

Group Actions and Singular Martingales II, The Recognition Problem

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Abstract. We continue our investigation in [RST] of a martingale formed by picking a measurable set A in a compact group G , taking random rotates of A , and considering measures of the resulting intersections, suitably normalized. Here we concentrate on the inverse problem of recognizing A from a small amount of data from this martingale. This leads to problems in harmonic analysis on G , including an analysis of integrals of products of Gegenbauer polynomials.

1 Introduction

In this paper we explore two circles of results in the area of harmonic analysis on compact groups. The first centers about a class of martingales, whose unusual properties came to light in [R1] and which have been studied further, from an ergodic theory and harmonic analysis perspective, in [RST]. The second concerns the decomposition of a tensor product of representations of the rotation group $SO(n)$ on spaces of spherical harmonics, with particular attention to the product of zonal functions. This second topic arises from our study of the first, though it takes a life of its own and can be appreciated independently of the martingale problem.

Let us first define the class of martingales we are treating. Let G be a compact group, with Haar measure m , normalized to have total mass 1. Let $\mathcal{Z} = G^\infty$, with product measure μ . Take $A \subset G$, measurable, with $m(A) > 0$. For $x = (x_1, x_2, \dots) \in \mathcal{Z}$, let

$$(1.1) \quad f_N^A(x) = \frac{m(\bigcap_{i=1}^N x_i A)}{m(A)^N}.$$

The numerator measures how much the N -fold translates of A overlap. The denominator is a normalizing factor, chosen so that $\int_{\mathcal{Z}} f_N^A d\mu = 1$.

It is not hard to see that $(f_N^A : N = 1, 2, 3, \dots)$ is a martingale. Indeed, this fact is just an inductive use of the well-known formula, a particular consequence of the Fubini Theorem, that $\int_G m(xA \cap B) dm(x) = m(A)m(B)$ for all measurable sets $A \subset G$ and $B \subset G$. This formula is basic in the study of random translations of sets and functions on groups. The study of random translations and the behavior of certain randomly constructed sets that arise from them has a long history. The book by Kahane [K] contains many interesting results of this type. One of the important themes in [K] is the interplay between the use of probability theory and harmonic analysis in the study of random translations. This same interplay occurs in [R1], [R2], [RST], and to some degree in this article, although the emphasis in this article is more on

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the harmonic analysis side of this interaction. We expect that the martingales (f_N^A) and their generalizations will have other roles to play, both in terms of probability theory and harmonic analysis, in the future in the study of random translations and multiple products. This article presents just one basic example of this phenomenon.

The following result was established in [RST]:

Proposition 1.1 *The martingale (f_N^A) defined by (1.1) has the property that, if $m(A) < 1$, then*

$$(1.2) \quad f_N^A \rightarrow 0, \quad \mu\text{-a.e. on } \mathcal{Z}.$$

Another way of putting this is that the martingale (f_N^A) determines a probability measure ν^A on \mathcal{Z} , and this measure and μ are mutually singular. In the course of analyzing ν^A , the authors developed techniques that led to some results on the extent to which (f_N^A) determines the set A . In particular, the following result was proven in [RST]:

Proposition 1.2 *If $A, B \subset G$ and $f_N^A \equiv f_N^B$ for all N , then there exists $y \in G$ such that*

$$(1.3) \quad B = Ay, \quad \text{modulo null sets.}$$

In this paper we pursue further the question of to what extent the sequence of functions f_N^A given by (1.1) determines the set A . More specifically, if $f_N^A \equiv f_N^B$ for a certain fixed N , what can we say about the relation between A and B ? As a preliminary, note that, when $N \geq 2$,

$$(1.4) \quad f_N^A(y, \dots, y) = m(A)^{-(N-1)},$$

so $m(A)$ is determined by f_N^A for any $N \geq 2$. We will show that in many cases an analogue of the conclusion of Proposition 1.2 can be established when $f_3^A \equiv f_3^B$.

In Section 2 we consider the case where G is a compact abelian group. We show in Proposition 2.1, necessarily using a method different from the one in [RST], that $f_3^A \equiv f_3^B$ yields the conclusion (1.3) provided the group Fourier transform $\hat{\chi}_A$ is nowhere vanishing (χ_A denoting the indicator function of A). We also consider cases when this hypothesis on $\hat{\chi}_A$ fails, and we provide a harmonic analysis proof of Proposition 1.2 in this case.

In Section 3 we obtain an extension of Proposition 2.1 for G compact and non-commutative, assuming the group Fourier transform $\hat{\chi}_A(\pi)$ is invertible for each irreducible unitary representation π of G . It turns out that this result is not a completely satisfactory generalization of the commutative case.

This point is brought home in Section 4. This section deals with $G = \text{SO}(n)$, with $n \geq 3$, and $A = p^{-1}(A')$, with $A' \subset S^{n-1}$, the unit sphere in \mathbb{R}^n , and $p: \text{SO}(n) \rightarrow S^{n-1}$ the natural projection. In such a case, $\hat{\chi}_A(\pi)$ is never invertible (except for the trivial representation). Nevertheless we produce conditions on A' such that when $A', B' \subset S^{n-1}$ then

$$(1.5) \quad f_3^A \equiv f_3^B \Rightarrow A' = B' \text{ or } A' = -B',$$

where $-B'$ is the image of B' under the antipodal map. For example, (1.5) is shown to hold whenever $\chi_{A'}$ has a nonzero projection onto each eigenspace of the Laplace operator; this is an analogue (or better, a replacement) of the invertibility condition mentioned above. We also obtain (1.5) in other situations, in particular in all cases where A' is symmetric, *i.e.*, $A' = -A'$. We also show that such an implication always holds when f_3 is replaced by f_4 .

Results of Section 4 depend on an analysis of $\hat{f}_N(D_{k_1}, \dots, D_{k_N})$, where D_k is the representation of $SO(n)$ on spherical harmonics of degree k . This analysis brings in the second topic mentioned above, which is developed in Section 5 of this paper. It concerns the decomposition of the representation $D_{k_1} \otimes \dots \otimes D_{k_N}$ of $SO(n)$, together with one extra piece of structure. Namely, we are interested in when the trivial representation D_0 is contained in this tensor product, and in addition,

$$(1.6) \quad \mathfrak{z}_{k_1} \otimes \dots \otimes \mathfrak{z}_{k_N} \text{ has a nontrivial component in } W_0,$$

where W_0 is the space on which the tensor product representation acts trivially, and \mathfrak{z}_k denotes the zonal harmonic of degree k on S^{n-1} . We show that if k_ν are positive integers satisfying $k_1 \leq \dots \leq k_N$, then (1.6) holds if and only if

$$(1.7) \quad k_1 + \dots + k_{N-1} \geq k_N, \quad \text{and} \quad k_1 + \dots + k_N \text{ is even.}$$

A key point in demonstrating the equivalence of (1.6) and (1.7) is to show that each one is equivalent to

$$(1.8) \quad \int_{S^{n-1}} \mathfrak{z}_{k_1}(x) \cdots \mathfrak{z}_{k_N}(x) \, d\nu(x) \neq 0.$$

This in turn is an integral of a product of Gegenbauer polynomials, analyzed in slightly greater generality in Lemma 5.2.

As we have mentioned, the material of Section 5 can be appreciated independently of the first topic. (One small exception: the proof of Proposition 5.5 makes use of Proposition 4.2.) One could proceed directly from here to Section 5.

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2 The Case of a Compact Abelian Group

In this section we assume G is a compact abelian group. Then its dual \hat{G} is a discrete abelian group. If we define the Fourier transform of $\varphi \in L^2(G)$:

$$(2.1) \quad \hat{\varphi}(\xi) = \int_G \varphi(y)\xi(y) \, dm(y),$$

then

$$(2.2) \quad \varphi(y) = \sum_{\xi \in \hat{G}} \hat{\varphi}(\xi) \overline{\xi(y)},$$

with convergence in norm in $L^2(G)$.

Let us set

$$(2.3) \quad \begin{aligned} g_{N+1}^A(x_1, \dots, x_N) &= m(A)^{N+1} f_{N+1}^A(x_1, \dots, x_N, 1) \\ &= m(A \cap x_1 A \cap \dots \cap x_N A), \end{aligned}$$

where 1 denotes the identity element of G . Then, for $\xi_\nu \in \hat{G}$,

$$(2.4) \quad \begin{aligned} \hat{g}_{N+1}^A(\xi_1, \dots, \xi_N) &= \int_{G^N} g_{N+1}^A(x_1, \dots, x_N) \xi_1(x_1) \cdots \xi_N(x_N) dm(x_1) \cdots dm(x_N) \\ &= \int_{G^{N+1}} \xi_1(x_1) \cdots \xi_N(x_N) \chi_A(x_1^{-1}y) \cdots \chi_A(x_N^{-1}y) \chi_A(y) \\ &\quad dm(y) dm(x_1) \cdots dm(x_N) \\ &= \hat{\chi}_A(\xi_1 + \cdots + \xi_N) \hat{\chi}_A(-\xi_1) \cdots \hat{\chi}_A(-\xi_N), \end{aligned}$$

where χ_A denotes the indicator function of A , and in our notation we are writing \hat{G} as an additive (abelian) group. In particular,

$$(2.5) \quad \hat{g}_2^A(\xi_1) = \hat{\chi}_A(\xi_1) \hat{\chi}_A(-\xi_1) = |\hat{\chi}_A(\xi_1)|^2.$$

By (1.4), given $N \geq 2$, $f_N^A \equiv f_N^B \Rightarrow g_N^A \equiv g_N^B$. In particular, if $f_2^A \equiv f_2^B$, then

$$(2.6) \quad |\hat{\chi}_B(\xi)| = |\hat{\chi}_A(\xi)|, \quad \forall \xi \in \hat{G}.$$

Thus

$$(2.7) \quad \hat{\chi}_B(\xi) = u(\xi) \hat{\chi}_A(\xi), \quad |u(\xi)| = 1,$$

with the factor $u(\xi)$ uniquely determined for all $\xi \in \hat{G}$ such that $\hat{\chi}_A(\xi) \neq 0$.

Suppose now that $f_{N+1}^A \equiv f_{N+1}^B$, so $\hat{g}_{N+1}^A \equiv \hat{g}_{N+1}^B$. If we use (2.4) for A and B , plug in (2.7), and cancel when permitted, we see that if $f_{N+1}^A \equiv f_{N+1}^B$ and if $\xi_\nu \in \hat{G}$ satisfy

$$(2.8) \quad \hat{\chi}_A(\xi_\nu) \neq 0, \quad \nu = 1, \dots, N, \text{ and } \hat{\chi}_A(\xi_1 + \cdots + \xi_N) \neq 0,$$

then

$$(2.9) \quad u(\xi_1 + \cdots + \xi_N) = u(\xi_1) \cdots u(\xi_N).$$

This allows us to prove the following uniqueness result:

Proposition 2.1 *Assume A and B are measurable subsets of a compact abelian group G and that $f_3^A \equiv f_3^B$. Also assume*

$$(2.10) \quad \hat{\chi}_A(\xi) \neq 0, \quad \text{for each } \xi \in \hat{G}.$$

Then there exists $y \in G$ such that $B = yA$, modulo sets of measure zero.

Proof If (2.10) holds, then (2.9) holds (with $N = 2$) for all $\xi_\nu \in \hat{G}$. Hence u is a character on \hat{G} , so, by Pontrjagin duality, there exists $y \in G$ such that $u(\xi) = \xi(y)$ for all $\xi \in \hat{G}$. Then

$$\begin{aligned} \hat{\chi}_{yA}(\xi) &= \int_G \chi_A(y^{-1}x_1)\xi(x_1) dm(x_1) \\ (2.11) \qquad &= \int_G \chi_A(x_1)\xi(yx_1) dm(x_1) \\ &= \xi(y)\hat{\chi}_A(\xi) = \hat{\chi}_B(\xi), \end{aligned}$$

for all $\xi \in \hat{G}$, so $\chi_{yA} = \chi_B$, m -a.e. on G . ■

Remark We make some comments on constructively producing the set A given the data f_3^A , or equivalently g_3^A . As we have seen, $\hat{\chi}_A(0) = m(A) = f_2^A(y, y)^{-1}$, and for $\xi \in \hat{G}$,

$$(2.12) \qquad \hat{\chi}_A(\xi) = \nu(\xi)|\hat{\chi}_A(\xi)| = \nu(\xi)\hat{g}_2^A(\xi)^{1/2},$$

where, for $\xi \in \Sigma_A = \{\xi \in \hat{G} : \hat{\chi}_A(\xi) \neq 0\}$, $\nu(\xi)$ is a complex number of absolute value 1. Furthermore, $\nu: \Sigma_A \rightarrow \mathbb{T}^1$ satisfies the following identities:

$$(2.13) \qquad \nu(\xi_1 + \dots + \xi_N) = \Phi_N(\xi_1, \dots, \xi_N)\nu(\xi_1) \cdots \nu(\xi_N),$$

whenever $\xi_1, \dots, \xi_N, \xi_1 + \dots + \xi_N \in \Sigma_A$, with

$$(2.14) \qquad \Phi_N(\xi_1, \dots, \xi_N) = \frac{\hat{g}_{N+1}^A(\xi_1, \dots, \xi_N)}{[\hat{g}_2^A(\xi_1) \cdots \hat{g}_2^A(\xi_N)\hat{g}_2^A(\xi_1 + \dots + \xi_N)]^{1/2}}.$$

It follows from the argument above that if $\Sigma_A = \hat{G}$ then ν is uniquely determined on \hat{G} by f_3^A , up to multiplication by a character $u(\xi) = \xi(y)$ on \hat{G} . We want to describe a constructive determination of $\nu(\xi)$. For simplicity we take $G = \mathbb{T}^1$, so $\hat{G} = \mathbb{Z}$.

If $|\hat{\chi}_A(k)| \neq 0$ for all $k \in \mathbb{Z}$, then we can pick any $\alpha \in \mathbb{T}^1$, set $\nu(1) = \alpha$, and define $\nu(k)$ inductively for $k \in \mathbb{Z}^+$ by

$$\begin{aligned} (2.15) \qquad \nu(2) &= \nu(1)^2\Phi_2(1, 1), \quad \nu(3) = \nu(1)\nu(2)\Phi_2(1, 2), \dots \\ \nu(k) &= \nu(1)\nu(k-1)\Phi_2(1, k-1), \end{aligned}$$

while $\nu(-k) = \overline{\nu(k)}$. Note that the construction uses only f_2^A and f_3^A , and of course knowledge of f_3^A determines f_2^A .

We now drop the hypothesis (2.10). Let us set

$$(2.16) \qquad \Sigma_A = \{\xi \in \hat{G} : \hat{\chi}_A(\xi) \neq 0\},$$

and let L_A be the subgroup of \hat{G} generated by Σ_A . If $L_A \neq \hat{G}$, consider

$$(2.17) \qquad H_A = \{y \in G : \xi(y) = 1, \forall \xi \in L_A\},$$

a closed subgroup of G . Note that, for $y \in G, \xi \in \hat{G}$,

$$(2.18) \quad \hat{\chi}_{yA}(\xi) = \xi(y)\hat{\chi}_A(\xi),$$

which equals $\hat{\chi}_A(\xi)$ if $y \in H_A$, so

$$(2.19) \quad y \in H_A \Rightarrow yA = A \text{ (modulo sets of measure zero)}.$$

Thus $A = p^{-1}(A')$ where $p: G \rightarrow G/H_A = G'$ is the natural projection and $A' = A/H_A$. Then the subgroup \hat{G}' spanned by $\Sigma_{A'}$ is all of \hat{G}' . Hence we can work under the hypothesis that

$$(2.20) \quad L_A = \hat{G}.$$

Suppose now that $f_{N+1}^A \equiv f_{N+1}^B$, with $N+1 \geq 2$. We have the following data to work with: a map

$$(2.21) \quad u: \Sigma_A \rightarrow \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\},$$

satisfying (2.7) and having the property

$$(2.22) \quad \xi_1, \dots, \xi_N, \xi_1 + \dots + \xi_N \in \Sigma_A \Rightarrow u(\xi_1 + \dots + \xi_N) = u(\xi_1) \cdots u(\xi_N).$$

Also we have $u(0) = 1$ and $\xi \in \Sigma_A \Rightarrow -\xi \in \Sigma_A$ and $u(-\xi) = u(\xi)^{-1}$. The following is a key result.

Lemma 2.2 *Assume $f_N^A \equiv f_N^B$ for all N . Then the map $u: \Sigma_A \rightarrow \mathbb{T}^1$ has a unique extension to a character on L_A .*

Proof Any $\zeta \in L_A$ can be written

$$(2.23) \quad \zeta = \xi_1 + \dots + \xi_M, \quad \xi_\nu \in \Sigma_A.$$

It is natural to try to set

$$(2.24) \quad u(\zeta) = u(\xi_1) \cdots u(\xi_M).$$

We need to check that if also

$$(2.25) \quad \zeta = \eta_1 + \dots + \eta_N, \quad \eta_\mu \in \Sigma_A,$$

then

$$(2.26) \quad u(\xi_1) \cdots u(\xi_M) = u(\eta_1) \cdots u(\eta_N).$$

If $\zeta \in \Sigma_A$ there is no problem, since (2.22) applies. In general, note that $-\eta_\mu \in \Sigma_A$ and

$$(2.27) \quad \xi_1 + \dots + \xi_M + (-\eta_1) + \dots + (-\eta_N) = 0 \in \Sigma_A,$$

so (2.22) implies

$$(2.28) \quad u(\xi_1) \cdots u(\xi_M)u(-\eta_1) \cdots u(-\eta_N) = u(0) = 1,$$

and we have (2.26). Thus (2.24) uniquely extends u to $u: L_A \rightarrow \mathbb{T}^1$, and it is routine to verify that this is a character on L_A . ■

With this in hand, we can give a harmonic analysis proof of Proposition 1.2, for G abelian, quite different from the ergodic theory proof given in [RST].

Proposition 2.3 *Let $A, B \subset G$, a compact abelian group. If $f_N^A \equiv f_N^B$ for all N then there exists $y \in G$ such that $B = yA$, modulo sets of measure zero.*

Proof As mentioned, we can reduce to the case when (2.20) holds. We know that (2.7) holds, for $u: \Sigma_A \rightarrow \mathbb{T}^1$ satisfying the conditions of Lemma 2.2. Hence (2.7) holds with u a character of \hat{G} , i.e., $u(\xi) = \xi(y)$ for some $y \in G$. This gives $B = yA$, as desired. ■

3 Results for Nonabelian Groups

We now extend our considerations to the case where G is a compact, noncommutative group. We still have (1.8), so $m(A)$ is determined by f_2^A , and we continue to define g_{N+1}^A by (2.3), where we denote the identity element of G by 1. The Fourier analysis used in the abelian case is extended as follows.

Let π be a unitary representation of G ; here and below we make the convention that all unitary representations considered are finite dimensional, in order to shorten our terminology. Given an integrable function g on G , we define

$$(3.1) \quad \hat{g}(\pi) = \int_G g(x)\pi(x) dm(x).$$

Now if π_k are unitary representations of G , we have

$$(3.2) \quad \begin{aligned} \hat{g}_{N+1}^A(\pi_{k_1}, \dots, \pi_{k_N}) &= \int_{G^N} g_{N+1}^A(x_1, \dots, x_N)\pi_{k_1}(x_1) \otimes \cdots \otimes \pi_{k_N}(x_N) \\ &\quad dm(x_1) \cdots dm(x_N) \\ &= \int_{G^{N+1}} \pi_{k_1}(x_1) \otimes \cdots \otimes \pi_{k_N}(x_N)\chi_A(x_1^{-1}y) \cdots \chi_A(x_N^{-1}y)\chi_A(y) \\ &\quad dm(y) dm(x_1) \cdots dm(x_N) \\ &= \hat{\chi}_A(\pi_{k_1} \otimes \cdots \otimes \pi_{k_N}) \cdot [\hat{\chi}_A(\pi_{k_1})^* \otimes \cdots \otimes \hat{\chi}_A(\pi_{k_N})^*]. \end{aligned}$$

In particular, for each unitary representation π of G ,

$$(3.3) \quad \hat{g}_2^A(\pi) = \hat{\chi}_A(\pi)\hat{\chi}_A(\pi)^*.$$

If one can say that $\hat{\chi}_A(\pi)$ is invertible, then the identity of g_2^A and g_2^B implies that

$$(3.4) \quad \hat{\chi}_B(\pi) = \hat{\chi}_A(\pi)u(\pi),$$

for a uniquely determined unitary factor $u(\pi)$. Using this we can prove the following result, which extends Proposition 2.1.

Proposition 3.1 *Assume $\hat{\chi}_A(\pi)$ is invertible for each irreducible unitary representation π of G . If $f_3^A \equiv f_3^B$ then there exists $g \in G$ such that $B = Ag$.*

Proof The hypotheses imply $\hat{\chi}_A(\pi)$ is invertible for each unitary representation of G , irreducible or not. Also we have $f_2^B \equiv f_2^A$ and $g_2^B \equiv g_2^A$, and (3.3) holds for each unitary representation π . Thus, associated to each unitary representation π of G on V_π there is a uniquely determined unitary map $u(\pi)$ on V_π such that (3.4) holds.

It is routine to verify several properties of the correspondence $\pi \mapsto u(\pi)$. For example,

$$(3.5) \quad u(\pi_1 \oplus \pi_2) = u(\pi_1) \oplus u(\pi_2).$$

Also, given unitary $T: V_\pi \rightarrow V$,

$$(3.6) \quad u(T\pi T^{-1}) = Tu(\pi)T^{-1}.$$

Furthermore, if π is a unitary representation of G on \mathbb{C}^n , with contragredient representation $\bar{\pi}$, whose matrix entries are the complex conjugates of those of π , then

$$(3.7) \quad \hat{\chi}_A(\bar{\pi}) = \overline{\hat{\chi}_A(\pi)},$$

and hence

$$(3.8) \quad u(\bar{\pi}) = \overline{u(\pi)}.$$

Next, given $g_3^B \equiv g_3^A$, the $N = 2$ case of (3.2) yields, for any unitary representations π_1 and π_2 of G ,

$$(3.9) \quad \hat{\chi}_B(\pi_1 \otimes \pi_2) \cdot [\hat{\chi}_B(\pi_1)^* \otimes \hat{\chi}_B(\pi_2)^*] = \hat{\chi}_A(\pi_1 \otimes \pi_2) \cdot [\hat{\chi}_A(\pi_1)^* \otimes \hat{\chi}_A(\pi_2)^*].$$

If we apply (3.4) with $\pi = \pi_1, \pi_2$, and $\pi_1 \otimes \pi_2$, and use the invertibility hypothesis, we obtain

$$(3.10) \quad u(\pi_1 \otimes \pi_2) = u(\pi_1) \otimes u(\pi_2)$$

for all unitary representations π_1 and π_2 of G .

Since the map $\pi \mapsto u(\pi)$ satisfies (3.5), (3.6), (3.8), and (3.10), we can apply the Tanaka duality theorem (cf. [C], p. 211) to conclude that there exists $g \in G$ such that

$$(3.11) \quad u(\pi) = \pi(g)$$

for all unitary representations π of G . Then (3.4) gives $B = Ag$, as desired. ■

The invertibility hypothesis on $\hat{\chi}_A(\pi)$ plays an important role in the proof of Proposition 3.1. However, as we will see in the next section, this invertibility condition is not satisfied in what is arguably the most important case.

4 The Spherical Case

Let us consider $G = \text{SO}(n)$, acting on the unit sphere $S^{n-1} = \text{SO}(n)/K$, with $K = \text{SO}(n-1)$, pictured as acting on the first $n-1$ coordinates of a point in \mathbb{R}^n . Assume $n \geq 3$. We have the projection $p: \text{SO}(n) \rightarrow S^{n-1}$. Take $A' \subset S^{n-1}$ and set $A = p^{-1}(A') \subset \text{SO}(n)$. We examine $\hat{\chi}_A(\pi_k)$ when π_k is an irreducible representation of $\text{SO}(n)$ arising in the natural action of $\text{SO}(n)$ on $L^2(S^{n-1})$. We recall these representations, on the eigenspaces of the Laplace operator Δ_S on S^{n-1} . If we set

$$(4.1) \quad \nu = \left(-\Delta_S + \left(\frac{n-2}{2} \right)^2 \right)^{1/2},$$

then

$$(4.2) \quad \text{Spec } \nu = \left\{ \nu_k = \frac{n-2}{2} + k : k = 0, 1, 2, \dots \right\},$$

and $\text{SO}(n)$ acts irreducibly on V_k , the ν_k -eigenspace of ν ; we denote this representation by D_k . As is well known, each space V_k has a one-dimensional subspace of vectors fixed by the action of K , known as zonal harmonics. Pick a zonal harmonic $\mathfrak{z}_k \in V_k$, of unit norm.

Now if $A = p^{-1}(A')$, then $\hat{\chi}_A(\pi)$ is nonzero for an irreducible representation π of $\text{SO}(n)$ only if π is equivalent to some D_k . Furthermore, since χ_A is invariant under the right action of K , we have $\hat{\chi}_A(D_k)X = 0$ for the action X of any element of the Lie algebra of K . Hence

$$(4.3) \quad v \in V_k, \quad v \perp \mathfrak{z}_k \Rightarrow \hat{\chi}_A(D_k)v = 0.$$

Hence, for $v \in V_k$, we have

$$(4.4) \quad \hat{\chi}_A(D_k)v = (v, \mathfrak{z}_k)\hat{\chi}_A(D_k)\mathfrak{z}_k = \mathfrak{z}_k(p_0)(v, \mathfrak{z}_k)P_k\chi_{A'},$$

where $p_0 = p(1) \in S^{n-1}$ is the ‘‘north pole’’ and P_k is the orthogonal projection of $L^2(S^{n-1})$ onto V_k . For the second identity, see (3.13) on p. 121 of [T]. As shown in (3.27) on p. 123 of [T], we have $\mathfrak{z}_k(p_0) = d_k^{1/2} = (\dim V_k)^{1/2}$, if we pick $\mathfrak{z}_k(p_0) > 0$. Using (4.4) we compute that, for $v \in V_k$,

$$(4.5) \quad \hat{\chi}_A(D_k)^*v = \mathfrak{z}_k(p_0)(v, \chi_{A'})\mathfrak{z}_k.$$

Hence, for $v \in V_k$,

$$(4.6) \quad \hat{\chi}_A(D_k)\hat{\chi}_A(D_k)^*v = d_k(v, \chi_{A'})P_k\chi_{A'}.$$

Also note that

$$(4.7) \quad \begin{aligned} \text{Tr } \hat{\chi}_A(D_k)\hat{\chi}_A(D_k)^* &= \text{Tr } \hat{\chi}_A(D_k)^*\hat{\chi}_A(D_k) \\ &= \|\hat{\chi}_A(D_k)\mathfrak{z}_k\|^2 \\ &= d_k^{1/2}\|P_k\chi_{A'}\|^2. \end{aligned}$$

We are ready to prove the following result.

Proposition 4.1 *Suppose $A = p^{-1}(A')$ and $B = p^{-1}(B')$ are subsets of $SO(n)$ satisfying $f_2^A \equiv f_2^B$. Assume $P_k\chi_{A'} \neq 0$. Then*

$$(4.8) \quad P_k\chi_{B'} = u_k P_k\chi_{A'}$$

for some $u_k = \pm 1$. Furthermore,

$$(4.9) \quad \hat{\chi}_B(D_k) = u_k \hat{\chi}_A(D_k).$$

Proof Recall that $f_2^A \equiv f_2^B \Rightarrow g_2^A \equiv g_2^B$. Using (3.3), (4.6) and (4.7), we see that $g_2^A = g_2^B$ implies that $P_k\chi_{A'}$ and $P_k\chi_{B'}$ span the same linear subspace of V_k and have the same norms. This gives (4.8), with $|u_k| = 1$, but also these are real-valued functions, so $u_k = \pm 1$. Finally, (4.9) follows from (4.8), via (4.4). ■

Looking at g_{N+1}^A for larger N , we see that, if $v_k \in V_k$,

$$(4.10) \quad \hat{g}_{N+1}^A(D_{k_1}, \dots, D_{k_N})v_{k_1} \otimes \dots \otimes v_{k_N} = (d_{k_1} \dots d_{k_N})^{1/2} (v_{k_1}, \chi_{A'}) \dots (v_{k_N}, \chi_{A'}) \hat{\chi}_A(D_{k_1} \otimes \dots \otimes D_{k_N})\mathfrak{z}_{k_1} \otimes \dots \otimes \mathfrak{z}_{k_N}.$$

If we assume $A = p^{-1}(A')$, $B = p^{-1}(B')$ and that (4.8) holds for all $k = k_\nu$, we have

$$(4.11) \quad \hat{g}_{N+1}^B(D_{k_1}, \dots, D_{k_N})v_{k_1} \otimes \dots \otimes v_{k_N} = (d_{k_1} \dots d_{k_N})^{1/2} u_{k_1} \dots u_{k_N} (v_{k_1}, \chi_{A'}) \dots (v_{k_N}, \chi_{A'}) \hat{\chi}_B(D_{k_1} \otimes \dots \otimes D_{k_N})\mathfrak{z}_{k_1} \otimes \dots \otimes \mathfrak{z}_{k_N}.$$

Let us assume

$$(4.12) \quad P_{k_\nu}\chi_{A'} \neq 0, \quad \forall k_\nu = k_1, \dots, k_N.$$

Then we can pick $v_{k_\nu} = P_{k_\nu}\chi_{A'}$ and deduce that, if (4.10) and (4.11) are equal, then

$$(4.13) \quad \hat{\chi}_B(D_{k_1} \otimes \dots \otimes D_{k_N})\mathfrak{z}_{k_1} \otimes \dots \otimes \mathfrak{z}_{k_N} = u_{k_1} \dots u_{k_N} \hat{\chi}_A(D_{k_1} \otimes \dots \otimes D_{k_N})\mathfrak{z}_{k_1} \otimes \dots \otimes \mathfrak{z}_{k_N}.$$

The representation $D_{k_1} \otimes \dots \otimes D_{k_N}$ splits into irreducibles, and we obtain the following result.

Proposition 4.2 *In the setting of Proposition 4.1, assume (4.12) holds, and assume*

$$(4.14) \quad f_{N+1}^A \equiv f_{N+1}^B.$$

If D_ℓ occurs in the representation $D_{k_1} \otimes \dots \otimes D_{k_N}$, and if $\mathfrak{z}_{k_1} \otimes \dots \otimes \mathfrak{z}_{k_N}$ has a nonzero component in W_ℓ , the subspace where $SO(n)$ acts like (copies of) D_ℓ , then

$$(4.15) \quad u_\ell = u_{k_1} \dots u_{k_N},$$

provided we also have $P_\ell\chi_{A'} \neq 0$.

It is useful to have a variant of Proposition 4.2, which we obtain via the following calculation:

$$\begin{aligned}
 \hat{f}_N^A(\pi_{k_1}, \dots, \pi_{k_N}) &= \int_{G^{N+1}} \pi_{k_1}(x_1) \otimes \dots \otimes \pi_{k_N}(x_N) \chi_A(x_1^{-1}y) \dots \chi_A(x_N^{-1}y) \\
 (4.16) \quad & \quad \quad \quad dm(y) dm(x_1) \dots dm(x_N) \\
 &= \hat{\chi}_G(\pi_{k_1} \otimes \dots \otimes \pi_{k_N}) \cdot [\hat{\chi}_A(\pi_{k_1})^* \otimes \dots \otimes \hat{\chi}_A(\pi_{k_N})^*].
 \end{aligned}$$

Here $\chi_G \equiv 1$, so $\hat{\chi}_G(\pi)$ is given by (4.16), i.e., $\hat{\chi}_G(\pi_{k_1} \otimes \dots \otimes \pi_{k_N})$ is the orthogonal projection of $V_{k_1} \otimes \dots \otimes V_{k_N}$ onto W_0 . The arguments leading to Proposition 4.2 also yield the following result.

Proposition 4.3 *In the setting of Proposition 4.1, assume*

$$(4.17) \quad f_N^A \equiv f_N^B, P_{k_\nu} \chi_{A'} \neq 0, \quad 1 \leq \nu \leq N.$$

Assume the following holds:

$$(4.18) \quad D_0 \text{ occurs in } D_{k_1} \otimes \dots \otimes D_{k_N}, \text{ and } \mathfrak{z}_{k_1} \otimes \dots \otimes \mathfrak{z}_{k_N} \text{ has a nonzero component in } W_0,$$

Then

$$(4.19) \quad 1 = u_{k_1} \dots u_{k_N}.$$

In Section 5 we will show that if k_ν are positive integers satisfying $k_1 \leq \dots \leq k_N$, then (4.18) holds if and only if

$$(4.20) \quad k_1 + \dots + k_{N-1} \geq k_N, \quad \text{and} \quad k_1 + \dots + k_N \text{ is even.}$$

Here we will use various special cases of this result to derive specific uniqueness results from Proposition 4.3. One special case is when

$$(4.21) \quad k_1 + \dots + k_{N-1} = k_N.$$

Using this we can establish the following analogue of Proposition 2.1; note that this result does not follow from Proposition 3.1.

Proposition 4.4 *Suppose we have $A', B' \subset S^{n-1}$, with inverse images $A, B \subset \text{SO}(n)$, $n \geq 3$. Assume $f_3^A \equiv f_3^B$, and assume*

$$(4.22) \quad P_k \chi_{A'} \neq 0, \quad \forall k \in \mathbb{Z}^+.$$

Then either $B' = A'$ or $B' = -A'$, the image of A' under the antipodal map.

Proof We apply Proposition 4.3 with $k_1 = 1, k_2 = k, k_3 = k + 1$, to obtain $P_k \chi_{B'} = u_k P_k \chi_{A'}$ for each $k \in \mathbb{Z}^+$, with $u_k = \pm 1$ and

$$(4.23) \quad 1 = u_1 u_k u_{k+1}, \quad \forall k \in \mathbb{Z}^+.$$

Thus

$$(4.24) \quad u_1 = 1 \Rightarrow u_k = 1, \quad u_1 = -1 \Rightarrow u_k = (-1)^k,$$

for all $k \in \mathbb{Z}^+$. Hence one of the following holds for all $k \in \mathbb{Z}^+$: either

$$(4.25) \quad P_k \chi_{B'} = P_k \chi_{A'} \quad \text{or} \quad P_k \chi_{B'} = (-1)^k P_k \chi_{A'}.$$

In the first case $B' = A'$ and in the second case $B' = -A'$. ■

The hypothesis (4.22) holds for “generic” $A' \subset S^{n-1}$. Just to give one family of examples, we mention the following. Given $r \in (-1, 1)$, let $A'_r = \{x \in S^{n-1} : r < x_n \leq 1\}$. Then the inner product $(\chi_{A'_r}, \mathfrak{z}_k)$ is real-analytic in r for each k and not identically zero for any k . It follows that, for all but countably many $r \in (-1, 1)$, $(\chi_{A'_r}, \mathfrak{z}_k) \neq 0$ for all k , so (4.22) holds for such $A = A'_r$.

Our next goal is to extend the scope of Proposition 4.4 to various classes of subsets of S^{n-1} for which the hypothesis (4.22) does not hold. In analogy with (2.16), given a measurable set $A' \subset S^{n-1}$, we set

$$(4.26) \quad \Sigma_A = \{k \in \mathbb{Z}^+ : P_k \chi_{A'} \neq 0\}.$$

We want to treat sets for which Σ_A is not all of \mathbb{Z}^+ .

For example, suppose $A' \subset S^{n-1}$ is symmetric, i.e., $A' = -A'$ (modulo null sets); equivalently, $\chi_{A'}$ is an even function. Clearly A' is symmetric if and only if Σ_A consists only of even integers. We will obtain a completely satisfactory extension of Proposition 4.4 to the symmetric setting, making use of the following special case of the equivalence of (4.18) and (4.20), namely (4.20) holds with $N = 3, k_1 \leq k_2 \leq k_3$, and

$$(4.27) \quad k_1 + k_2 \geq k_3, \quad k_1 + k_2 + k_3 \text{ even}.$$

Hence, given $f_3^A \equiv f_3^B$, we can apply Proposition 4.3 and conclude that

$$(4.28) \quad k_1, k_2, k_3 \in \Sigma_A, \quad \text{satisfying (4.27)} \Rightarrow u_{k_1} u_{k_2} u_{k_3} = 1.$$

In particular, we have

$$(4.29) \quad k \in \Sigma_A, \quad k \text{ even} \Rightarrow u_k = 1, \text{ if } f_3^A \equiv f_3^B.$$

This immediately implies the following.

Proposition 4.5 *Let $A', B' \subset S^{n-1}$ be measurable and assume A' is symmetric. Then*

$$(4.30) \quad f_3^A \equiv f_3^B \Rightarrow B' = A'.$$

The result (4.29) holds whether or not A' is symmetric, but it is not always effective. For example, we say $A' \subset S^{n-1}$ is anti-symmetric provided $-A' = S^{n-1} \setminus A'$ (modulo null sets); equivalently, $\chi_{A'} - 1/2$ is an odd function. Thus A' is anti-symmetric if and only if all the nonzero elements of Σ_A are odd integers. Clearly the results (4.28)–(4.29) are not effective when A' is anti-symmetric.

Next we use the special case of (4.18) \Leftrightarrow (4.20) that arises when $N = 4$ and

$$(4.31) \quad k_1 \leq k_2 = k_3 = k_4, \quad k_1 + k_2 \text{ even.}$$

Using this, we can establish the following.

Proposition 4.6 *If $A', B' \subset S^{n-1}$ are measurable and $f_4^A \equiv f_4^B$, then either $B' = A'$ or $B' = -A'$.*

Proof As in Proposition 4.5, we have $u_k = 1$ for all even k . Meanwhile, since Proposition 4.3 is applicable when $N = 4$ and (4.31) holds, we have

$$(4.32) \quad k_1, k_2 \in \Sigma_A, \quad k_1 = k_2 \pmod 2 \Rightarrow u_{k_1} = u_{k_2}^3 = u_{k_2}.$$

In particular,

$$(4.33) \quad k_1, k_2 \in \Sigma_A, \quad \text{both odd} \Rightarrow u_{k_1} = u_{k_2}.$$

Hence either $u_k = 1$ for all odd k or $u_k = -1$ for all odd $k \in \Sigma_A$, and in these respective cases we have $B' = A'$ or $B' = -A'$. ■

5 Decomposition of Products of Zonal Harmonics

As before, D_k denotes the representation of $SO(n)$ on V_k , the k -th eigenspace of the Laplace-Beltrami operator on S^{n-1} , obtained by restricting the natural action of $SO(n)$ on $L^2(S^{n-1})$, and $\mathfrak{z}_k \in V_k$ the zonal harmonic (of unit L^2 -norm). We prove the following.

Proposition 5.1 *Let k_ν be positive integers, satisfying $k_1 \leq \dots \leq k_N$. The following conditions are equivalent.*

$$(5.1) \quad k_1 + \dots + k_{N-1} \geq k_N, \quad \text{and} \quad k_1 + \dots + k_N \text{ is even,}$$

$$(5.2) \quad D_{k_1} \otimes \dots \otimes D_{k_N} \text{ contains } D_0 \text{ and } \mathfrak{z}_{k_1} \otimes \dots \otimes \mathfrak{z}_{k_N} \text{ has a nontrivial component in } W_0 \text{ (the subspace of } V_{k_1} \otimes \dots \otimes V_{k_N} \text{ on which } SO(n) \text{ acts trivially),}$$

$$(5.3) \quad Z_{k_1 \dots k_N}(x_1, \dots, x_N) = \int_{SO(n)} \mathfrak{z}_{k_1}(g^{-1}x_1) \dots \mathfrak{z}_{k_N}(g^{-1}x_N) dm(g)$$

is not identically zero,

$$(5.4) \quad \int_{S^{n-1}} \mathfrak{z}_{k_1}(x) \cdots \mathfrak{z}_{k_N}(x) \, d\nu(x) \neq 0.$$

The equivalence of (5.1) with (5.2) is what was used in Section 4, but the other two conditions arise naturally in establishing this equivalence.

To begin this chain of equivalences, we note that generally, if π is a unitary representation of a compact group G on V_π , the orthogonal projection of $\mathfrak{z} \in V_\pi$ onto the subspace where G acts trivially is given by

$$(5.5) \quad P_0 \mathfrak{z} = \int_G \pi(g) \mathfrak{z} \, dm(g).$$

Identifying the tensor product space $V_{k_1} \otimes \cdots \otimes V_{k_N}$ with a space of functions on $(S^{n-1})^N$, we are looking at the function given in (5.3), so we see that (5.2) \Leftrightarrow (5.3). Furthermore, if we evaluate $Z_{k_1 \cdots k_N}(x_1, \dots, x_N)$ on the diagonal and integrate we obtain the integral in (5.4), so clearly (5.4) \Rightarrow (5.3).

To analyze the integrals in (5.3)–(5.4) further, we recall that, if $x = (x_1, \dots, x_n) \in S^{n-1} \subset \mathbb{R}^n$, then

$$(5.6) \quad \mathfrak{z}_k(x) = b_{nk} C_k^{(n-1)/2}(x_n),$$

where b_{nk} are nonzero constants and $C_k^\alpha(t)$ are Gegenbauer polynomials, given by

$$(5.7) \quad (1 - 2tr + r^2)^{-\alpha} = \sum_{k=0}^{\infty} C_k^\alpha(t) r^k.$$

See, e.g., Chapter 4 of [T]. The Gegenbauer polynomials have the following properties for each $\alpha > 0$. The function $C_k^\alpha(t)$ is a polynomial of degree k in t , even if k is even and odd if k is odd. The set $\{C_j^\alpha(t) : 0 \leq j \leq k\}$ spans the linear space \mathcal{P}_k of polynomials of degree k in t , and it is an orthogonal basis with respect to the inner product

$$(5.8) \quad \langle p, q \rangle_\alpha = \int_{-1}^1 p(t) \overline{q(t)} (1 - t^2)^{\alpha-1/2} \, dt.$$

In particular, the equivalence of (5.1) and (5.4) is a consequence of the following result.

Lemma 5.2 *Given $\alpha > 0$ and positive integers $k_1 \leq \cdots \leq k_N$, then*

$$(5.9) \quad \int_{-1}^1 C_{k_1}^\alpha(t) \cdots C_{k_N}^\alpha(t) (1 - t^2)^{\alpha-1/2} \, dt \neq 0$$

if and only if (5.1) holds.

Proof The necessity of (5.1) for (5.9) is straightforward. In fact the integrand in (5.9) has the parity of $k_1 + \dots + k_N$, so the integral surely vanishes if this sum is odd. Furthermore the integral is equal to

$$(5.10) \quad \langle C_{k_1}^\alpha \cdots C_{k_{N-1}}^\alpha, C_{k_N}^\alpha \rangle_\alpha,$$

the inner product given by (5.8). But $C_{k_1}^\alpha(t) \cdots C_{k_{N-1}}^\alpha(t)$ is a polynomial of degree $k_1 + \dots + k_{N-1}$ while $C_{k_N}^\alpha(t)$ is orthogonal to all polynomials of degree $< k_N$.

To establish the reverse implication in Lemma 5.2, we begin with the following identity, which holds whenever $\alpha > 0$ and $k_1 \leq k_2 \leq k_3$ are positive integers satisfying

$$(5.11) \quad k_1 + k_2 \geq k_3, \quad \text{and} \quad k_1 + k_2 + k_3 \text{ even.}$$

Namely, as shown in [Vil], pp. 490–491,

$$(5.12) \quad \int_{-1}^1 C_{k_1}^\alpha(t) C_{k_2}^\alpha(t) C_{k_3}^\alpha(t) (1-t^2)^{\alpha-1/2} dt = \frac{2^{1-2\alpha} \pi \Gamma(\ell+2\alpha) \Gamma(\ell-k_1+\alpha) \Gamma(\ell-k_2+\alpha) \Gamma(\ell-k_3+\alpha)}{\Gamma(\alpha)^4 \Gamma(\ell+\alpha+1) \Gamma(\ell-k_1+1) \Gamma(\ell-k_2+1) \Gamma(\ell-k_3+1)}.$$

Note that under these hypotheses all nine arguments of the gamma function are positive numbers, so all the numbers in (5.12) are *positive*. Equivalently, we have

$$(5.13) \quad C_{k_1}^\alpha(t) C_{k_2}^\alpha(t) = \sum_{\ell \in \mathcal{S}(k_1, k_2)} \sigma_{k_1, k_2}^\alpha(\ell) C_\ell^\alpha(t),$$

where

$$(5.14) \quad \mathcal{S}(k_1, k_2) = \{ \ell \in \mathbb{Z} : |k_1 - k_2| \leq \ell \leq k_1 + k_2 \text{ and } \ell = k_1 + k_2 \pmod{2} \},$$

and $\sigma_{k_1, k_2}^\alpha(\ell) > 0$ whenever $\ell \in \mathcal{S}(k_1, k_2)$ and $\alpha > 0$. (A formula for these coefficients is written down in [Vil], p. 491.) It follows inductively that

$$(5.15) \quad C_{k_1}^\alpha(t) \cdots C_{k_{N-1}}^\alpha(t) = \sum_{\ell \in \mathcal{S}(k_1, \dots, k_{N-1})} \sigma_{k_1, \dots, k_{N-1}}^\alpha(\ell) C_\ell^\alpha(t),$$

where

$$(5.16) \quad \mathcal{S}(k_1, \dots, k_{N-1}) = \{ \ell \in \mathbb{Z} : \mu(k_1, \dots, k_{N-1}) \leq \ell \leq \nu(k_1, \dots, k_{N-1}) \text{ and } \ell = k_1 + \dots + k_{N-1} \pmod{2} \},$$

for certain integers $\mu(k_1, \dots, k_{N-1})$ and $\nu(k_1, \dots, k_{N-1})$, which will be discussed below, and $\sigma_{k_1, \dots, k_{N-1}}^\alpha(\ell) > 0$ whenever $\ell \in \mathcal{S}(k_1, \dots, k_{N-1})$. To prove the lemma, it remains to show that, when $k_1 \leq \dots \leq k_N$ and (5.1) holds,

$$(5.17) \quad k_N \in \mathcal{S}(k_1, \dots, k_{N-1}).$$

To proceed, it is clear from (5.13)–(5.14) that, when $N - 1 \geq 2$,

$$(5.18) \quad \nu(k_1, \dots, k_{N-1}) = \nu(k_1 + \dots + k_{N-2}) + k_{N-1},$$

and hence

$$(5.19) \quad \nu(k_1, \dots, k_{N-1}) = k_1 + \dots + k_{N-1}.$$

Furthermore, we have

$$(5.20) \quad \mu(k_1, \dots, k_{N-1}) = \min\{|k_{N-1} - \lambda| : \lambda \in \mathcal{S}(k_1, \dots, k_{N-2})\} \leq k_{N-1}.$$

The results (5.19)–(5.20) readily yield (5.17) when $k_1 \leq \dots \leq k_N$ and (5.1) holds, so Lemma 5.2 is proven. ■

At this point we have (5.1) \Leftrightarrow (5.4) \Rightarrow (5.3) \Leftrightarrow (5.2), hence (5.1) \Rightarrow (5.2), which is enough to prove the results of Section 4, but we push on with the rest of the proof of Proposition 5.1.

Since the representations D_k all have real-valued characters, it follows from the orthogonality relations that $D_{k_1} \otimes \dots \otimes D_{k_N}$ contains D_0 if and only if $D_{k_1} \otimes \dots \otimes D_{k_{N-1}}$ contains D_{k_N} . Hence part of the implication (5.2) \Rightarrow (5.1) is contained in the following.

Lemma 5.3 *Let k_i be positive integers. If $k_1 + \dots + k_{N-1} < k_N$, then $D_{k_1} \otimes \dots \otimes D_{k_{N-1}}$ does not contain D_{k_N} .*

Proof We can identify the representation space V_k of D_k with the space of harmonic polynomials on \mathbb{R}^n , homogeneous of degree k . Let $X \in \mathfrak{so}(n)$ be such that $\exp \theta X$ is rotation through angle θ in the $x_1 x_2$ -plane. Then $D_k(\exp tX) = \exp tX_k$, and $(1/i)X_k$ has spectrum in $[-k, k]$ and contains these endpoints. It follows that $D_{k_1}(\exp tX) \otimes \dots \otimes D_{k_{N-1}}(\exp tX) = \exp tY$, where $(1/i)Y$ has spectrum contained in $[-(k_1 + \dots + k_{N-1}), k_1 + \dots + k_{N-1}]$. Now if D_{k_N} is contained in this tensor product, k_N must be contained in this spectrum, so Lemma 5.3 is proven. ■

The following result suffices to complete the proof of Proposition 5.1.

Lemma 5.4 *Let k_1, \dots, k_N be positive integers. If $k_1 + \dots + k_N$ is odd, then the function $Z_{k_1 \dots k_N}$ defined by (5.3) is identically zero.*

Proof Define $\tau \in \text{SO}(n)$ by $\tau(x_1, x_2, x_3, \dots, x_n) = (-x_1, x_2, x_3, \dots, x_{n-1}, -x_n)$. Then

$$(5.21) \quad \begin{aligned} Z_{k_1 \dots k_N}(x_1, \dots, x_N) &= \int_{\text{SO}(n)} \mathfrak{z}_{k_1}(\tau g^{-1} x_1) \cdots \mathfrak{z}_{k_N}(\tau g^{-1} x_N) dm(g) \\ &= (-1)^{k_1 + \dots + k_N} Z_{k_1 \dots k_N}(x_1, \dots, x_N), \end{aligned}$$

the first identity by right-invariance of Haar measure on $SO(n)$ and the second by

$$(5.22) \quad \mathfrak{z}_k(\tau x) = (-1)^k \mathfrak{z}_k(x),$$

which follows from (5.6) and the observation about the parity of $C_k^\alpha(t)$. ■

Proposition 5.1 bears directly on Proposition 4.3. On the other hand, Proposition 4.2 dealt with cases where D_ℓ is contained in $D_{k_1} \otimes \cdots \otimes D_{k_N}$ and

$$(5.23) \quad \mathfrak{z}_{k_1} \otimes \cdots \otimes \mathfrak{z}_{k_N} \text{ has a nonzero component in } W_\ell,$$

the space where $D_{k_1} \otimes \cdots \otimes D_{k_N}$ acts like copies of D_ℓ . The following result implies that the scope of Proposition 4.2 is not larger than that of Proposition 4.3.

Proposition 5.5 *If (5.23) holds, then $\mathfrak{z}_\ell \otimes \mathfrak{z}_{k_1} \otimes \cdots \otimes \mathfrak{z}_{k_N}$ has a nonzero component in W_0 , the space where $D_\ell \otimes D_{k_1} \otimes \cdots \otimes D_{k_N}$ acts trivially.*

Note that the hypothesis (5.23) is equivalent to the statement that

$$(5.24) \quad \int_{SO(n)} \chi_\ell(g) \mathfrak{z}_{k_1}(g^{-1}x_1) \cdots \mathfrak{z}_{k_N}(g^{-1}x_N) dm(g) = Y_{\ell k_1 \dots k_N}(x_1, \dots, x_N)$$

is not identically zero, where $\chi_\ell(g) = \text{Tr } D_\ell(g)$. Note also that we can apply Lemma 5.3 to show that if (5.23) holds then the largest number in $\{\ell, k_1, \dots, k_N\}$ is not greater than the sum of the rest of these numbers. To prove Proposition 5.5, it remains to show that if (5.23) holds then $\ell + k_1 + \cdots + k_N$ must be even.

This is straightforward if n is even. Then $-I$ belongs to $SO(n)$, and we have

$$(5.25) \quad \begin{aligned} Y_{\ell k_1 \dots k_N}(x_1, \dots, x_N) &= \int_{SO(n)} \chi_\ell(-g) \mathfrak{z}_{k_1}(-g^{-1}x_1) \cdots \mathfrak{z}_{k_N}(-g^{-1}x_N) dm(g) \\ &= (-1)^{\ell+k_1+\dots+k_N} Y_{\ell k_1 \dots k_N}(x_1, \dots, x_N), \end{aligned}$$

the first identity holding since $g \mapsto -g$ is measure-preserving on $SO(n)$ for n even and the second since

$$(5.26) \quad \chi_\ell(-g) = (-1)^\ell \chi_\ell(g), \quad \mathfrak{z}_k(-x) = (-1)^k \mathfrak{z}_k(x).$$

If n is odd the argument involving (5.25) does not work, but the following argument works for all $n \geq 3$. This argument makes essential use of Proposition 4.2, and it is the one place in this section where we depend on results from Section 4.

To proceed, pick a measurable set $A' \subset S^{n-1}$ satisfying hypothesis (4.22). As noted in Section 4, such sets exist in great profusion. Set $B' = -A'$, so $f_N^A \equiv f_N^B$ for all N . We see that the numbers u_k are well defined by (4.8) for all $k \in \mathbb{Z}^+$, and in fact $u_k = +1$ for k even and -1 for k odd, i.e., $u_k = (-1)^k$. Now suppose (5.23) holds. By Proposition 4.2 we can deduce that $u_\ell = u_{k_1} \cdots u_{k_N}$, i.e., $(-1)^\ell = (-1)^{k_1+\dots+k_N}$. This forces $\ell + k_1 + \cdots + k_N$ to be even and completes the proof of Proposition 5.5.

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