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Action at a distance, and the transmission of stress by isotropic elastic solid media.

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INTRODUCTION.

§ 1. The mutual action of two electrified bodies was regarded by Maxwell as transmitted by a medium. According to him the stress in the medium* consists of a "tension like a rope" along the lines of electrical force whose intensity per unit of area is $R^2/8\pi$, where R is the resultant electric intensity, and of a pressure numerically equal to this in all orthogonal directions. Maxwell's remarks are somewhat vague but his notation is strongly suggestive of an elastic solid medium. It has, however, been pointed out by Minchin† that Maxwell's stress system would not in an ordinary elastic solid give origin to strains consistent with the "equations of compatibility" which the theory of elastic solids supplies. Considerable interest still attaches to the theory of an elastic solid medium propagating stresses equivalent to the action between distant bodies of forces varying inversely as the square of the distance. For in the first place, it has been pointed out that the stress system given by Maxwell does not constitute a unique solution‡ of his equations; and, in the second place, it has been suggested that some medium must exist for the transmission of gravitational forces. The statical problem of the propagation of gravitational forces by an isotropic elastic medium has been treated by Minchin.§ His treatment how-

* *Electricity and Magnetism*, 3rd edition, Art. 106.

† *Treatise on Statics*, vol. II., 3rd edition, pp. 451-3.

‡ Minchin l.c., or Maxwell's *Electricity and Magnetism*, 3rd edition, Art. 110 footnote.

§ Minchin l.c., pp. 454-8.

ever neglects a certain surface condition. I have thus thought it worth while to consider the problem independently, employing the ordinary surface conditions. The first part of the paper is devoted more especially to the electrostatic problem, but the elastic solid problem is essentially the same throughout.

§ 2. The ordinary "action at a distance" theory regards a charge of electricity as a very thin superficial layer which repels a charge of the same sign and attracts one of opposite sign with a force varying inversely as the square of the distance. In interpreting this as an elastic solid problem the most obvious plan is to regard the layer as a thin shell of elastic material containing and surrounded by other elastic material, the layer being the only material exercising what we may term "gravitational forces". Supposing initially there are no gravitational forces and no strains anywhere in the medium, the endowment of the layer with the gravitational forces gives origin to a system of stresses required to keep the medium in equilibrium. These stresses reversed in sign would be those which a theory such as Maxwell's would substitute for the action at a distance of the thin layer. In the electrostatic problem the layer must be supposed extremely thin while at the same time the surface density is finite. In order to avoid the risk of unduly limiting the problem the layer is regarded here as of a different elastic material from that either inside or outside it. It is assumed, however, that the material of the layer is not wholly incompressible but satisfies the ordinary elastic solid equations, and that its elastic constants are neither infinitely great nor infinitely small compared to those of an adjacent medium. The assumption is also made that all the media are isotropic.

§ 3. In all the cases treated here the applied forces, and so the strains and stresses, are functions only of the distance r from a fixed point. The displacement u at every point is along the radius vector, and the dilatation Δ is given by

$$\Delta = \frac{du}{dr} + \frac{2u}{r} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1).$$

In an isotropic medium whose density is ρ and elastic constants m, n, in the notation of Thomson and Tait's *Natural Philosophy*, the bodily equations of equilibrium reduce in such a case to the one equation

$$(m+n)\frac{d\Delta}{dr}+\rho\frac{dV}{dr}=0 \quad \dots \quad \dots \quad \dots \quad (2),$$

where V, supposed a function of r only, is the potential of the bodily forces.

In the most general case considered here V is of the type

$$Vr^2+V'/r,$$

where V and V′ are constants, and the complete solution of (2) is

$$\Delta=A-\frac{\rho}{m+n}(Vr^2+V'r^{-1}) \quad \dots \quad \dots \quad (3),$$

where A is an arbitrary constant.

Substituting for Δ in (1) we deduce as the complete value of u

$$u=\tfrac{1}{3}Ar+Br^{-2}-\frac{\rho}{m+n}\{\tfrac{1}{5}Vr^3+\tfrac{1}{2}V'\} \quad \dots \quad \dots \quad (4),$$

where B is a second arbitrary constant.

The stress system consists of a principal stress along r and two other principal stresses perpendicular to r. The latter two are equal and may be supposed to act along any two mutually orthogonal directions in the plane perpendicular to r. Employing the notation introduced by Professor Pearson,* we shall denote the stress along the radius by $\widehat{rr}$, and employ $\widehat{\theta\theta}$ for the stress in any perpendicular direction, or what we may call the *transverse* stress.

The relations between the stresses and strains are

$$\left.\begin{aligned}\widehat{rr}&=(m-n)\Delta+2n\frac{du}{dr},\\ \widehat{\theta\theta}&=(m-n)\Delta+2n\frac{u}{r}\end{aligned}\right\} \quad \dots \quad \dots \quad (5).$$

The ordinary three surface conditions satisfied by the stresses reduce to one, viz:

$$\widehat{rr}=\text{radial surface force per unit of surface} \quad \dots \quad (6).$$

If the surface be "free", or acted on by no forces, then $\widehat{rr}$ must vanish over it. At a common surface of two media $\widehat{rr}$ must be continuous. This condition appears to be considered unnecessary by Prof. Minchin. In place of it he omits what is equivalent to the constant A in (4), on the ground that the corresponding term contributes

* Todhunter and Pearson's "*History of Elasticity*", Vol. I., p. 321.

nothing to the "gravitative action" on the element of the medium (l.c., p. 454). A further obvious condition at a common surface of two media is the continuity of the displacements, in this case of the radial displacement.

§ 4. When we attempt to picture to ourselves the state of matters close to the interface of two different media we encounter a difficulty which has occurred to several writers. Regarding the media as composed of molecules, the molecules of one of the media when close to the interface may be acted on by the molecules of the other medium, even supposing there is no mixing of the media. Thus it seems not unlikely there may be a narrow debateable ground wherein the relations between stress and strain show a gradual transition from the equations that hold inside the one medium to those that hold inside the other. The thickness of this transition zone must probably be a very small quantity, and in ordinary elastic solid problems its existence or nonexistence may be of little importance. In such applications, however, as to a hypothetical electrostatic medium, in which the gravitating layer is supposed extremely thin, the possibility of such a "modified action" ought to be present to the mind of the reader. The modified action, if appreciable, might affect the entire nature of the solution, so far at least as concerns the strains and stresses in the layer itself. While such a possibility may affect our attitude towards the solution it does not justify our dispensing with elastic solid surface conditions while applying elastic solid internal equations. As Professor Minchin employs the same elastic constants for space outside and inside the gravitating body there would appear no reason for supposing any modified action in the cases he treats, and thus his neglect of the continuity in the value of the radial stress must have some other explanation. This neglect leads Professor Minchin to the conclusion that "*the stress of the ether is discontinuous at the surface of the body*" (l.c., p. 455). This may be true though it presents serious difficulties, but it does not flow from the ordinary elastic solid theory.

§ 5. Before passing to our special problems we may employ the fundamental equations already given to show, in an elementary way, that Maxwell's stresses can not exist in any ordinary isotropic elastic medium.

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The electrostatic force R in the air outside a spherical surface, over which a charge Q is uniformly distributed, is given by $R = Q/r^2$, and so Maxwell's radial stress would be $Q^2/(8\pi r^4)$. But supposing no bodily forces to act we find from (3), (4), and (5),

$$\widehat{rr} = (m - \tfrac{1}{3}n)A - 4nr^{-3}B.$$

If the medium extend to infinity A vanishes, but in any case the term involving the negative power of r depends on r^{-3} and not on r^{-4} as Maxwell's theory requires.

Electrostatic Medium, Single Layer.

§ 6. Our first problem deals with three isotropic media, the surfaces separating which are concentric spheres. The inmost material is a core of radius e whose elastic constants are m, n; while the outmost material extends from $r = c$ to $r = \infty$ and has elastic constants m_2, n_2. Between these is a layer of material whose elastic constants are m_1, n_1 and density ρ, the particles of which *repel* one another with a force varying inversely as the square of the distance. The media are supposed in an unstrained state before the gravitational force commences to act, and our object is to find the strains and stresses in the state of final equilibrium under the action of the gravitational forces.

No bodily forces exist except in the layer where they answer to a potential

$$V = \tfrac{2}{3}\pi\rho(r^2 + 2e^3r^{-1}) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7).$$

The expressions for the dilatation and displacement may be derived from (3) and (4). Thus, employing A, A_1, B_1, B_2 as arbitrary constants, we have

in the core

$$\left.\begin{aligned} \Delta &= A, \\ u &= \tfrac{1}{3}rA \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (8),$$

in the layer

$$\left.\begin{aligned} \Delta_1 &= -\tfrac{2}{3}\pi\frac{\rho^2}{m_1+n_1}(r^2 + 2e^3r^{-1}), \\ u_1 &= -\tfrac{2}{3}\pi\frac{\rho^2}{m_1+n_1}(\tfrac{1}{5}r^3 + e^3) + \tfrac{1}{3}rA_1 + r^{-2}B_1 \end{aligned}\right\} \qquad \ldots \qquad (9),$$

outside the layer

$$\left.\begin{aligned} \Delta_2 &= 0, \\ u_2 &= r^{-2}B_2 \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (10).$$

The suffixes $_1$, $_2$ distinguish quantities referring to the layer and external medium respectively from those referring to the core.

The strains must not be infinite at the origin and should vanish at infinity, so no negative powers of r are admitted in (8) and no positive powers in (10). From (5) we find for the radial stresses in the three media

$$\widehat{rr} = (m - \tfrac{1}{3}n)\text{A} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11),$$

$$\widehat{rr}_1 = -\tfrac{2}{3}\pi\frac{\rho^2}{m_1+n_1}\{\tfrac{1}{5}r^2(5m_1+n_1)+2r^{-1}e^3(m_1-n_1)\}$$
$$+(m_1-\tfrac{1}{3}n_1)\text{A}_1 - 4r^{-3}n_1\text{B}_1, \quad \ldots \quad (12),$$

$$\widehat{rr}_2 = -4r^{-3}n_2\text{B}_2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13).$$

The surface conditions are the continuity of the displacement and radial stress. The equations embodying these conditions are easily written down, and their solution may be effected without serious difficulty on the lines adopted in treating the more complicated problem of § 8. It is thus sufficient to record the results we require without giving the algebraical work. Suppose then for shortness that

$$\frac{c^3}{e^3}(3m-n+4n_1)(3m_1-n_1+4n_2)$$
$$-4(n_1-n_2)\{3m_1-n_1-(3m-n)\} = \text{D} \quad \ldots \quad (14),$$

and merely record the values of A and B_2, viz.,

$$\text{A.D} = 2\pi\rho^2(c-e)^2[c^2e^{-3}(c+2e)(3m_1-n_1+4n_2)$$
$$+\tfrac{4}{5}(n_1-n_2)e^{-3}(2c^3+4c^2e+6ce^2+3e^3)] \quad \ldots \quad (15),$$

$$\text{B}_2.\text{D} = 2\pi\rho^2(c-e)^2[\tfrac{1}{5}c^3e^{-3}(2c^3+4c^2e+6ce^2+3e^3)(3m-n+4n_1)$$
$$+c^2(c+2e)\{3m_1-n_1-(3m-n)\}] \quad \ldots \quad (16).$$

§ 7. In the case which presents an analogy to the electrostatic problem $(c-e)/e$ is very small. For it, retaining only lowest powers of $(c-e)/e$, we find

$$\left.\begin{aligned} \text{A} &= 6\pi\rho^2(c-e)^2/(3m-n+4n_2), \\ \text{B}_2 &= 2\pi\rho^2(c-e)^2e^3/(3m-n+4n_2) \end{aligned}\right\} \quad \ldots \quad (17).$$

Now suppose that, however small $c-e$ may be,

$$\rho(c-e) = \sigma,$$

where σ is finite. Then putting

$$m - \tfrac{1}{3}n = k \quad \ldots \quad \ldots \quad \ldots \quad (18),$$

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so that k is the *bulk modulus* in the core, and substituting from (17) in (8), (10), and (5) we find

in the core

$$\left.\begin{aligned} u/r = \frac{du}{dr} = 2\pi\sigma^2/(3k+4n_2), \\ \widehat{rr} = \widehat{\theta\theta} = 6\pi\sigma^2 k/(3k+4n_2) \end{aligned}\right\} \quad \ldots \quad (19),$$

and outside the layer

$$\frac{u_2}{r} = -\tfrac{1}{2}\frac{du_2}{dr} = \frac{\widehat{rr_2}}{-4n_2} = \frac{\widehat{\theta\theta_2}}{2n_2} = \frac{2\pi\sigma^2}{3k+4n_2}\left(\frac{e}{r}\right)^3 \quad \ldots \quad (20).$$

The resultant per unit of surface of the radial forces exerted by the two media on the intervening layer being F, measured inwards, we have to the present degree of approximation, for all values of k or n_2,

$$\mathrm{F} \equiv (\widehat{rr})_c - (\widehat{rr_2})_c = 2\pi\sigma^2 \quad \ldots \quad \ldots \quad (21).*$$

The stresses in the medium on Maxwell's theory ought, as already explained, to be numerically equal but of opposite sign to those just found. Thus the action of the elastic medium is seen by (21) to supply the well known value for the electric force exerted on itself by a charged surface. The real fact is that both the transverse and radial stresses in the layer are only of the same order of magnitude as the stresses outside it, and so, to the present degree of approximation, F alone must suffice to balance the mutual repulsion existing between the elements of the layer. Thus (21) ought to be regarded rather as a partial verification of the accuracy of our work than as affording any support to the theory of an elastic medium.

While, as we have just seen, there is a difference between the values of the radial stresses at the *two* surfaces of the thin layer, *no discontinuity* such as Professor Minchin's treatment leads to *is found at either surface.* We shall not examine the stresses in the layer at present, but shall do so in treating the gravitational problem, and shall then show how the radial stress varies in a continuous way throughout the entire thickness.

It should be noticed that, to the present degree of approximation, the strains and stresses in the core and outside the layer are independent of the magnitude of the elastic constants in the layer,

* The suffixes c, c outside the brackets indicate the radii of the surfaces where the respective stresses are measured.

provided these constants be as originally assumed, neither very great nor very small compared to those in the other media.

The radial stress outside the layer is numerically double the transverse stress, and not equal to it as in Maxwell's theory. In the core the principal stresses are all equal and their values are everywhere the same. Not only are there in general stresses in the core but their magnitude depends partly on the external medium. Conversely by (20) the stresses in the external medium are partly dependent on the elastic properties of the core. These results are strikingly different from those observed in electrostatics, where the electric force, and so Maxwell's stress, vanishes inside the charged surface, and where the force outside does not depend on the internal dielectric. The only obvious way of getting rid of these discrepancies is to assume k/n_2 negligible.

Supposing the same media inside and outside the layer, this would require the medium to offer very great resistance to torsion but very small resistance to change of volume. Such properties, so far as my knowledge goes, have never been observed in actual experiments. The hypothesis is thus a very extreme one, but the value it supplies for the stresses outside the layer, viz.

$$\left.\begin{aligned}\widehat{rr}_2 &= -2\pi\sigma^2(e/r)^3,\\ \widehat{\theta\theta}_2 &= \quad\pi\sigma^2(e/r)^3\end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (22),$$

are so simple as to merit attention. These with their signs reversed bear a certain resemblance to Maxwell's stresses, whose values for a surface density σ are

$$\text{radial tension} = \text{transverse pressure} = 2\pi\sigma^2(e/r)^4 \quad \ldots \quad (23),$$

but the law of force is of course different.

Electrostatic Medium, Two Layers.

§ 8. In the electrostatic problem lines of force run from a charged surface to an oppositely charged. Thus there may seem a radical difference between the elastic problem last treated and that of a charged spherical surface. A closer approach to the conditions of the electrical problem would seem to be the elastic problem of two thin layers with properties such as those of the single layer of our last problem.

Let us suppose then that a spherical layer whose surfaces are of radii $OE = e$ and $OC = c$ has a density $+\rho_1$, while an outer layer whose surfaces are of radii $OB = b$ and $OA = a$ has a density $-\rho_2$, and let the densities elsewhere be negligible. We shall for simplicity suppose the elastic constants of both layers to be m_1, n_1, while everywhere else the elastic constants are m, n. It is assumed that m_1, n_1 are neither very great nor very small compared to m, n, so that the ordinary surface conditions may apply. The medium outside the outer layer extends to infinity. Positive matter is supposed to repel positive and attract negative and conversely.

In the inner layer at a distance r from the centre the bodily force is directed outwards and answers to the potential

$$V_1 = \tfrac{2}{3}\pi\rho_1 (r^2 + 2e^3 r^{-1}) \quad \ldots \quad \ldots \quad (24),$$

while in the outer layer the resultant outwardly directed bodily force answers to the potential

$$V_2 = \tfrac{2}{3}\pi\{2\rho_1(c^3 - e^3)r^{-1} + \rho_2(r^2 + 2b^3 r^{-1})\} \quad \ldots \quad (25).$$

The solution of the bodily equations in terms of arbitrary constants is as follows, quantities referring to the several media being distinguished by suffixes,

from centre O to E

$$\left.\begin{aligned} \Delta &= A, \\ u &= \tfrac{1}{3} r A, \\ \widehat{rr} &= (m - \tfrac{1}{3}n)A \end{aligned}\right\} \quad \ldots \quad \ldots \quad (26),$$

from E to C

$$\left.\begin{aligned} \Delta_1 &= -\tfrac{2}{3}\pi \frac{\rho_1^2}{m_1 + n_1}(r^2 + 2e^3 r^{-1}) + A_1, \\ u_1 &= -\tfrac{2}{3}\pi \frac{\rho_1^2}{m_1 + n_1}(\tfrac{1}{5}r^3 + e^3) + \tfrac{1}{3} r A_1 + r^{-2} B_1, \\ \widehat{rr}_1 &= -\tfrac{2}{3}\pi \frac{\rho_1^2}{m_1 + n_1}\{\tfrac{1}{5}r^2(5m_1 + n_1) + 2e^3 r^{-1}(m_1 - n_1)\} \\ &\qquad + (m_1 - \tfrac{1}{3}n_1)A_1 - 4r^{-3} n_1 B_1 \end{aligned}\right\} \quad \ldots \quad (27),$$

from C to B

$$\left.\begin{aligned} \Delta_2 &= A_2 \quad , \\ u_2 &= \tfrac{1}{3} r A_2 + r^{-2} B_2, \\ \widehat{rr}_2 &= (m - \tfrac{1}{3}n)A_2 - 4r^{-3} n B_2 \end{aligned}\right\} \quad \ldots \quad (28),$$

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from B to A

$$\left.\begin{aligned}\Delta_3 &= -\tfrac{2}{3}\pi\frac{\rho_2}{m_1+n_1}\{\rho_2(r^2+2b^3r^{-1})+2\rho_1(c^3-e^3)r^{-1}\}+A_3,\\ u_3 &= -\tfrac{2}{3}\pi\frac{\rho_2}{m_1+n_1}\{\rho_2(\tfrac{1}{5}r^3+b^3)+\rho_1(c^3-e^3)\}+\tfrac{1}{3}rA_3+r^{-2}B_3,\\ \widehat{rr}_3 &= -\tfrac{2}{3}\pi\frac{\rho_2}{m_1+n_1}[\rho_2\{\tfrac{1}{5}r^2(5m_1+n_1)+2b^3r^{-1}(m_1-n_1)\}\\ &\qquad+2\rho_1(c^3-e^3)r^{-1}(m_1-n_1)]+(m_1-\tfrac{1}{3}n_1)A_3\\ &\qquad-4r^{-3}n_1B_3\end{aligned}\right\}\quad\ldots\quad(29),$$

outside A

$$\left.\begin{aligned}\Delta_4 &= 0,\\ u_4 &= r^{-2}B_4,\\ \widehat{rr}_4 &= -4r^{-3}nB_4\end{aligned}\right\}\quad\ldots\quad\ldots\quad(30).$$

The continuity of the radial displacement and stress at the common surfaces leads to the following equations :

over $r=e$

$$A = -2\pi\frac{\rho_1^2}{m_1+n_1}(\tfrac{1}{5}e^2+e^2)+A_1+3e^{-3}B_1 \quad\ldots\quad (31),$$

$$(3m-n)A = -2\pi\frac{\rho_1^2}{m_1+n_1}\{\tfrac{1}{5}e^2(5m_1+n_1)+2e^2(m_1-n_1)\}$$
$$+(3m_1-n_1)A_1-12e^{-3}n_1B_1 \quad\ldots\quad (32),$$

over $r=c$

$$A_2+3c^{-3}B_2 = -2\pi\frac{\rho_1^2}{m_1+n_1}(\tfrac{1}{5}c^2+e^3c^{-1})+A_1+3c^{-3}B_1 \quad\ldots\quad (33),$$

$$(3m-n)A_2-12c^{-3}nB_2 = -2\pi\frac{\rho_1^2}{m_1+n_1}\{\tfrac{1}{5}c^2(5m_1+n_1)+2e^3c^{-1}(m_1-n_1)\}$$
$$+(3m_1-n_1)A_1-12c^{-3}n_1B_1 \quad\ldots\quad\ldots\quad (34),$$

over $r=b$

$$A_2+3b^{-3}B_2 = -2\pi\frac{\rho_2}{m_1+n_1}\{\rho_2(\tfrac{1}{5}b^2+b^2)+\rho_1(c^3-e^3)b^{-1}\}$$
$$+A_3+3b^{-3}B_3 \quad\ldots\quad\ldots\quad (35),$$

$$(3m-n)A_2-12b^{-3}nB_2 = -2\pi\frac{\rho_2}{m_1+n_1}[\rho_2\{\tfrac{1}{5}b^2(5m_1+n_1)$$
$$+2b^2(m_1-n_1)\}+2\rho_1(c^3-e^3)b^{-1}(m_1-n_1)]$$
$$+(3m_1-n_1)A_3-12b^{-3}n_1B_3 \quad\ldots\quad\ldots\quad (36),$$

over $r = a$

$$3a^{-3}B_4 = -2\pi\frac{\rho_2}{m_1+n_1}\{\rho_2(\tfrac{1}{5}a^2 + b^3a^{-1}) + \rho_1(c^3 - c^3)a^{-1}\}$$
$$+ A_3 + 3a^{-3}B_3 \quad \ldots \quad \ldots \quad (37),$$

$$-12a^{-3}nB_4 = -2\pi\frac{\rho_2}{m_1+n_1}[\rho_2\{\tfrac{1}{5}a^2(5m_1+n_1) + 2b^3a^{-1}(m_1-n_1)\}$$
$$+ 2\rho_1(c^3 - e^3)a^{-1}(m_1-n_1)] + (3m_1-n_1)A_3 - 12a^{-3}n_1B_3 \quad \ldots \quad (38).$$

The above eight equations suffice to determine the eight arbitrary constants of the solution.

From (31) and (32)

$$(3m - n + 4n_1)A = -6\pi\rho_1^2e^2 + 3(m_1+n_1)A_1 \quad \ldots \quad \ldots \quad (39),$$

$$\{3m_1 - n_1 - (3m-n)\}A = -\tfrac{6}{5}\pi\rho_1^2e^2 + 9e^{-3}(m_1+n_1)B_1 \quad \ldots \quad (40);$$

from (33) and (34)

$$(3m-n+4n_1)A_2 + 12c^{-3}(n_1-n)B_2 = -2\pi\rho_1^2(c^3+2e^3)c^{-1}$$
$$+ 3(m_1+n_1)A_1 \quad \ldots \quad \ldots \quad \ldots \quad (41),$$

$$\{3m_1 - n_1 - (3m-n)\}A_2 + 3c^{-3}(3m_1 - n_1 + 4n)B_2$$
$$= \tfrac{2}{5}\pi\rho_1^2(2c^3 - 5e^3)c^{-1} + 9c^{-3}(m_1+n_1)B_1 \quad \ldots \quad (42);$$

from (35) and (36)

$$(3m-n+4n_1)A_2 + 12b^{-3}(n_1-n)B_2$$
$$= -2\pi\rho_2\{3\rho_2b^2 + 2\rho_1(c^3-e^3)b^{-1}\} + 3(m_1+n_1)A_3 \quad \ldots \quad (43),$$

$$\{3m_1 - n_1 - (3m-n)\}A_2 + 3b^{-3}(3m_1 - n_1 + 4n)B_2$$
$$= -2\pi\rho_2\{\tfrac{3}{5}\rho_2b^2 + \rho_1(c^3-e^3)b^{-1}\} + 9b^{-3}(m_1+n_1)B_3 \quad \ldots \quad (44);$$

from (37) and (38)

$$12a^{-3}(n_1-n)B_4 = -2\pi\rho_2\{\rho_2(a^2 + 2b^3a^{-1}) + 2\rho_1(c^3-e^3)a^{-1}\}$$
$$+ 3(m_1+n_1)A_3 \quad \ldots \quad \ldots \quad \ldots \quad (45),$$

$$3a^{-3}(3m_1 - n_1 + 4n)B_4 = 2\pi\rho_2a^{-1}\{\tfrac{1}{5}\rho_2(2a^3 - 5b^3) - \rho_1(c^3 - e^3)\}$$
$$+ 9a^{-3}(m_1+n_1)B_3 \quad \ldots \quad \ldots \quad \ldots \quad (46);$$

from (39) and (41) eliminating A_1

$$(3m-n+4n_1)(A_2 - A) + 12c^{-3}(n_1-n)B_2$$
$$= -2\pi\rho_1^2c^{-1}(c-e)^2(c+2e) \quad \ldots \quad \ldots \quad (47);$$

from (40) and (42) eliminating B_1

$$\{3m_1 - n_1 - (3m - n)\}(c^3A_2 - e^3A) + 3(3m_1 - n_1 + 4n)B_2$$
$$= \tfrac{2}{5}\pi\rho_1^2(c-e)^2(2c^3 + 4c^2e + 6ce^2 + 3e^3) \quad \ldots \quad \ldots \quad (48);$$

from (43) and (45) eliminating A_3

$$(3m - n + 4n_1)A_2 + 12(n_1 - n)(b^{-3}B_2 - a^{-3}B_4)$$
$$= 2\pi\rho_2\frac{(a-b)}{a}\{\rho_2(a-b)(a+2b) - 2\rho_1 b^{-1}(c-e)(c^2+ce+e^2)\} \quad \ldots \quad (49);$$

from (44) and (46) eliminating B_3

$$b^3\{3m_1 - n_1 - (3m - n)\}A_2 + 3(3m_1 - n_1 + 4n)(B_2 - B_4)$$
$$= 2\pi\rho_2(a-b)\{-\tfrac{1}{3}\rho_2(a-b)(2a^3 + 4a^2b + 6ab^2 + 3b^3)$$
$$+ \rho_1(c-e)(a+b)(c^2+ce+e^2)\} \quad \ldots \quad \ldots \quad \ldots \quad (50).$$

The equations (47)–(50) are true whatever be the thickness of the layers, and the determination from them of A, A_2, B_2, B_4 presents no difficulty apart from the length of the expressions. When these four constants are determined the other four, A_1, B_1, A_3, B_3 may easily be found by means of (39), (40), (45), (46).

§ 9. For the electrostatic problem we shall confine our attention to the case when the layers are very thin, *i,e.*, when $(c-e)/e$ and $(a-b)/a$ are very small. For this case putting

$$\rho_1(c-e) = \sigma_1, \ \rho_2(a-b) = \sigma_2 \quad \ldots \quad \ldots \quad (51),$$

we easily find from (47)-(50), retaining only lowest powers of σ_1 and σ_2,

$$A = \frac{2\pi}{m+n}\{\sigma_1^2 - 2\sigma_1\sigma_2(c/b)^2 + \sigma_2^2\} \quad \ldots \quad \ldots \quad (52),$$

$$A_2 = \frac{2\pi}{m+n}\{\sigma_2^2 - 2\sigma_1\sigma_2(c/b)^2\} \quad \ldots \quad \ldots \quad \ldots \quad (53),$$

$$B_2 = \tfrac{2}{3}\frac{\pi}{m+n}\sigma_1^2c^3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (54),$$

$$B_4 = \tfrac{2}{3}\frac{\pi}{m+n}\{\sigma_1^2c^3 - 2\sigma_1\sigma_2c^2b + \sigma_2^2b^3\} \quad \ldots \quad \ldots \quad (55).$$

Substituting these values, and denoting the bulk modulus outside the layers by k as before, we find:

from O to E

$$\frac{u}{r} = \frac{1}{3k}\widehat{rr} = \tfrac{2}{3}\frac{\pi}{m+n}\{\sigma_1^2 - 2\sigma_1\sigma_2(c/b)^2 + \sigma_2^2\} \quad \ldots \quad (56),$$

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from C to B

$$\left.\begin{aligned} u_2 &= \tfrac{2}{3}\frac{\pi}{m+n}\left[r\{\sigma_2 - 2\sigma_1\sigma_2(c/b)^2\} + r^{-2}c^3\sigma_1^2\right], \\ \frac{du_2}{dr} &= \tfrac{2}{3}\frac{\pi}{m+n}\{\sigma_2^2 - 2\sigma_1\sigma_2(c/b)^2 - 2\sigma_1^2(c/r)^3\}, \\ \widehat{rr}_2 &= \frac{2\pi}{m+n}\left[k\{\sigma_2^2 - 2\sigma_1\sigma_2(c/b)^2\} - \tfrac{4}{3}n(c/r)^3\right] \end{aligned}\right\} \quad \ldots \quad (57),$$

outside A

$$\frac{u_4}{r} = -\tfrac{1}{2}\frac{du_4}{dr} = -\frac{\widehat{rr}_4}{4n} = \tfrac{2}{3}\frac{\pi}{m+n}r^{-3}\{\sigma_1^2c^3 - 2\sigma_1\sigma_2c^2b + \sigma_2^2b^3\} \quad \ldots \quad (58).$$

All these expressions are independent of the elastic constants of the material in the thin layers.

The inwardly directed resultant of the forces exerted by the two adjacent media on the inner layer per unit of surface is to the present degree of approximation

$$(\widehat{rr})_c - (\widehat{rr}_2)_c = 2\pi\sigma_1^2 \quad \ldots \quad \ldots \quad (59)$$

while that on the outer layer is

$$(\widehat{rr}_2)_b - (\widehat{rr}_4)_a = 2\pi\{\sigma_2^2 - 2\sigma_1\sigma_2(c/b)^2\} \quad \ldots \quad (60).$$

These values supply a partial verification of the accuracy of our work, for the transverse stress in either layer is only of the same order of magnitude as the radial stress, and to the present degree of approximation the resultant action of the adjacent media must balance the gravitational force.

This is obviously true of the inner layer on which no gravitational force is exerted by the other. Again, the outer layer exerts on itself a force $2\pi\sigma_2^2$ outwards per unit of surface, while the inner layer contributes a force $(4\pi\sigma_1c^2)\sigma_2/b^2$ inwards.

§ 10. In the problem analogous to the electrostatic problem, where the charges on the two surfaces are equal and opposite, we are to put

$$4\pi c^2\sigma_1 = 4\pi b^2\sigma_2 = Q \quad \ldots \quad \ldots \quad (61),$$

so that Q answers to the charge on the positively electrified surface.

Making this substitution and using (18), we get

from O to E

$$\frac{u}{r} = \frac{\widehat{rr}}{3k} = \frac{Q^2}{8\pi}\,\frac{b^4 - c^4}{b^4c^4}/(3k+4n) \quad \ldots \quad \ldots \quad (62),$$

from C to B

$$\left.\begin{aligned} u_2 &= \frac{Q^2}{8\pi}\frac{1}{3k+4n}\left(-\frac{r}{b^4}+\frac{1}{cr^2}\right), \\ \widehat{rr}_2 &= -\frac{Q^2}{8\pi}\frac{1}{3k+4n}\left(\frac{3k}{b^4}+\frac{4n}{cr^3}\right), \\ \widehat{\theta\theta}_2 &= -\frac{Q^2}{8\pi}\frac{1}{3k+4n}\left(\frac{3k}{b^4}-\frac{2n}{cr^3}\right) \end{aligned}\right\} \quad \ldots \quad (63),$$

outside A

$$\frac{u_4}{r} = -\frac{\widehat{rr}_4}{4n} = \frac{\widehat{\theta\theta}_4}{2n} = \frac{Q^2}{8\pi}\frac{1}{3k+4n}\frac{b-c}{bcr^3} \quad \ldots \quad (64).$$

In this case the right hand side of (60) equals $-2\pi\sigma_2^2$; or the resultant of the actions of the adjacent media on the outer layer is numerically the same as if the inner layer did not exist, but is directed outwards.

The strains and stresses in the core will not vanish even approximately unless either $(b-c)/c$ be very small, *i.e.*, the layers very close together, or else k/n be negligible. While outside the outer layer the strains and stresses will be negligible only if $(b-c)/c$ be very small.

In the medium between the two layers the radial stress is always a pressure, but the transverse stress may be a pressure or a tension according to circumstances. It will be everywhere a tension if

$$n > \tfrac{3}{2}k(c/b) \quad \ldots \quad \ldots \quad \ldots \quad (65),$$

and everywhere a pressure if

$$n < \tfrac{3}{2}k(c/b)^4 \quad \ldots \quad \ldots \quad \ldots \quad (66).$$

The former case includes that in which k/n is negligible. When this is so we have between the layers

$$-\widehat{rr}_2 = 2\widehat{\theta\theta}_2 = Q^2/(8\pi cr^3) \quad \ldots \quad \ldots \quad (67),$$

outside the outer layer

$$-\widehat{rr}_4 = 2\widehat{\theta\theta}_4 = Q^2(b-c)/(8\pi bcr^3) \quad \ldots \quad (68).$$

For the strain and stress in a medium propagating electrostatic action the signs of all the above expressions are to be reversed. According to Maxwell's theory the stresses should vanish except between the two charged surfaces, and there we should have a radial tension $Q^2/(8\pi r^4)$ and an equal pressure in all orthogonal directions. The nearest approach to coincidence with his theory is thus when

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the distance apart of the two layers is very small compared to the radius of either surface.

For the case when the radius of the outer surface becomes infinite, while the total distribution over it remains numerically equal to that on the inner surface, we put $(c/b)=0$ in (62) and (63) while regarding r as finite. The results so obtained agree with those already found in (19) and (20) for a single layer when $n_2=n$, so that the existence of the layer at infinity is of no consequence.

Gravitational Medium, Single Layer.

§ 11. We now pass to the gravitational problem and consider first a single layer of radii e, c, density ρ, and elastic constants m_1, n_1, containing and surrounded by elastic media. To shorten the expressions we shall suppose the external and internal media the same and possessed of elastic constants m, n. In this case the layer being self-attractive we must change the sign of all terms containing ρ^2 or σ^2 in equations (9)–(23). Putting

$$\frac{c^3}{e^3}(3m-n+4n_1)(3m_1-n_1+4n)-4(n_1-n)\{3m_1-n_1-(3m-n)\}=\mathrm{D} \quad \ldots \quad (69),$$

we easily find from the surface conditions

$$\mathrm{A.D}=-2\pi\rho^2(c-e)^2e^{-3}[c^2(c+2e)(3m_1-n_1+4n)+\tfrac{4}{5}(n_1-n)(2c^3+4c^2e+6ce^2+3e^3)] \quad \ldots \quad (70),$$

$$\mathrm{B_2.D}=-\tfrac{2}{3}\pi\rho^2(c-e)^2[c^2(c+2e)\{3m_1-n_1-(3m-n)\}+\tfrac{1}{5}(c/e)^3(2c^3+4c^2e+6ce^2+3e^3)(3m-n+4n_1)] \quad \ldots \quad (71),$$

$$\mathrm{A_1.D}=-2\pi\rho^2e^{-3}\Big[c^2(c^3+2e^3)(3m-n+4n_1)+\tfrac{4}{5}\frac{n_1-n}{m_1+n_1}\Big\{3e^5\Big(2(3m-n)-5m_1+3n_1\Big)-c^2(c^3+5e^3)(3m-n+4n_1)\Big\}\Big] \quad \ldots \quad \ldots \quad \ldots \quad (72),$$

$$\mathrm{B_1.D}=-\tfrac{2}{3}\pi\rho^2c^2\Bigg[\Big\{c^3+2e^3-\tfrac{4}{5}(c^3+5e^3)\frac{n_1-n}{m_1+n_1}\Big\}\{3m_1-n_1-(3m-n)\}+\tfrac{3}{5}ce^2\,\frac{3m_1-n_1+4n}{m_1+n_1}\,\{2(3m-n)-5m_1+3n_1\}\Bigg] \quad \ldots \quad (73).$$

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Substituting the value of A in (8) and that of B_2 in (10) we have the displacements in the core and outside the layer. Again substituting for A_1 and B_1 in (9) and changing the sign of ρ^2 in the particular solutions we have the displacement in the thin layer. The solution so obtained is in all respects complete.

Assuming the bulk modulus and rigidity positive quantities, we may easily prove that D is essentially positive and A essentially negative. Thus by (5) and (8) the three principal stresses at every point in the core consist of three equal pressures of constant value. If $3m_1 - n_1 > 3m - n$ the value of B_2 is essentially negative, but if $(3m - n)/(3m_1 - n_1)$ be large and $c - e$ be not very small B_2 may be positive. We conclude from (5) and (10) that outside the layer the radial stress is always opposite in sign to and numerically double of the transverse stress; the radial stress is necessarily a tension if the layer have a larger bulk modulus,—*i.e.*, is less compressible—than the other medium, but if the layer be considerably more compressible than the other medium, and be neither of unusually great rigidity nor extremely thin, the radial stress may be a pressure.

§ 12. To enter into details in the general case would involve dealing with very cumbrous algebraical expressions. I shall thus consider only a special case, which sufficiently illustrates the nature of the results. Thus let

$$n_1 = n,$$

or suppose that the layer has the same rigidity as the other medium and differs from it only in compressibility. In this case we find:

in the core

$$u/r = \widehat{rr}/(3m - n) = -\tfrac{2}{9}\pi\frac{\rho^2}{m+n}(c-e)^2\frac{c+2e}{c} \qquad \dots (74),$$

in the layer

$$u_1 = \tfrac{2}{3}\pi\frac{\rho^2}{m_1+n}\left[\tfrac{1}{5}r^3 + e^3 - \tfrac{1}{3}(r/c)(c^3+2e^3)\right.$$
$$\left. - \frac{e^3}{cr^2}\,\frac{1}{m+n}\{\tfrac{1}{5}ce^2(m+n) + \tfrac{1}{3}(c-e)^2(c+2e)(m_1-m)\}\right] \dots (75),$$

$$\widehat{rr_1} = \tfrac{2}{9}\frac{\pi\rho^2}{m_1+n}\left[(c-r)\left(\frac{2e^3}{cr} - c - r\right)(3m_1 - n) + \tfrac{4}{5}\frac{(r-e)^2}{r^3}(2r^3\right.$$
$$\left. + 4r^2e + 6re^2 + 3e^3)n + 4\frac{e^3}{cr^3}(c-e)^2(c+2e)\frac{n(m_1-m)}{m+n}\right] \dots (76),$$

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outside the layer

$$\frac{u_2}{r} = -\frac{\widehat{rr_2}}{4n} = -\frac{2}{9}\frac{\pi\rho^2}{m_1+n}\frac{(c-e)^2}{r^3}\left\{\frac{1}{5}(2c^3+4c^2e+6ce^2+3e^3) + \frac{e^3(c+2e)}{c}\frac{m_1-m}{m+n}\right\} \ldots (77).$$

From (76) we find for the radial stresses at the inner and outer surfaces of the layer the respective values

$$(\widehat{rr_1})_e = -\frac{2}{9}\pi\rho^2(c-e)^2\frac{c+2e}{c}\frac{3m-n}{m+n} \quad \ldots \quad \ldots \quad \ldots \quad (78),$$

$$(\widehat{rr_1})_b = \frac{8}{9}\pi\rho^2\frac{n}{m_1+n}\frac{(c-e)^2}{c^3}\left\{\frac{1}{5}(2c^3+4c^2e+6ce^2+3e^3) + e^3\frac{(c+2e)}{c}\frac{m_1-m}{m+n}\right\} \ \ldots \ (79).$$

Comparing (74) with (78) and (77) with (79) we see that our solution gives, as it ought, complete continuity in the value of the radial stress.

§ 13. To examine in detail the solution for a layer of any thickness would occupy too much space. When the layer is very thin the first approximations to the displacements, strains, and stresses in the core and outside the layer are given by (19) and (20), when $-\sigma^2$ is replaced by $(\rho h)^2$. The results so obtained apply whether the media outside and inside the layer are the same or not, and so are in one way more general than the results (74)–(77). Their degree of approximation is not however sufficiently close to show the variation of the strains and stresses throughout the layer. This variation may be satisfactorily illustrated by the special case, $n_1 = n$, so we shall confine our attention to it. Thus taking (75) let

$$c - e = h, \ r - e = \xi,$$

so that h is the thickness of the layer and ξ the distance of a point in it from the inner surface, and expand in powers of the small quantities h/e and ξ/e to any required degree of approximation. For our present purpose we may content ourselves with

$$u_1 = -\frac{2}{3}\pi\rho^2\frac{h^2e}{m+n}\left\{1 - \frac{2}{3}\frac{h}{e} - 2\frac{\xi}{e}\frac{m_1-m}{m_1+n} + \frac{\xi}{e}\left(1 - \frac{\xi^2}{h^2}\right)\frac{m+n}{m_1+n}\right\} \ldots (80).$$

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As first approximations we find for the principal strains and stresses throughout the layer:

$$\left.\begin{aligned} u_1/r &= -\tfrac{2}{3}\pi\rho^2\frac{h^2}{m+n}, \\ \frac{du_1}{dr} &\equiv \frac{du_1}{d\xi} = -\tfrac{2}{3}\pi\rho^2\frac{h^2}{m+n}\left\{1 - 3\frac{m_1-m}{m_1+n} - 3\frac{\xi^2}{h^2}\,\frac{m+n}{m_1+n}\right\}, \\ \widehat{rr}_1 &= 2\pi\rho^2\left\{\xi^2 - \tfrac{1}{3}h^2\frac{3m-n}{m+n}\right\}, \\ \widehat{\theta\theta}_1 &= -\tfrac{4}{3}\pi\rho^2h^2\frac{n}{m+n}\left\{1 + \tfrac{3}{2}\left(1 - \frac{\xi^2}{h^2}\right)\frac{(m_1-n)(m+n)}{n(m_1+n)}\right\} \end{aligned}\right\} \quad (81).$$

Thus the transverse strain is always a compression and if $m_1 > n$, as appears the case in all satisfactory experiments, the transverse stress is always a pressure. The radial strain is a compression at the inner surface if $\quad 3m - n > 2(m_1 - n),$

which will be the case unless the layer be much less compressible than the adjacent media. The radial strain is algebraically greatest at the outer surface where it is always an extension. The radial stress is always a pressure at the inner surface, a tension at the outer, and varies continuously throughout the thickness.

The maxima values $\bar{s}$ and $\bar{S}$ of the greatest strain* and the stress-difference*—*i.e.*, the difference between the algebraically greatest and least stresses at a point—occur at the outer surface, and we have

$$\left.\begin{aligned} \bar{s} &= \tfrac{4}{3}\pi\rho^2h^2/(m+n), \\ \bar{S} &= 4\pi\rho^2h^2n/(m+n) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (82).$$

It is easy to prove that the dilatation vanishes over the outer surface and elsewhere is negative, so that the volume occupied by the layer is reduced.

As first approximations for the increments in the radius e and thickness h of the layer we have

$$\left.\begin{aligned} \delta e/e &= -\tfrac{2}{3}\pi\rho^2h^2/(m+n), \\ \delta h/h &= \quad \tfrac{4}{3}\pi\rho^2\frac{h^2}{m+n}\,\frac{m_1-m}{m_1+n} \end{aligned}\right\} \quad \ldots \quad (83).\dagger$$

* The magnitude of one or other of these quantities is frequently regarded as measuring the "tendency to rupture" in the material. See *Philosophical Magazine*, September 1891, pp. 239-242.

† The letter δ denotes the increment of the quantity denoted by the following letter.

Thus the radius of the shell is always reduced. The thickness is increased or diminished according as the shell is less or more compressible than the adjacent media.

Gravitational Medium, Two Layers.

§ 14. The case of two gravitating layers in which the elastic constants have the values m_1, n_1, while in the surrounding media they have the values m, n, may be deduced from the treatment of the two electrostatic layers by changing the signs of ρ_1^2, ρ_2^2 but not of $\rho_1\rho_2$, where ρ_1, ρ_2 denote the densities of the inner and outer layers, the radii of whose surfaces in ascending order of magnitude are e, c, b, a. We shall only glance at the case when the thicknesses

$$c - e = h_1, \quad a - b = h_2$$

are very small. Putting

$$\rho_1 h_1 = \sigma_1, \quad \rho_2 h_2 = \sigma_2,$$

we find from (56), (57), (58) as first approximations, employing k as before for the bulk modulus outside the layers,

in the core

$$\frac{u}{r} = \frac{\widehat{rr}}{3k} = -\tfrac{2}{3}\frac{\pi}{m+n}\{\sigma_1^2 + 2\sigma_1\sigma_2(c/b)^2 + \sigma_2^2\} \quad \ldots \ (84),$$

between the layers

$$u_2 = -\tfrac{2}{3}\frac{\pi}{m+n}[r\{\sigma_2^2 + 2\sigma_1\sigma_2(c/b)^2\} + c^3r^{-2}\sigma_1^2] \quad \ldots \ (85),$$

$$\widehat{rr}_2 = -\tfrac{2}{3}\frac{\pi}{m+n}[3k\{\sigma_2^2 + 2\sigma_1\sigma_2(c/b)^2\} - 4n(c/r)^3\sigma_1^2] \ \ldots \ (86),$$

outside the outer layer

$$\frac{-u_4}{r} = \frac{\widehat{rr}_4}{4n} = \tfrac{2}{3}\frac{\pi r^{-3}}{m+n}\{\sigma_1^2c^3 + 2\sigma_1\sigma_2c^2b + \sigma_2^2b^3\} \quad \ldots \ (87).$$

To a first approximation we have

$$u/r = (u_2/r)_c, \text{ and } (u_2/r)_b = (u_4/r)_a.$$

Thus, since the radial displacement is continuous, we deduce that the transverse strain in each layer has a nearly constant value, that value being given for the inner layer by (84) and for the outer layer by the value of u_4/r in (87) when $r = a$. We also know the values of the radial stress over the surfaces of the layers from the fact that there is no discontinuity in that stress. All the strains and stresses in the present case vary as the squares or products of the thicknesses of the layers.

In the previous gravitational problems the stresses we have found are those which maintain equilibrium when forces at a distance act. If we suppose the stresses reversed we get the stress system required to propagate gravitational forces in a hypothetical medium. These reversed stresses are to be regarded as residing in the medium and it is this medium and not any sensible substance to which the elastic constants of the solution belong. How such stresses may be excited, or what connection there may exist between the medium and sensible matter does not come within the scope of the present enquiry.

Single Gravitating Shell.

§ 15. The problems previously considered are of a speculative nature referring to the action of some medium to be classed under the general title "ether". To prevent misconception we add the solution of the corresponding problems in their relation to the actual visible matter of which the spherical layers are composed.

The first problem then is that of a spherical shell of ordinary isotropic material, say of density ρ and elastic constants m, n, existing alone in space, acted on by no surface forces and no bodily forces other than its own gravitation. The potential of the bodily forces in the shell, supposing its radii e and c, is given by

$$V = -\tfrac{2}{3}\pi\rho(r^2 + 2e^3r^{-1}).$$

The solution in arbitrary constants is analogous to (27), but the constants must now be determined from the conditions that the radial stress vanishes over both surfaces. It is unnecessary to record the values found for the constants. The displacement is

$$u = \tfrac{2}{3}\frac{\pi\rho^2}{m+n}\Big[\tfrac{1}{5}r^3 + e^3 - \frac{r}{(c^3-e^3)(3m-n)}\{\tfrac{1}{5}(c^5-e^5)(5m+n) + 2e^3(c^2-e^2)(m-n)\} - \frac{r^{-2}}{2(c^3-e^3)n}\{\tfrac{1}{10}e^3c^3(c^2-e^2)(5m+n) - e^5c^2(c-e)(m-n)\}\Big] \quad \ldots \quad (88).$$

The strains and stresses may easily be deduced. The radial stress it will be found vanishes, as it ought, over both surfaces.

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§ 16. The most interesting case for comparison with the previous problems is that of a very thin shell. For it we find, with our previous notation,

$$\left.\begin{aligned} u &= -\frac{\pi\rho^2 he^2(m+n)}{2n(3m-n)}\left\{1+\tfrac{4}{3}\frac{h}{e}\,\frac{m}{m+n}-2\frac{\xi}{e}\,\frac{m-n}{m+n}\right\}, \\ \widehat{rr} &= -2\pi\rho^2\xi(h-\xi)\left\{1+\tfrac{4}{3}\frac{h}{e}\,\frac{m}{m+n}-\tfrac{2}{3}\frac{\xi}{e}\,\frac{m+3n}{m+n}\right\} \end{aligned}\right\} \quad \ldots \quad (89).$$

Comparing (89) with (81) we see that in both cases the radial stress is of order h^2, but the radial displacement in (81) is also of order h^2 whereas in (89) it is of order h and so enormously greater. In the case of the "ether" media the contraction of the layer was opposed by the contiguous media, and it was the action of the latter media that sustained the gravitational forces. In the present case the forces opposing the contraction of the shell are derived solely from the elastic stresses in itself, and to produce transverse stresses of sufficient intensity for this end requires a contraction of very much greater magnitude than in the previous case. For purposes of comparison it will suffice to confine our attention to the first approximation in the present case. We shall employ the same notation as before, and in addition shall put

$$n(3m-n)/m = \mathrm{E},$$
$$(m-n)/2m = \eta,$$
$$4\pi\rho h = g,$$

so that E is Young's modulus and η Poisson's ratio for the shell, while g is the acceleration at its outer surface due to the gravitational forces. We easily find

$$\left.\begin{aligned} u/r = \delta e/e &= -\tfrac{1}{4}g\rho e\frac{1-\eta}{\mathrm{E}}, \\ \frac{du}{dr} = \delta h/h &= \tfrac{1}{2}g\rho e\frac{\eta}{\mathrm{E}}, \\ \widehat{rr} &= -\tfrac{1}{2}g\rho e\frac{\xi(h-\xi)}{eh}, \\ \overline{\mathrm{S}} &= -\widehat{\theta\theta} = \tfrac{1}{4}g\rho e \end{aligned}\right\} \quad \ldots \quad \ldots \quad (90).$$

Strictly $\overline{\mathrm{S}}$ is the difference between the radial and the transverse stresses, but the former is negligible compared to the latter. The largeness of the transverse stress is perhaps the most striking feature of the solution. To show that to the present degree of approximation it balances the gravitational forces, we employ the known result that when a spherical membrane of radius e contains a gas at pressure p the requisite tension T in the membrane is given by

$$p = 2\mathrm{T}/e.$$

In other words the resultant of the tensions is a normal force $2\mathrm{T}/e$ directed inwards. Thus if instead of tensions in a membrane we have transverse pressures in an elastic shell whose intensity is $-\widehat{\theta\theta}$, or T′, over unit area, and so for the entire thickness $\mathrm{T}'h$ per unit arc of surface, their resultant is an outwardly directed force whose value is $2\mathrm{T}'h/e$ or by (90) is $2\pi(\rho h)^2$. But this is numerically equal and oppositely directed to the gravitational action of the shell on itself.

From (90) we see that the strains and the transverse stress are all to a first approximation constant throughout the thickness. Assuming Poisson's ratio positive, the radial strain is everywhere an extension and the thickness of the shell is increased. The transverse strain is always a compression and the radius of the shell is diminished. The radial and transverse stresses are both pressures. The former is to a first approximation a maximum at the mid thickness and diminishes numerically as we approach either surface.

§ 17. As an idea of the actual magnitude of the stresses may be of use, let us consider the approximate value of $\overline{\mathrm{S}}$ in a shell of the radius of the earth's outer surface, due to its own gravitation only. Let g' be the value of "gravity" at the surface of a solid sphere of the same radius e and density ρ as the shell, then

$$g' = \tfrac{4}{3}\pi\rho e,$$
$$\overline{\mathrm{S}} = \tfrac{3}{4}g'\rho h.$$

At the earth's surface a cubic foot of water weighs about $62\frac{1}{2}$ lbs., and thus if the specific gravity of the shell be equal to the mean value of the earth, say 5·5, we deduce for the approximate value of the maximum stress-difference in a shell i miles thick, and 4000 miles radius,

$$\overline{\mathrm{S}} = (4{\cdot}2)i \text{ tons weight per square inch.}$$

The conditions are of course totally different from those existing in the outer layer of a gravitating *solid* sphere.

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Two Gravitating Shells.

§ 18. The case of two concentric shells of ordinary matter acted on solely by their gravitational forces may be solved in a similar way. In the inner shell, which is assumed not to touch the outer, the solution is exactly the same as in the previous case because the forces exerted by the outer shell are nil. The forces on the outer shell arise partly from its mutual gravitation, partly from the attraction of the inner shell. The consequences of the former set of forces we know already, and so, as strains are superposable when kept within the limits to which the mathematical theory applies, we need now investigate only the action of the inner shell on the outer.

The forces exerted by a shell of radii e, c and density ρ_1 on an external shell are derived from the potential

$$\tfrac{4}{3}\pi\rho_1(c^3-e^3)/r = \mathrm{M}/r \text{ say.}$$

The solution thus obtained for the action solely of the inner shell on the outer is

$$u = -\tfrac{1}{2}\frac{\mathrm{M}\rho_2}{m+n}\left\{1 - 2r\frac{a^2-b^2}{a^3-b^3}\,\frac{m-n}{3m-n} + \frac{a^2b^2}{r^2}\,\frac{a-b}{a^3-b^3}\,\frac{m-n}{2n}\right\} \quad \ldots (91).$$

From this we find for any thickness of shell

$$\frac{du}{dr} = \mathrm{M}\rho_2\frac{1}{a^2+ab+b^2}\,\frac{m-n}{m+n}\left(\frac{a+b}{3m-n}+\frac{a^2b^2}{2nr^3}\right) \quad \ldots \quad (92),$$

$$\frac{u_a-u_b}{a-b} = \frac{\delta h}{h} = \tfrac{3}{4}\mathrm{M}\rho_2\frac{a+b}{a^2+ab+b^2}\,\frac{m-n}{n(3m-n)} \quad \ldots \quad \ldots \quad (93),$$

$$\widehat{rr} = -\mathrm{M}\rho_2\frac{(a-r)(r-b)(ab+ar+br)}{r^3(a^2+ab+b^2)}\,\frac{m-n}{m+n} \quad \ldots \quad (94).$$

The radial stress is thus everywhere a pressure; it vanishes of course over both surfaces. The radial strain is everywhere an extension.

The displacement, and thence the strains and stresses in the outer shell, due to its mutual gravitation, may be deduced from (88) by replacing ρ, e, c, by ρ_2, b, a respectively.

When the shell is very thin let

$$a-b=h, \quad r-b=\xi,$$

and let
$$\tfrac{4}{3}\pi\rho_1\frac{(c^3-e^3)}{a^2} = g_1, \quad 4\pi\rho_2 h_2 = g_2,$$

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so that g_1 and g_2 are respectively the accelerations at the outer surface of the outer shell due to the attraction of the inner shell and its own gravitation. Then to a first approximation the complete values for the strains and stresses in the outer shell are as follows:—

$$\left.\begin{aligned} u/r &= \delta a/a = -\tfrac{1}{4}(g_2+2g_1)\rho_2 a(1-\eta)/\mathrm{E}, \\ \frac{du}{dr} &= \delta h/h = \tfrac{1}{2}(g_2+2g_1)\rho_2 a\eta/\mathrm{E}, \\ \widehat{rr} &= -\tfrac{1}{2}g_2\rho_2 a\frac{\xi(h-\xi)}{ah}, \\ \overline{\mathrm{S}} &= -\widehat{\theta\theta} = \tfrac{1}{4}(g_2+2g_1)\rho_2 a \end{aligned}\right\} \quad \ldots \quad (95).$$

The intensity of the actual bodily force in the outer shell varies regularly from g_1 at the inner to g_1+g_2 at the outer surface. Thus the above results show that to a first approximation the strains and the transverse stress in the shell are the same as if the bodily forces had at every point of the thickness a constant value equal to the mean of the actual values. The value of the radial stress depends even to a first approximation on the law of distribution of the bodily forces, but this stress is negligible compared to the transverse stress. So far as concerns the results (95) the inner shell may be a solid core or a shell of any thickness. The only limitation is that the two shells must not be in contact.

The Elements of Quaternions (*Second Paper*).

Discussion of the Proofs of the Laws of the Quaternionic Algebra.

[*Abstract.*]

By Dr William Peddie.

Three main laws regulate the treatment of ordinary algebraic quantities. These are the Associative Law, the Distributive Law, and the Commutative Law. If a, b, c, ..., represent quantities dealt with in the algebra, the associative law of multiplication asserts that $a(bc)=(ab)c$, where the brackets have the usual meaning that the quantity within them is to be regarded as a single quantity: the distributive law of multiplication asserts that $(a+b)(c+d)=ac+bc+ad+bd$: and the commutative law gives $ab=ba$. With regard to addition, the associative law asserts that $(a+b)+c=a+(b+c)$: and the commutative law gives $a+b=b+a$.