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A MONOTONICITY THEOREM AND A BERNOULLI-L'HOSPITAL–OSTROWSKI RULE

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ABSTRACT. It is proved that a function is nondecreasing if it is Baire one and Darboux and fulfills Lusin's condition (N), and if its derivative is non-negative for almost every point at which the function is derivable. Using this result, a process to formulate various results on the existence and the valuation of indeterminate forms via various monotonicity theorems is illustrated. In particular, the ordinary Bernoulli–L'Hospital rule and some of its variations obtained recenty by A. M. Ostrowski are generalized.

1. **A monotonicity theorem.** There are various monotonicity theorems scattered in the literature. Bruckner's book [1] (cf. also [2]) gives a very clear and flavored discussion of certain interesting results, and also provides a good list of references for many other known results. See also the recent interesting work by Thomson [9]. Here we prove the following result, which is well-known for continuous functions (cf. Saks [8], page 286).

THEOREM 1. *Let F be a real-valued function having the following properties on an interval:*
(i) *F is Baire one and Darboux;*
(ii) *F fulfills Lusin's condition (N);*
(iii) *$F'(x)\geq 0$ at almost every x at which F is derivable.*
Then F is nondecreasing (and hence also continuous by the Darboux property) on the interval.

Proof. First, note that F, being measurable (as it is Baire one) and fulfilling Lusin's condition (N), must fulfill Banach's condition (T_2) (cf. Theorem IV in Ellis [3]). Now, suppose that, to the contrary, F is not nondecreasing on the

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interval. Then there exist a and b in the interval such that $a<b$ and $F(b)<F(a)$. Let

$$N=\{x : x \text{ is in } [a, b] \text{ and } 0 \geq F'(x) \geq -\infty\}.$$

Then by Theorem 2.2 in [1], page 178, we have

$$|F(N)| \geq F(a)-F(b)>0.$$

Denote $E_1=\{x : F'(x)=0\}$, $E_2=\{x : F'(x)=-\infty\}$ and $E_3=N \sim (E_1 \cup E_2)$. Then $|E_3|=0$ by condition (iii) since at each point of E_3 the function F is derivable; and $|E_2|=0$ by the theorem in [8], page 270; and $|F(E_1)|=0$ by the theorem in [8], page 227. Then, by Lusin's condition (N) we have $|F(E_3)|=|F(E_2)|=0$, and hence $|F(N)|=|F(E_1)|+|F(E_2)|+|F(E_3)|=0$, a contradiction.

REMARK 1. *If* the "$\geq$" in (iii) is replaced by "$>$", we conclude that F is *increasing* on the interval. For, a nondecreasing function F, if it is not increasing, will be constant on a subinterval, and then one will have $F'=0$ on a set of positive measure.

REMARK 2. Theorem 1 fails to hold if condition (i) is weakened to that F is *either* Baire one *or* Darboux. *Example* 1: Let $F(x)=1$ when $x=0$, and $F(x)=0$ otherwise. Then F is Baire one and satisfies both conditions (ii) and (iii) on $[0, 1]$, but F is not nondecreasing on $[0, 1]$. *Example* 2: Define F on $[0, 1]$ as follows: for each contiguous interval (a, b) of the Cantor set, let F be defined so that its graph on the *closed* interval $[a, b]$ is the line segment joining the points $(a, -1)$ and $(b, 1)$; and on the set of all other points in $[0, 1]$, the function F is defined to be zero. Then F is Darboux and satisfies (ii) and (iii) on $[0, 1]$, but F is *not* nondecreasing there.

REMARK 3. As an interesting consequence of Theorem 1, we have the following result generalizing one of Banach's theorems ([9], page 286) from continuous functions to Baire one and Darboux functions.

COROLLARY. *Any real-valued function F which is Baire one and Darboux and fulfills the condition (N) on a nondegenerate interval, is derivable at every point of a set of* positive *measure.*

Proof. Suppose to the contrary that the set of points at which F is derivable is of measure zero. Then all the conditions in Theorem 1 are satisfied by the function F and $-F$, and hence F is both nondecreasing and nonincreasing on the interval. Therefore, the function F is constant on the interval, and hence is derivable everywhere, a contradiction.

REMARK 4. Observe that in the proof of Theorem 1, the condition (N) is used to show that the function F fulfills Banach's condition (T_2) and the sets $F(E_2)$ and $F(E_3)$ are of measure zero. Therefore, we have the following more general result.

THEOREM 1′. *Suppose that F is a real-valued function which is Baire one and Darboux, and fulfills Banach's condition* (T_2) *on an interval, and suppose that* $|F(E)|=0$, *where* $E=\{x:0>F'(x)\geq-\infty\}$. *Then F is nondecreasing and continuous on the interval.*

REMARK 5. Theorem 1 has been presented by the author in the Real Analysis Symposium at Syracuse, New York (1981) under the title "On Baire one Darboux functions with Lusin's condition (N)." We were happy to learn in the symposium that Professor K. Garg has obtained some more general versions of the above results as he mentioned in his lecture on "A new notion of derivatives" (cf. [5]).

2. **A generalization of Bernoulli–L'Hospital–Ostrowski rules.** Recently, Ostrowski [7] has obtained some interesting variations of the classical Bernoulli–L'Hospital rule. However, Ostrowski's result and the classical one are not comparable. Here, we will illustrate how to use a monotonicity theorem to formulate a result which is more general than both Ostrowski's result and the classical one. Throughout the rest of the section, let $-\infty\leq a<b\leq+\infty$ and let f, g be real-valued functions defined on (a, b), and all the limit processes involved will be taken as $x\to a+$ unless otherwise stated. First, we prove the following.

LEMMA. *Suppose that either*

(I) $\lim g(x)=-\infty$

or

(II) *g is increasing on (a, b) and* $\lim g(x)=\lim f(x)=0$. *Then for each real number q*

$$\limsup[f(x)/g(x)]\leq q$$

provided that $qg-f$ is nondecreasing on (a, c) for some c in (a, b).

Proof. Since $qg-f$ is nondecreasing on (a, c), one has

$$f(\beta)-f(\alpha)\leq q[g(\beta)-g(\alpha)] \tag{1}$$

for all α, β in (a, c) with $\alpha<\beta$. Hence

$$[f(\beta)-f(\alpha)]/[g(\beta)-g(\alpha)]\leq q \tag{2}$$

for all such α and β provided that g is increasing. If (II) holds, then it follows from (2) that one has

$$f(\beta)/g(\beta)\leq q \tag{3}$$

for all β in (a, c). If (I) holds, then for a *fixed* β in (a, c), there exists δ in (a, β) such that $g(\alpha)<0$ for all α in (a, δ); and by dividing both sides of (1) by

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$g(\alpha)<0$, one has

$$f(\alpha)/g(\alpha)\le q-q[g(\beta)/g(\alpha)]+f(\beta)/g(\alpha) \tag{4}$$

for all α in (a,δ). Letting $\beta\to a^+$ in (3) or $\alpha\to a^+$ in (4), we have $\limsup[f(x)/g(x)]\le q$, completing the proof.

THEOREM 2. *Suppose that either* $\lim g(x)=-\infty$ *or* $\lim g(x)=\lim f(x)=0$, *and suppose that* $g'(x)>0$ *for almost every* x *at which* g *is derivable, and let* $E=\{x:$ *both* f *and* g *are derivable at* x *and* $g'(x)\ne 0\}$. *Then for each set* $A\subset E$ *with* $|E\sim A|=0$, *one has*

$$\limsup[f(x)/g(x)]\le A-\limsup[f'(x)/g'(x)]\equiv r$$

provided that g *and* $qg-f$ *(for each* $q>r$ *when* $r\ne+\infty$*) are Baire one and Darboux, and fulfill the condition* (N) *on* (a,c) *for some* c *in* (a,b). *[Here* A-$\limsup[f'(x)/g'(x)]$ *denotes the ordinary* $\limsup[f'(x)/g'(x)]$ *except that* x *is restricted to the set* A.*]*

Proof. Note that applying Theorem 1 to the function g, one concludes that g is increasing on (a,c). Hence g is derivable almost everywhere on (a,c), and then $0<g'(x)<+\infty$ for almost all x in (a,c). If A-$\limsup[f'(x)/g'(x)]=+\infty$, we are done. Hence we assume that A-$\limsup[f'(x)/g'(x)]\equiv r<+\infty$, and let q be any fixed real number greater than r. Then we have c' in (a,c) such that

$$f'(x)/g'(x)\le q$$

for almost all x in $A\cap(a,c')$. Since $g'(x)>0$ for almost all x in (a,c), we see that $(qg-f)'(x)\ge 0$ for all x in (a,c') at which $qg-f$ is derivable. Therefore, we can apply Theorem 1 to the function $qg-f$ on the interval (a,c') and conclude that $qg-f$ is nondecreasing on (a,c'). It then follows from the lemma above that

$$\limsup[f(x)/g(x)]\le q.$$

But since $q>r$ is arbitrary, we conclude that

$$\limsup[f(x)/g(x)]\le r,$$

completing the proof.

COROLLARY. *Suppose that the following conditions are satisfied:*

(H1) *either* $\lim f(x)=\lim g(x)=0$ *or* $\lim g(x)=-\infty$;

(H2) f, g *are* ACG *on* (a,b);

(H3) $g'(x)>0$ *for almost every* x *at which* g *is derivable.*

And let $E=\{x:$ *both* f *and* g *are derivable and* $g'(x)>0\}$. *Then for each set* $A\subset E$ *with* $|E\sim A|=0$ *one has*

$$A-\liminf[f'(x)/g'(x)]\le\liminf[f(x)/g(x)]$$
$$\le\limsup[f(x)/g(x)]\le A-\limsup[f'(x)/g'(x)].$$

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(*Hence for such set A one has*

$$\lim[f(x)/g(x)] = A - \lim[f'(x)/g'(x)]$$

provided that A-$\lim[f'(x)/g'(x)]$ *exists.*)

Proof. Note that a linear combinaton of ACG functions is also ACG. Furthermore, an ACG function is continuous according to its definition given in Saks' book [8], and hence is Baire one and Darboux. Thus, this corollary is an easy consequence of Theorem 2.

REMARK 5. Similar results hold when conditions "$g'>0$ and $\lim g(x) = -\infty$" are replaced by conditions "$g'<0$ *and lim* $g(x) = +\infty$". Also, similar results can easily be formulated when the limit processes are taken as $x \to b-$.

REMARK 6. The corollary is apparently a generalization of one of Otrowski's results in [7]. His other two results can similarly be generalized. Also, the classical Bernoulli–L'Hospital rule is a special case of the corollary since a differentiable function is ACG and since a derivative has the Darboux property.

REMARK 7. Some generalizations of the L'Hospital rule have been obtained in [6] by using other known monotonicity theorems. Note that as M. Evans has pointed out in [4] and in a letter to the author that in the theorem stated in the abstract in [6], the author forgot to indicate explicitly that the functions there should be approximately continuous (cf. the discussion (A) on page 319 in [6]). Without this assumption, the theorem stated there fails to hold true as the following example shows: Let $F(x) = x$, and let

$$G(x)\begin{cases} 2x & \text{when } x \text{ is rational,} \\ = x & \text{when } x \text{ is irrational.} \end{cases}$$

Then $\text{esslim}_{x\to 0+}[F'_{ap}(x)/G'_{ap}(x)] = 1$, but $\lim_{x\to 0+}[F(x)/G(x)]$ does not exist. (This example is also due to M. Evans, to whom the author expresses many thanks.) Although the assumption that the functions should be approximately continuous cannot be neglected, it can somehow be weakened. In fact, Theorem 2 established here shows that this assumption can be replaced by that certain linear combinations of the functions involved should be Baire one and Darboux. The above example also shows that the condition of being Baire one cannot be neglected in Theorem 2. We will leave the interested readers to construct examples to show that neither the Darboux property nor the condition (N) can be omitted there.

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