similar other relations, together with the use of the equation (4.1), we can express any associated series

$${}_3\Phi_2\left[\begin{matrix}a+l,\ b+m,\ c+n\ ;\ x\\ d+p,\ e+s\end{matrix}\right]$$

in terms of Φ, $\vartheta\Phi$ and $\vartheta^2\Phi$. Thus between any four relations of this type we can eliminate Φ, $\vartheta\Phi$ and $\vartheta^2\Phi$ to get a linear homogeneous relation between four associated series of the type ${}_3\Phi_2$, with polynomial coefficients.

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