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Linear Maps Transforming the Unitary Group

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Abstract. Let $U(n)$ be the group of $n \times n$ unitary matrices. We show that if ϕ is a linear transformation sending $U(n)$ into $U(m)$, then m is a multiple of n, and ϕ has the form

$$A \mapsto V[(A \otimes I_s) \oplus (A^t \otimes I_r)]W$$

for some $V, W \in U(m)$. From this result, one easily deduces the characterization of linear operators that map $U(n)$ into itself obtained by Marcus. Further generalization of the main theorem is also discussed.

1 Main Result

Denote by M_n the algebra of $n \times n$ complex matrices. Let $U(n)$ be the group of $n \times n$ unitary matrices. The purpose of this note is to prove the following result.

Theorem 1 *Suppose $\phi\colon M_n \to M_m$ is a linear transformation satisfying $\phi\big(U(n)\big) \subseteq U(m)$. Then m is a multiple of n and*

$$\phi(A) = V[(A \otimes I_s) \oplus (A^t \otimes I_r)]W$$

for some fixed $V, W \in U(m)$.

For any linear map $\phi\colon M_n \to M_m$ satisfying $\phi\big(U(n)\big) \subseteq U(m)$, one can replace it by the mapping ψ of the form $A \mapsto \phi(I_n)^{-1}\phi(A)$. Then $\psi\colon M_n \to M_m$ is linear, unital, *i.e.*, $\psi(I_n) = I_m$, and satisfies $\psi\big(U(m)\big) \subseteq U(n)$. Using this observation, one sees that Theorem 1 is equivalent to the following.

Theorem 2 *Let $\phi\colon M_n \to M_m$ be a unital linear transformation satisfying $\phi\big(U(n)\big) \subseteq U(m)$. Then m is a multiple of n and*

$$\phi(A) = V[(A \otimes I_s) \oplus (A^t \otimes I_r)]V^{-1} \tag{1}$$

for some fixed $V \in U(m)$.

By Theorems 1 and 2, one easily deduces the following result of Marcus [5].

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Corollary 3 *A linear operator ϕ on M_n satisfying $\phi\big(U(n)\big) \subseteq U(n)$ must be of the form*

$$A \mapsto VAW \quad \text{or } A \mapsto VA^tW$$

for some $V, W \in U(n)$. If, in addition, we assume that ϕ is unital, then ϕ is an (algebra) automorphism or anti-automorphism.

Let $\mathrm{GL}(m)$ be the group of $m \times m$ invertible matrices. By a result of Auerbach [1] (see [3] for an elementary proof), if G is a bounded subgroup of $\mathrm{GL}(m)$, then there exists a positive definite matrix $P \in M_m$ such that $PGP^{-1} \subseteq U(m)$. So, if $\phi\colon M_n \to M_m$ satisfies $\phi\big(U(n)\big) \subseteq G$ for a bounded subgroup G of $\mathrm{GL}(m)$, then we may apply Theorem 1 to the mapping $A \mapsto P\phi(A)P^{-1}$ to determine the structure of ϕ. Thus, we have the following corollary.

Corollary 4 *Suppose $\phi\colon M_n \to M_m$ is a linear transformation such that $\phi\big(U(n)\big) \subseteq G$, where G is a bounded subgroup of $\mathrm{GL}(m)$. Then m is a multiple of n and*

$$\phi(A) = LV[(A \otimes I_s) \oplus (A^t \otimes I_r)]L^{-1} \tag{2}$$

for some fixed $L \in \mathrm{GL}(m)$ and $V \in U(m)$.

If we just assume that $\phi\big(U(n)\big) \subseteq \mathrm{GL}(m)$, the conclusion of Corollary 4 will not hold as shown by the following example.

Example 5 Consider the unital linear $\phi\colon M_2 \to M_2$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & ib \\ c & d \end{pmatrix}.$$

One readily checks that $\phi\big(U(2)\big) \subseteq \mathrm{GL}(2)$. However, ϕ does not preserve the rank of matrices, and hence is not of the form (2) with $L \in \mathrm{GL}(2)$ and $V \in U(2)$.

Marcus and Purves [6, Theorem 2.1] showed that Corollary 3 is valid if we replace $U(n)$ by $\mathrm{GL}(n)$. One may wonder whether Theorem 1 or Theorem 2 is valid if we replace $U(m)$ and $U(n)$ by $\mathrm{GL}(m)$ and $\mathrm{GL}(n)$, respectively. This is not true as shown by the following example, which is a slight modification of [2, Example 4.3 C].

Example 6 Consider the unital linear map $\phi\colon M_2 \to M_6$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} aI_3 & bI_3 \\ cI_3 & dI_3 \end{pmatrix} + 0_3 \oplus \begin{pmatrix} 0 & b & 0 \\ c & 0 & -b \\ 0 & c & 0 \end{pmatrix}.$$

One readily checks that $\det\big(\phi(A)\big) = \det(A)^3$, and hence $\phi\big(\mathrm{GL}(2)\big) \subseteq \mathrm{GL}(6)$. However, $\phi\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$ is not similar to $-I_3 \oplus I_3$. Hence, ϕ is not of the form (1) with $V \in \mathrm{GL}(6)$.

2 Proof of Theorem 2

Let $X = [1] \oplus -I_{n-1}$. Since $Y = \phi(X)$ and $\phi(0.6I + 0.8iX) = 0.6I + 0.8iY$ are unitary, it follows that Y is both hermitian and unitary. So we can further assume that $Y = I_k \oplus -I_{m-k}$; otherwise, replace ϕ by a mapping of the form $A \mapsto W^*\phi(A)W$ for some $W \in U(m)$ such that $W^*\phi(X)W = Y$. We always assume that

$$\phi(I_n) = I_m \quad \textit{and} \quad \phi([1] \oplus -I_{n-1}) = I_k \oplus -I_{m-k} \tag{3}$$

in the rest of the proof. Our result will follow once we establish the following.

Assertion *There exist $V \in U(m)$ and nonnegative integers r and s with $r+s = k$ such that $V\phi(A)V^*$ is a block matrix $(A_{ij})_{1\le i,j\le n}$, where $A_{ij} = a_{ij}I_s \oplus a_{ji}I_r$ for all $1 \le i$, $j \le n$.*

We prove the Assertion by induction on $n \ge 2$. When $n = 2$, consider the matrix $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that $\phi(T)$, $\phi(0.6I + 0.8iT)$ and $\phi\big(0.6([1] \oplus [-1]) + 0.8T\big)$ are all unitary, which is possible if and only if $k = m - k$, *i.e.* $m = 2k$, and $\phi(T) = \left(\begin{smallmatrix} 0 & U \\ U^* & 0 \end{smallmatrix}\right)$ for some unitary matrix $U \in U(k)$. We can further assume that $U = I_k$; otherwise, replace ϕ by the mapping $A \mapsto (U^* \oplus I)\phi(A)(U \oplus I)$. Next, consider $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then $\phi(S)$, $\phi(0.6I + 0.8S)$ and $\phi\big(0.6([1] \oplus [-1]) + 0.8iS\big)$ are all unitary, which is possible if and only if $\phi(S) = \left(\begin{smallmatrix} 0 & V \\ -V^* & 0 \end{smallmatrix}\right)$. Since $\phi(0.6T \pm 0.8iS)$ are also unitary, we see that V is hermitian. We can further assume that $V = I_s \oplus -I_{k-s}$; otherwise, replace ϕ by a mapping of the form $A \mapsto (W^* \oplus W^*)\phi(A)(W \oplus W)$, where $W \in U(m/2)$ satisfies $W^*VW = I_s \oplus -I_{k-s}$. As a result, the modified mapping is of the asserted form with $V = I_m$.

Now, suppose the Assertion is true for $n = p \ge 2$, and consider $n = p + 1$. By (3), we have

$$\phi([1] \oplus 0_p) = I_k \oplus 0_{m-k}.$$

Moreover, for any $U \in U(p)$ and any $\mu \in \mathbf{C}$ with $|\mu| = 1$, we have $\phi([1] \oplus \mu U) \in U(m)$. It follows that $\phi([1] \oplus U) = I_k \oplus \bar{\phi}(U) \in U(m)$. By induction assumption, there exist $W \in U(m-k)$ and integers l and s such that $m - k = pl$, and for any $A = (a_{ij}) \in M_p$ we have $\bar{\phi}(A) = W(A_{ij})W^*$, where $A_{ij} = a_{ij}I_s \oplus a_{ji}I_{l-s}$ for all $1 \le i$, $j \le p$. We may assume that $W = I_{m-k}$; otherwise, replace ϕ by the mapping $A \mapsto (I_k \oplus W^*)\phi(A)(I_k \oplus W)$. Thus, for any $A = (a_{ij}) \in M_p$, we have

$$\phi([1] \oplus A) = I_k \oplus (A_{ij}), \quad A_{ij} = a_{ij}I_s \oplus a_{ji}I_{l-s}. \tag{4}$$

Now, for $X = 0_p \oplus [1]$, we have

$$\phi(X) = 0_{m-l} \oplus I_l.$$

We can apply the previous argument to $\phi(U \oplus [1])$ for $U \in U(p)$ and conclude that there exist $V \in U(m-l)$ and integers u, v such that $m - l = pu$, and for any $B = (b_{ij}) \in M_p$

$$\phi(B \oplus [1]) = V(B_{ij})V^* \oplus I_l, \quad B_{ij} = b_{ij}I_v \oplus b_{ji}I_{u-v}. \tag{5}$$

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Next, consider $X = [1] \oplus 0_{p-1} \oplus [1]$. By (4) and (5), we see that

$$\phi(X) = V[I_u \oplus 0_{m-l-u}]V^* \oplus I_l = I_k \oplus 0_{m-k-l} \oplus I_l.$$

Hence $u = k$ and $V = V_1 \oplus U_2$ for some $V_1 \in U(k)$, $U_2 \in U(m-l-k)$. Moreover, from $m - k = pl$ and $m - l = pu$, we have $k = l$ and $m = k(p+1)$.

Let $E_{ij} \in M_{p-1}$ be the matrix with an 1 at the (i,j)-th position and 0 elsewhere. By considering $\phi(X)$ with $X = [1] \oplus E_{ii} \oplus [1]$, we see that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$ for some $V_1, \ldots V_p \in U(k)$. By considering $\phi(X)$ for $X = [1] \oplus E_{ij} + E_{ji} \oplus [1]$, we see that $V_2 = V_3 = \cdots = V_p$. By considering $[1] \oplus E_{ij} \oplus [1]$, we see that $v = s$ and $V_2 = Y_1 \oplus Y_2$ for some $Y_1 \in U(s)$, $Y_2 \in U(k-s)$. We may now assume that $V = I_m$; otherwise, replace ϕ by the mapping

$$A \mapsto [V_1 \oplus (I_p \otimes V_2)]^* \phi(A)[V_1 \oplus (I_p \otimes V_2)].$$

Hence, (4) and (5) hold with $V = I_m$; so $\phi(A) = (A_{ij})$ where $A_{ij} = a_{ij} I_s \oplus a_{ji} I_{k-s}$ if $(i,j) \neq (1, p+1)$ or $(p+1, 1)$.

Now, apply the previous argument to $\phi(C)$ for those matrices $C \in M_{p+1}$ such that $c_{2j} = c_{i2} = 0$ for $i \neq 2 \neq j$ and $c_{22} = 1$. We see that there exists $X, Y \in U(k)$ so that

$$A_{1,p+1} = X(a_{1,p+1} I_s \oplus a_{p+1,1} I_{k-s})Y^* \quad \text{and} \quad A_{p+1,1} = Y(a_{p+1,1} I_s \oplus a_{1,p+1} I_{k-s})X^*.$$

The rest of our proof is to show that X and Y may be assumed to be I_k. To this end, let

$$U = \begin{pmatrix} 0.6 & 0 \cdots 0 & 0.8 \\ -0.8 & 0 \cdots 0 & 0.6 \\ 0 & & 0 \\ \vdots & I_{p-1} & \vdots \\ 0 & & 0 \end{pmatrix} \in U(p+1).$$

Then $\phi(U) \in U(m)$. The submatrix of $\phi(U)$ formed by the first $2k$ rows equals

$$\begin{pmatrix} 0.6 I_k & 0 \cdots 0 & X[0.8 I_s \oplus 0_{k-s}]Y^* \\ -0.8 I_s \oplus 0_{k-s} & * \cdots * & 0.6 I_s \oplus 0_{k-s} \end{pmatrix}$$

and has orthonormal row vectors. Therefore $X[I_s \oplus 0_{k-s}]Y^* = I_s \oplus 0_{k-s}$. Next, considering U^*, we have $X[0_s \oplus I_{k-s}]Y^* = 0_s \oplus I_{k-s}$. Thus for $(i,j) = (1, p+1)$ or $(p+1, 1)$, we also have $A_{i,j} = a_{ij} I_s \oplus a_{ji} I_{k-s}$. The proof of our Assertion is hereby completed, and the theorem follows. ■

Note Added in Proof Professor Peter Šemrl pointed out that Theorem 2 can also be proved by establishing the following.

Lemma 7 *If $\phi\colon M_n \to M_m$ is a unital linear map satisfying $\phi\big(U(n)\big) \subseteq U(m)$ then $\phi(H^2) = \phi(H)^2$ for any Hermitian $H \in M_n$.*

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Proof Suppose $H \in M_n$ is Hermitian. Then

$$e^{itH} = I + itH - t^2H^2/2 + \cdots \quad \text{and} \quad \phi(e^{itH}) = I + it\phi(H) - t^2\phi(H^2)/2 + \cdots$$

are unitary. Thus,

$$\begin{aligned} I &= \phi(e^{itH})\phi(e^{itH})^* \\ &= \left(I + it\phi(H) - t^2\phi(H^2)/2 + \cdots\right)\left(I - it\phi(H)^* - t^2\phi(H^2)^*/2 + \cdots\right). \end{aligned}$$

Comparing the coefficients of t, we see that $i\phi(H) - i\phi(H)^* = 0$, *i.e.*, $\phi(H)$ is Hermitian. Now, comparing the coefficient at t^2, we see that $-\phi(H^2)/2 + \phi(H)^2 - \phi(H^2)/2 = 0$, *i.e.*, $\phi(H^2) = \phi(H)^2$. ■

Once this is done, one can follow the proof in [4, Corollary 4.3], which depends on Noether-Skolem Theorem, to conclude that ϕ is of the asserted form. In any event, our proof is more straight forward and self-contained.

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References

[1] H. Auerbach, *Sur les groupes bornés de substitutions linéaires*. C. R. Acad. Sci. Paris **195**(1932), 1367–1369.

[2] E. Christensen, *On invertibility preserving linear mappings, simultaneous triangularization and Property L*. Linear Algebra Appl. **301**(1999), 153-170.

[3] E. Deutsch and H. Schneider, *Bounded groups and norm-hermitian matrices*. Linear Algebra Appl. **9**(1974), 9–27.

[4] A. Guterman, C. K. Li and P. Šemrl, *Some general techniques on linear preserver problems*. Linear Algebra Appl. **315**(2000), 61–81.

[5] M. Marcus, *All linear operators leaving the unitary group invariant*. Duke Math. J. **26**(1959), 155–163.

[6] M. Marcus and R. Purves, *Linear transformations on algebras of matrices II: The invariance of the elementary symmetric functions*. Canad. J. Math. **11**(1959), 383–396.

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