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On smooth perturbations of Chebyshëv polynomials and $\bar{\partial}$-Riemann–Hilbert method

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Abstract. $\bar{\partial}$-extension of the matrix Riemann–Hilbert method is used to study asymptotics of the polynomials $P_n(z)$ satisfying orthogonality relations

$$\int_{-1}^{1} x^l P_n(x)\frac{\rho(x)dx}{\sqrt{1-x^2}} = 0, \quad l \in \{0,\dots,n-1\},$$

where $\rho(x)$ is a positive m times continuously differentiable function on $[-1,1]$, $m \geq 3$.

1 Main results

In this note, we are interested in the asymptotic behavior of monic polynomials $P_{n,i}(x)$, $\deg(P_{n,i}) = n$, dependent on a parameter $i \in \{1,2,3,4\}$, satisfying orthogonality relations

$$\int_{-1}^{1} x^l P_{n,i}(x)\frac{\rho(x)|v_i(x)|dx}{\sqrt{1-x^2}} = 0, \quad l \in \{0,\dots,n-1\}, \tag{1.1}$$

where $\rho(x)$ is a positive and smooth function on $[-1,1]$ and

$$v_1(z) \equiv 1, \quad v_2(z) = z^2-1, \quad v_3(z) = z+1, \quad \text{and} \quad v_4(z) = z-1.$$

That is, $P_{n,i}(z)$ are smooth perturbations of the Chebyshëv polynomials of the ith kind. Besides polynomials themselves, we are also interested in the asymptotic behavior of their recurrence coefficients. That is, numbers $a_{n,i} \in [0,\infty)$ and $b_{n,i} \in (-\infty,\infty)$ such that

$$xP_{n,i}(x) = P_{n+1,i}(x) + b_{n,i}P_{n,i}(x) + a_{n,i}^2 P_{n-1,i}(x).$$

To describe the results, let $w(z) := \sqrt{z^2-1}$ be the branch analytic in $\mathbb{C}\backslash[-1,1]$ such that $w(z)/z \to 1$ as $z \to \infty$. The Szegő function of the weight $\rho(x)$ is defined by

$$S(z) := \exp\left\{\frac{w(z)}{2\pi \mathrm{i}}\int_{-1}^{1}\frac{\log\rho(x)}{z-x}\frac{dx}{w_+(x)}\right\}, \quad z \in \overline{\mathbb{C}}\backslash[-1,1], \tag{1.2}$$

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which is an analytic and nonvanishing function in the domain of its definition satisfying

$$S_+(x)S_-(x) = \rho^{-1}(x), \quad x \in [-1,1]. \tag{1.3}$$

Since $\rho(x)$ is positive, it holds that $S_+(x) = \overline{S_-(x)}$ for $x \in [-1,1]$, and, utilizing the full power of Plemelj–Sokhotski formulae, (1.3) can be strengthen to

$$\sqrt{\rho(x)}S_\pm(x) = e^{\pm\mathrm{i}\theta(x)}, \quad \theta(x) := \frac{\sqrt{1-x^2}}{2\pi} ⨍_{-1}^{1} \frac{\log\rho(t)}{t-x}\frac{dt}{\sqrt{1-t^2}}, \tag{1.4}$$

where $⨍$ is the integral in the sense of the principal value. Further, let

$$\varphi(z) := z + w(z) \tag{1.5}$$

be the conformal map of $\overline{\mathbb{C}}\backslash[-1,1]$ onto $\mathbb{C}\backslash\{z : |z| \geq 1\}$ such that $\varphi(z)/z \to 2$ as $z \to \infty$. One can readily verify that

$$\varphi_\pm(x) = x \pm \mathrm{i}\sqrt{1-x^2} = e^{\pm\mathrm{i}\arccos(x)}, \quad x \in [-1,1]. \tag{1.6}$$

Finally, we explicitly define the Szegő functions of the weights $|v_i(x)|$. Namely, set

$$\begin{cases} S_1(z) :\equiv 1, & S_3(z) := \big(\varphi(z)/(z+1)\big)^{1/2}, \\ S_2(z) := \varphi(z)/w(z), & S_4(z) := \big(\varphi(z)/(z-1)\big)^{1/2}, \end{cases} \tag{1.7}$$

where the square roots are principal and one needs to notice that the images of $\overline{\mathbb{C}}\backslash[-1,1]$ under $(z+1)/\varphi(z)$ and $(z-1)/\varphi(z)$ are domains symmetric with respect to conjugation whose intersections with the real line are equal to $(0,2)$ (so the square roots are indeed well defined). These functions satisfy

$$S_{i+}(x)S_{i-}(x) = |S_{i\pm}(x)|^2 = 1/|v_i(x)|, \quad x \in (-1,1). \tag{1.8}$$

Observe also that $S_1(\infty) = 1$, $S_2(\infty) = 2$, and $S_3(\infty) = S_4(\infty) = \sqrt{2}$. Moreover, one can readily deduce from (1.6) and (1.8) that

$$S_{i\pm}(x) = \frac{e^{\pm\mathrm{i}\theta_i(x)}}{\sqrt{|v_i(x)|}}, \quad \begin{cases} \theta_1(x) :\equiv 0, & \theta_2(x) := \arccos(x) - \frac{\pi}{2}, \\ \theta_3(x) := \frac{1}{2}\arccos(x), & \theta_4(x) := \frac{1}{2}\arccos(x) - \frac{\pi}{2}. \end{cases} \tag{1.9}$$

Recall that the modulus of continuity of a continuous function $f(x)$ on $[-1,1]$ is given by

$$\omega(f;h) := \max_{|x-y|\leq h,\ x,y\in[-1,1]} |f(x) - f(y)|.$$

Theorem 1.1 *Assume that $\rho(x)$ is a strictly positive m times continuously differentiable function on $[-1,1]$ for some $m \geq 3$. Set*

$$\varepsilon_n := \frac{\log n}{n^m}\omega\left((1/\rho)^{(m)}; 1/n\right).$$

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Then it holds for any $i \in \{1,2,3,4\}$ *that*

$$P_{n,i}(z) = (1 + O(\varepsilon_n))\,\frac{(S_iS)(z)}{(S_iS)(\infty)}\left(\frac{\varphi(z)}{2}\right)^n$$

uniformly on closed subsets of $\overline{\mathbb{C}}\setminus[-1,1]$ *and*

$$P_{n,i}(x) = \frac{\cos\big(n\arccos(x) + \theta(x) + \theta_i(x)\big) + O(\varepsilon_n)}{2^{n-1}(S_iS)(\infty)\sqrt{\rho(x)|v_i(x)|}}$$

uniformly on $[-1,1]$. *Moreover, it also holds for any* $i \in \{1,2,3,4\}$ *that*

$$a_{n,i} = 1/2 + O(\varepsilon_n) \quad \textit{and} \quad b_{n,i} = O(\varepsilon_n).$$

The above results are not entirely new. It is well known [18, Theorem 11.5] that perturbed first and second kind Chebyshëv polynomials can be expressed via orthogonal polynomials on the unit circle with respect to the weight $\rho(\frac{1}{2}(\tau + 1/\tau))$. Then using [17, Corollary 5.2.3], that in itself is an extension of ideas from [5], and Geronimus relations, see [17, Theorem 13.1.7], one can show that

$$\sum (n+1)^\gamma \big(|a_{n,1} - 1/2| + |b_{n,1}|\big) < \infty$$

for any $\gamma \in (0, m-1)$ and $m \geq 2$, which is consistent with Theorem 1.1. What is novel in this note is the method of proof. While the Baxter–Simon argument relies on the machinery of Banach algebras, we follow the approach of Fokas et al. [11, 12] connecting orthogonal polynomials to matrix Riemann–Hilbert problems and then utilizing the nonlinear steepest descent method of Deift and Zhou [9]. The main advantages of this approach are the ability to get full asymptotic expansions for analytic weights of orthogonality [8, 15] and its indifference to positivity of such weights [1, 2, 6]. However, here we deal with non-analytic densities by elaborating on the idea of extensions with controlled $\bar\partial$-derivative introduced by Miller and McLaughlin [16] and adapted to the setting of Jacobi-type polynomials by Baratchart and Yettselev [4].

2 Weight extension

Given $r > 1$, let $E_r := \{z : |\varphi(z)| < r\}$. The boundary ∂E_r is an ellipse with foci ± 1.

Proposition 2.1 *Let* $\rho(x)$ *and* ε_n *be as in Theorem 1.1. For each* $r > 1$ *and* $n > 2m$ *there exists a continuous function* $\ell_{n,r}(z) = l_n(z) + L_{n,r}(z)$, $z \in \mathbb{C}$, *such that*

$$\ell_{n,r}(x) = \rho^{-1}(x), \quad x \in [-1,1],$$

where $l_n(z)$ *is a polynomial of degree at most* n *satisfying*

$$\mathrm{supp}_{x\in[-1,1]}|l_n(x)| \leq C'_\rho$$

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for some constant C'_ρ independent of n, while $L_{n,r}(z)$ and $\bar\partial L_{n,r}(z)$ are continuous functions in $\mathbb{C}$ supported by $\overline{E}_r$ (in particular, $L_{n,r}(z) = 0$ for $z \notin E_r$) and

$$\frac{|\bar\partial L_{n,r}(z)|}{\sqrt{|1-z^2|}} \le C''_\rho \frac{n\varepsilon_n}{\log n}, \qquad z \in \overline{E}_r$$

for some constant C''_ρ independent of n and r, where $\bar\partial := (\partial_x + \mathrm{i}\partial_y)/2$, $z = x + \mathrm{i}y$.

Proof It follows from [14, Theorem 9] that for each $n > 2m$, there exists a polynomial $l_n(z)$ of degree at most n such that

$$\left|\left(\rho^{-1}(x)\right)^{(k)} - l_n^{(k)}(x)\right| \le C_{m,k}(1-x^2)^{\frac{m-k}{2}} n^{k-m} E_{n-m}\left(\left(\rho^{-1}\right)^{(m)}\right)$$

for all $x \in [-1,1]$ and each $k \in \{0,\dots,m\}$, where $C_{m,k}$ is a constant that depends only m and k and $E_j(f)$ is the error of best uniform approximation on the interval $[-1,1]$ of a continuous function $f(x)$ by algebraic polynomials of degree at most j. Furthermore, it was shown by Timan, see [14, Equation (3)], that

$$\begin{aligned} E_{n-m}(f) &\le C_1\omega\left(f;\frac{\sqrt{1-x^2}}{n-m} + \frac{1}{(n-m)^2}\right) \le C_1\omega\left(f;\frac{2}{n-m}\right) \\ &\le C_1\omega\left(f;\frac{4}{n}\right) \le 4C_1\omega\left(f;\frac{1}{n}\right) \end{aligned}$$

for some absolute constant C_1, where we used that $n > 2m$ and $\omega(f;2h) \le 2\omega(f;h)$ (in what follows, we understand that all constants C_j might depend on $\rho(x)$, but are independent of n). Set

$$\lambda_n(x) := \frac{\rho^{-1}(x) - l_n(x)}{\sqrt{1-x^2}}, \qquad x \in [-1,1].$$

It then holds that $\lambda_n(x)$ is a continuous function on $[-1,1]$ that satisfies $\|\lambda_n\| \le C_3\varepsilon_n/\log n$, where $\|\cdot\|$ is the uniform norm on $[-1,1]$. Since $m \ge 3$, it also holds that

$$\lambda'_n(x) = \frac{\left(\rho^{-1}(x)\right)' - l'_n(x)}{\sqrt{1-x^2}} + x\frac{\rho^{-1}(x) - l_n(x)}{\sqrt{(1-x^2)^3}}$$

is a continuous function on $[-1,1]$ that satisfies $\|\lambda'_n\| \le C_4 n\varepsilon_n/\log n$ (this is exactly the place where condition $m \ge 3$ is used). Extend $\lambda_n(x)$ by zero to the whole real line. As the numerator of $\lambda_n(x)$ together with its first and second derivatives vanishes at ± 1, $\lambda'_n(x)$ also extends continuously by zero to the whole real line. The following construction is standard, see [10, Proof of Theorem 3.67]. Define

$$\Lambda_n(z) := \frac{1}{|y|}\int_0^{|y|} \lambda_n(x+t)dt, \qquad z = x + \mathrm{i}y,$$

which, due to continuity of $\lambda_n(x)$, is a continuous function in $\mathbb{C}$ satisfying $\Lambda_n(x) = \lambda_n(x)$ on the real line and $|\Lambda_n(z)| \le \|\lambda_n\|$ in the complex plane. Similarly,

$$\left|\partial_x \Lambda_n(z)\right| = \left|\frac{1}{|y|}\int_0^{|y|} \lambda'_n(x+t)dt\right| \le \|\lambda'_n\|$$

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and the function $\partial_x \Lambda_n(z)$, which is given by the integral within the absolute value in the above equation, is also continuous in $\mathbb{C}$. Furthermore, we have that

$$\begin{aligned}|\partial_y \Lambda_n(z)| &= \left|\frac{1}{y^2}\int_0^{|y|}\big(\lambda_n(x+t)-\lambda_n(x+|y|)\big)dt\right| \\ &\le \|\lambda_n'\|\int_0^{|y|}\frac{|y|-t}{y^2}dt = \frac{\|\lambda_n'\|}{2},\end{aligned}$$

and is also a continuous function in $\mathbb{C}$. Altogether, since $\bar\partial = (\partial_x + \mathrm{i}\partial_y)/2$, it holds that $\bar\partial\Lambda_n(z)$ is a continuous function in $\mathbb{C}$ that satisfies $|\bar\partial\Lambda_n(z)| \le \|\lambda_n'\|$ in the complex plane. Let $\psi_r(z)$ be any real-valued continuous function with continuous partial derivatives that is equal to one on $[-1,1]$ and is equal to zero in the complement of E_r. Define

$$L_{n,r}(z) := \mathrm{i}w(z)\Lambda_n(z)\psi_r(z)\begin{cases}-1, & \mathrm{Im}(z)\ge 0,\\ 1, & \mathrm{Im}(z)<0.\end{cases}$$

Since $w_\pm(x) = \pm\mathrm{i}\sqrt{1-x^2}$ for $x\in[-1,1]$ and $\Lambda_n(x)=0$ for $x\notin(-1,1)$, it holds that $L_{n,r}(z)$ is a continuous function in $\mathbb{C}$ that is supported by $\overline{E}_r$ and is equal to $\rho^{-1}(x) - l_n(x)$ for $x\in[-1,1]$. Furthermore, since $\bar\partial(\Lambda_n(z)\psi_n(z))$ is continuous in $\mathbb{C}$ and vanishes for $z = x\notin(-1,1)$ while $w_+(x) = -w_-(x)$ for $x\in(-1,1)$, $\bar\partial L_{n,r}(z)$ is also continuous in $\mathbb{C}$. Moreover, it holds that

$$\begin{aligned}|\bar\partial L_{n,r}(z)| &= \sqrt{|1-z^2|}\,\big|\bar\partial(\Lambda_n(z)\psi_r(z))\big| \\ &\le C_5\sqrt{|1-z^2|}\,\big(|\Lambda_n(z)| + |\bar\partial\Lambda_n(z)|\big) \\ &\le C_6\sqrt{|1-z^2|}\,\frac{n\varepsilon_n}{\log n}, \quad z\in\overline{E}_r.\end{aligned}$$

Finally, observe that polynomials $l_n(x)$ approximate $\rho^{-1}(x)$ on $[-1,1]$ and therefore have uniformly bounded above uniform norms. The claim of the proposition now follows by setting $\ell_{n,r}(z) := l_n(z) + L_{n,r}(z)$ for $l_n(z)$ and $L_{n,r}(z)$ as above. ■

3 Proof of Theorem 1.1

3.1 Initial Riemann–Hilbert problem

Notice that the functions $v_i(x)$ and $|v_i(x)|$ are either equal to each other or differ by a sign when $x\in[-1,1]$. So, we can equally use $v_i(x)$ in (1.1) without changing the polynomials $P_{n,i}(x)$.

Denote by $R_{n,i}(z)$ the function of the second kind associated with $P_{n,i}(z)$. That is,

$$R_{n,i}(z) := \frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}\frac{P_{n,i}(x)}{x-z}\frac{\rho(x)v_i(x)dx}{w_+(x)}, \tag{3.1}$$

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which is a holomorphic function in $\overline{\mathbb{C}}\backslash[-1,1]$. It follows from Plemelj–Sokhotski formulae, [13, Chapter I.4.2], that

$$R_{n,i+}(x) - R_{n,i-}(x) = P_{n,i}(x)\frac{\rho(x)v_i(x)}{w_+(x)}, \quad x \in (-1,1),$$

and, see [13, Chapter I.8.4], that

$$R_{n,i}(z) = O\left(|z-a|^{\alpha_{a,i}}\right) \quad \text{as} \quad \mathbb{C}\backslash[-1,1] \ni z \to a \in \{-1,1\},$$

where $\alpha_{a,i} = 0$ if $v_i(a) = 0$ and $\alpha_{a,i} = -1/2$ otherwise. Moreover, we get from (1.1) that

$$R_{n,i}(z) = \frac{1}{m_{n,i}z^n} + O\left(\frac{1}{z^{n+1}}\right) \quad \text{as} \quad z \to \infty$$

for some finite constant $m_{n,i}$. Consider the following Riemann–Hilbert problem for 2×2 matrix functions (RHP-$\boldsymbol{Y}$):

(a) $\boldsymbol{Y}(z)$ is analytic in $\mathbb{C}\backslash[-1,1]$ and $\lim_{z\to\infty}\boldsymbol{Y}(z)z^{-n\sigma_3} = \boldsymbol{I}$;

(b) $\boldsymbol{Y}(z)$ has continuous traces on $(-1,1)$ that satisfy

$$\boldsymbol{Y}_+(x) = \boldsymbol{Y}_-(x)\begin{pmatrix}1 & \frac{\rho(x)v_i(x)}{w_+(x)} \\ 0 & 1\end{pmatrix}; \text{ and}$$

(c) $\boldsymbol{Y}(z)$ behaves like

$$\boldsymbol{Y}(z) = O\begin{pmatrix}1 & |z-a|^{\alpha_{a,i}} \\ 1 & |z-a|^{\alpha_{a,i}}\end{pmatrix} \quad \text{as} \quad \mathbb{C}\backslash[-1,1] \ni z \to a \in \{-1,1\}.$$

The following lemma is well known [15, Theorem 2.4].

Lemma 3.1 RHP-$\boldsymbol{Y}$ *is uniquely solvable by*

(3.2)
$$\boldsymbol{Y}(z) = \begin{pmatrix}P_{n,i}(z) & R_{n,i}(z) \\ m_{n-1,i}P_{n-1,i}(z) & m_{n-1,i}R_{n-1,i}(z)\end{pmatrix}.$$

3.2 Opening of the lens

Fix $1 < r < R$ and orient ∂E_R clockwise. Set

(3.3)
$$\boldsymbol{X}(z) := \begin{cases}\boldsymbol{Y}(z)\begin{pmatrix}1 & 0 \\ -\frac{w(z)\ell_{n,r}(z)}{v_i(z)} & 1\end{pmatrix}, & \text{in} \quad E_R\backslash[-1,1], \\ \boldsymbol{Y}(z), & \text{in} \quad \mathbb{C}\backslash\overline{E}_R,\end{cases}$$

where $\ell_{n,r}(z)$ is the extension of $\rho^{-1}(x)$ constructed in Proposition 2.1. Observe that

$$\ell_{n,r}(s) = l_n(s), \; s \in \partial E_R, \quad \text{and} \quad \bar\partial\ell_{n,r}(z) = \bar\partial L_{n,r}(z), \; z \in \overline{E}_r,$$

since $L_{n,r}(z)$ is supported by $\overline{E}_r$ and $l_n(z)$ is analytic (in fact, is a polynomial). It is trivial to verify that $\boldsymbol{X}(z)$ solves the following $\bar\partial$-Riemann–Hilbert problem ($\bar\partial$RHP-$\boldsymbol{X}$):

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(a) $\boldsymbol{X}(z)$ is continuous in $\mathbb{C}\setminus([-1,1]\cup\partial E_R)$ and $\lim_{z\to\infty}\boldsymbol{X}(z)z^{-n\sigma_3}=\boldsymbol{I}$;
(b) $\boldsymbol{X}(z)$ has continuous traces on $(-1,1)\cup\partial E_R$ that satisfy

$$\boldsymbol{X}_+(s)=\boldsymbol{X}_-(s)\begin{cases}\begin{pmatrix}0 & \frac{\rho(s)v_i(s)}{w_+(s)}\\ -\frac{w_+(s)}{\rho(s)v_i(s)} & 0\end{pmatrix} & \text{on} \quad s\in(-1,1),\\ \begin{pmatrix}1 & 0\\ \frac{w(s)l_n(s)}{v_i(s)} & 1\end{pmatrix} & \text{on} \quad s\in\partial E_R;\end{cases}$$

(c) $\boldsymbol{X}(z)$ has the same behavior near ±1 as $\boldsymbol{Y}(z)$, see RHP-$\boldsymbol{Y}$(c); and
(d) $\boldsymbol{X}(z)$ deviates from an analytic matrix function according to

$$\bar\partial\boldsymbol{X}(z)=\boldsymbol{X}(z)\begin{pmatrix}0 & 0\\ -\frac{w(z)\bar\partial L_{n,r}(z)}{v_i(z)} & 0\end{pmatrix}.$$

One can readily verified that the following lemma holds, see [4, Lemma 6.4].

Lemma 3.2 $\bar\partial$RHP-$\boldsymbol{X}$ *and* RHP-$\boldsymbol{Y}$ *are simultaneously solvable and the solutions are connected by* (3.3).

3.3 Model Riemann–Hilbert problem

In this subsection we present the solution of the following Riemann–Hilbert problem (RHP-$\boldsymbol{N}$):

(a) $\boldsymbol{N}(z)$ is analytic in $\mathbb{C}\setminus[-1,1]$ and $\lim_{z\to\infty}\boldsymbol{N}(z)z^{-n\sigma_3}=\boldsymbol{I}$;
(b) $\boldsymbol{N}(z)$ has continuous traces on $(-1,1)$ that satisfy

$$\boldsymbol{N}_+(x)=\boldsymbol{N}_-(s)\begin{pmatrix}0 & \frac{\rho(x)v_i(x)}{w_+(x)}\\ -\frac{w_+(x)}{\rho(x)v_i(x)} & 0\end{pmatrix};\text{and}$$

(c) $\boldsymbol{N}(z)$ has the same behavior near ±1 as $\boldsymbol{Y}(z)$, see RHP-$\boldsymbol{Y}$(c).

Recall the definition of the functions $S_i(z)$ in (1.7). Define $S_*(z):=S_i(z)$ when $i\in\{1,3\}$ and $S_*(z):=\mathrm{i}S_i(z)$ when $i\in\{2,4\}$. Then it follows from (1.8) that

$$S_{*+}(x)S_{*-}(x)=1/v_i(x), \quad x\in(-1,1).$$

Let $S(z)$ and $\varphi(z)$ be given by (1.2) and (1.5), respectively. It follows from (1.3) and (1.6) that

$$(S_*S\varphi^n)_-^{\sigma_3}(x)\begin{pmatrix}0 & \frac{\rho(x)v_i(x)}{w_+(x)}\\ -\frac{w_+(x)}{\rho(x)v_i(x)} & 0\end{pmatrix}(S_*S\varphi^n)_+^{-\sigma_3}(x)=\begin{pmatrix}0 & 1/w_+(x)\\ -w_+(x) & 0\end{pmatrix}$$

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for $x \in (-1,1)$. It also can be readily verified with the help of (1.6) that

$$\begin{pmatrix} 1 & \frac{1}{w_+(x)} \\ \frac{1}{2\varphi_+(x)} & \frac{\varphi_+(x)}{2w_+(x)} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{w_-(x)} \\ \frac{1}{2\varphi_-(x)} & \frac{\varphi_-(x)}{2w_-(x)} \end{pmatrix} \begin{pmatrix} 0 & 1/w_+(x) \\ -w_+(x) & 0 \end{pmatrix}$$

for $x \in (-1,1)$. Therefore, RHP-$\boldsymbol{N}$ is solved by $\boldsymbol{N}(z) = \boldsymbol{CM}(z)$, where

$$\text{(3.4)} \qquad \boldsymbol{C} := (2^n S_* S)^{-\sigma_3}(\infty) \quad \text{and} \quad \boldsymbol{M}(z) := \begin{pmatrix} 1 & \frac{1}{w(z)} \\ \frac{1}{2\varphi(z)} & \frac{\varphi(z)}{2w(z)} \end{pmatrix} (S_* S\varphi^n)^{\sigma_3}(z).$$

3.4 Analytic approximation

To solve $\bar\partial$RHP-$\boldsymbol{X}$, we first solve its analytic version. That is, consider the following Riemann–Hilbert problem (RHP-$\boldsymbol{A}$):

(a) $\quad \boldsymbol{A}(z)$ is analytic in $\mathbb{C}\backslash([-1,1] \cup \partial E_R)$ and $\lim_{z\to\infty} \boldsymbol{A}(z) z^{-n\sigma_3} = \boldsymbol{I}$ and

(b,c) $\quad \boldsymbol{A}(z)$ satisfies $\bar\partial$RHP-$\boldsymbol{X}$(b,c).

Lemma 3.3 *For all n large enough there exists a matrix $\boldsymbol{Z}(z)$, analytic in $\overline{\mathbb{C}}\backslash\partial E_R$ and satisfying*

$$\boldsymbol{Z}(z) = \boldsymbol{I} + \boldsymbol{O}\left(R_*^{-n}\right)$$

uniformly in $\overline{\mathbb{C}}$ for any $r < R_* < R$, such that $\boldsymbol{A}(z) = \boldsymbol{CZ}(z)\boldsymbol{M}(z)$ solves RHP-$\boldsymbol{A}$.

Proof Assume that there exists a matrix $\boldsymbol{Z}(z)$ that is analytic in $\overline{\mathbb{C}}\backslash\partial E_R$, is equal to $\boldsymbol{I}$ at infinity, and satisfies

$$\boldsymbol{Z}_+(s) = \boldsymbol{Z}_-(s)\boldsymbol{M}(s)\begin{pmatrix} 1 & 0 \\ \frac{w(s)l_n(s)}{v_i(s)} & 1 \end{pmatrix}\boldsymbol{M}^{-1}(s), \quad s \in \partial E_R.$$

It can be readily verified that $\boldsymbol{A}(z) = \boldsymbol{CZ}(z)\boldsymbol{M}(z)$ solves RHP-$\boldsymbol{A}$. To show that such $\boldsymbol{Z}(z)$ does indeed exist, observe that

$$\det \boldsymbol{M}(z) = \frac{\varphi(z)}{2w(z)} - \frac{1}{2\varphi(z)w(z)} \equiv 1$$

in the entire complex plane and that

$$v_i(z)S_*^2(z) = (-1)^{i-1}\varphi^{k_i}(z), \quad z \notin [-1,1],$$

straight by the definition of $S_i(z)$ in (1.7), where $k_1 = 0$, $k_2 = 2$, and $k_3 = k_4 = 1$. Thus,

$$\text{(3.5)} \quad \boldsymbol{M}(s)\begin{pmatrix} 1 & 0 \\ \frac{w(s)l_n(s)}{v_i(s)} & 1 \end{pmatrix}\boldsymbol{M}^{-1}(s) = \boldsymbol{I} + \frac{(-1)^{i-1}l_n(s)}{w(s)S^2(s)\varphi^{2n+k_i}(s)}\begin{pmatrix} \frac{1}{2}\varphi(s) & -1 \\ \frac{1}{4}\varphi^2(s) & -\frac{1}{2}\varphi(s) \end{pmatrix}$$

for $s \in \partial E_R$. It follows from the very definition of E_R that $|\varphi(s)| = R$ for $s \in \partial E_R$. Moreover, since $\deg(l_n) \le n$ and the uniform norms on $[-1,1]$ of these polynomials are bounded by C'_ρ, see Proposition 2.1, it holds that

$$|l_n(s)| \leq C'_\rho |\varphi(s)|^n = C'_\rho R^n, \quad s \in \partial E_R,$$

by the Bernstein–Walsh inequality. Hence, we can conclude that the jump of $\boldsymbol{Z}(z)$ on ∂E_R can be estimated as $\boldsymbol{I} + \boldsymbol{O}(R^{-n})$. It now follows from [7, Theorem 7.18 and Corollary 7.108] that such $\boldsymbol{Z}(z)$ does exist, is unique, and has continuous traces on ∂E_R whose L^2-norms with respect to the arclength measure are of size $O(R^{-n})$. This yields the desired pointwise estimate of $\boldsymbol{Z}(z)$ locally uniformly in $\overline{\mathbb{C}}\setminus\partial E_R$. Next, observe that the jump of $\boldsymbol{Z}(s)$ is analytic around ∂E_R and therefore we can vary the value of R. Since the solutions corresponding to different values of R are necessarily analytic continuations of each other, the desired uniform estimate follows from the locally uniform ones for any fixed $R_* < R$ and $R' > R$. ∎

3.5 An auxiliary estimate

Denote by dA the area measure and by $\mathcal{K}$ the Cauchy area operator acting on integrable functions on $\mathbb{C}$, i.e.,

$$\mathcal{K}f(z) = \frac{1}{\pi}\iint \frac{f(s)}{z-s}dA. \tag{3.6}$$

Lemma 3.4 *Let $u(z)$ be a bounded function supported on $\overline{E}_r$. Then*

$$\left\|\mathcal{K}\left(u|\varphi|^{-2n}\right)\right\| \leq C_r \frac{\log n}{n}\|u\|,$$

where $\|\cdot\|$ is the essential supremum norm and the constant C_r is independent of n.

Proof Observe that the integrand is a bounded compactly supported function and therefore its Cauchy area integral is Hölder continuous in $\mathbb{C}$ with any index $\alpha < 1$, see [3, Theorem 4.3.13]. Moreover, since the integral is analytic in $\overline{\mathbb{C}}\setminus\overline{E}_r$, the maximum of its modulus is achieved on $\overline{E}_r$. Notice also that it is enough to prove the claim of the lemma only for $u(z) = \chi_{E_r}(z)$, the indicator function of E_r.

Let $z \in \overline{E}_r$. Observe that $\varphi(s) = \tau$ when $s = \frac{1}{2}(\tau + 1/\tau)$. Write $z = \frac{1}{2}(\xi + 1/\xi)$. Then

$$\begin{aligned}\left|\mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}}\right)(z)\right| &\leq \frac{1}{\pi}\iint_{E_r}\frac{1}{|z-s|}\frac{dA}{|\varphi(s)|^{2n}} \\ &= \frac{1}{\pi}\iint_{1<|\tau|<r}\frac{|\tau^2-1|^2}{|(\xi-\tau)(1-1/(\tau\xi))|}\frac{dA}{|\tau|^{2n+4}}.\end{aligned}$$

Partial fraction decomposition now yields

$$\begin{aligned}\left|\mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}}\right)(z)\right| &\leq \frac{1}{\pi}\iint_{1<|\tau|<r}\left|\frac{\xi}{\tau-\xi}+\frac{\tau}{\tau-1/\xi}\right|\frac{|\tau^2-1|}{|\tau|^{2n+4}}dA \\ &\leq \frac{2r^3}{\pi}\iint_{1<|\tau|<r}\left(\frac{1}{|\tau-\xi|}+\frac{1}{|\tau-1/\xi|}\right)\frac{dA}{|\tau|^{2n+4}}.\end{aligned}$$

Write $\tau = \rho e^{i\theta}$ and $\xi = \rho_* e^{i\theta_*}$. Then

$$\begin{aligned}|\tau - \xi| &= \sqrt{(\rho-\rho_*)^2 + 4\rho\rho_* \sin^2\left(\frac{\theta-\theta_*}{2}\right)} \\ &\geq \frac{1}{\sqrt{2}}\left(|\rho-\rho_*| + \sqrt{\rho\rho_*}\left|2\sin\left(\frac{\theta-\theta_*}{2}\right)\right|\right) \\ &\geq C\big(|\rho-\rho_*| + |\theta-\theta_*|\big)\end{aligned}$$

for some constant $C < 1/\sqrt{2}$, where on the last step we used inequalities $\rho\rho_* \geq 1$ and $\min_{[-\pi/2,\pi/2]} |\sin x/x| > 0$. Since $\rho/\rho_* \geq 1/r$, the constant C can be adjusted so that

$$|\tau - 1/\xi| \geq C\big(|\rho - 1/\rho_*| + |\theta+\theta_*|\big) \geq C\big(|\rho-\rho_*| + |\theta+\theta_*|\big)$$

is true as well. By going to polar coordinates and applying the above estimates we get that

$$\begin{aligned}\left|\mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}}\right)(z)\right| &\leq \frac{4r^3}{\pi C}\int_1^r\left(\int_0^\pi \frac{d\theta}{|\rho-\rho_*|+\theta}\right)\frac{d\rho}{\rho^{2n+3}} \\ &= \frac{4r^3}{\pi C}\left(\int_{I_1} + \int_{I_2}\right)\log\left(1+\frac{\pi}{|\rho-\rho_*|}\right)\frac{d\rho}{\rho^{2n+3}} =: S_1 + S_2,\end{aligned}$$

where $I_1 = (1,r) \cap \{\rho : |\rho-\rho_*| < \pi/n\}$ and $I_2 = (1,r)\setminus I_1$. Then

$$\begin{aligned}S_1 &\leq \frac{8r^3}{\pi C}\int_0^{\pi/n}\log\left(1+\frac{\pi}{\rho}\right)d\rho = \frac{8r^3}{C}\int_{n+1}^\infty \frac{\log t\,dt}{(t-1)^2} \\ &= \frac{8r^3}{C}\left(\frac{\log(n+1)}{n} + \int_{n+1}^\infty \frac{dt}{t(t-1)}\right) \leq \frac{8r^3}{C}\frac{\log(n+1)+1}{n}.\end{aligned}$$

Finally, it holds that

$$S_2 \leq \frac{8r^3\log(n+1)}{\pi C}\int_1^\infty \frac{d\rho}{\rho^{2n+3}} = \frac{4r^3}{\pi C}\frac{\log(n+1)}{n+1},$$

which finishes the proof of the lemma. ∎

3.6 $\bar\partial$-Problem

Consider the following $\bar\partial$-problem ($\bar\partial$P-$\boldsymbol{D}$):

(a) $\boldsymbol{D}(z)$ is a continuous matrix function on $\overline{\mathbb{C}}$ and $\boldsymbol{D}(\infty) = \boldsymbol{I}$ and
(b) $\boldsymbol{D}(z)$ satisfies $\bar\partial \boldsymbol{D}(z) = \boldsymbol{D}(z)\boldsymbol{W}(z)$, where

$$\boldsymbol{W}(z) := \boldsymbol{Z}(z)\boldsymbol{M}(z)\begin{pmatrix} 0 & 0 \\ -w(z)\bar\partial L_{n,r}(z)/v_i(z) & 0\end{pmatrix}\boldsymbol{M}^{-1}(z)\boldsymbol{Z}^{-1}(z).$$

Notice that $\boldsymbol{W}(z)$ is supported by $\overline{E}_r$ and therefore $\boldsymbol{D}(z)$ is necessarily analytic in the complement of $\overline{E}_r$.

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Lemma 3.5 *The solution of* $\bar\partial$P-$\boldsymbol{D}$ *exists for all n large enough and it holds uniformly in* $\overline{\mathbb{C}}$ *that*

$$\boldsymbol{D}(z) = \boldsymbol{I} + \boldsymbol{O}(\varepsilon_n).$$

Proof As explained in [4, Lemma 8.1], solving $\bar\partial$P-$\boldsymbol{D}$ is equivalent to solving an integral equation

$$\boldsymbol{I} = (\mathcal{I} - \mathcal{K}_{\boldsymbol{W}})\boldsymbol{D}(z)$$

in the space of bounded matrix functions on $\mathbb{C}$, where $\mathcal{I}$ is the identity operator and $\mathcal{K}_{\boldsymbol{W}}$ is the Cauchy area operator (3.6) acting component-wise on the product $\boldsymbol{m}(s)\boldsymbol{W}(s)$ for a bounded matrix function $\boldsymbol{m}(z)$. If $\|\mathcal{K}_{\boldsymbol{W}}\|$, the operator norm of $\mathcal{K}_{\boldsymbol{W}}$, is less than $1-\varepsilon$, $\varepsilon \in (0,1)$, then $(\mathcal{I} - \mathcal{K}_{\boldsymbol{W}})^{-1}$ exists as a Neumann series and

$$\boldsymbol{D}(z) = (\mathcal{I} - \mathcal{K}_{\boldsymbol{W}})^{-1}\boldsymbol{I} = \boldsymbol{I} + \boldsymbol{O}_\varepsilon(\|\mathcal{K}_{\boldsymbol{W}}\|)$$

uniformly in $\overline{\mathbb{C}}$ (it also holds that $\boldsymbol{D}(z)$ is Hölder continuous in $\mathbb{C}$). It follows from Lemma 3.4 that to estimate $\|\mathcal{K}_{\boldsymbol{W}}\|$, we need to estimate L^∞-norms of the entries of $\boldsymbol{W}(z)$. To this end, similarly to (3.5), we get that

$$\boldsymbol{W}(z) = \frac{(-1)^i\bar\partial L_{n,r}(z)}{w(z)S^2(z)\varphi^{2n+k_i}(z)}\boldsymbol{Z}(z)\begin{pmatrix} \frac12\varphi(z) & -1 \\ \frac14\varphi^2(z) & -\frac12\varphi(z)\end{pmatrix}\boldsymbol{Z}^{-1}(z), \quad z \in \overline{E}_r.$$

Using Proposition 2.1 and Lemma 3.3 we can conclude that entries of $\boldsymbol{W}(z)$ are continuous functions on $\mathbb{C}$ supported by $\overline{E}_r$ with absolute values bounded above by $C_\rho|\varphi(z)|^{-2n}n\varepsilon_n/\log n$ for some constant C_ρ independent of n. Hence, $\|\mathcal{K}_{\boldsymbol{W}}\| = O(\varepsilon_n)$ as claimed. ∎

3.7 Asymptotic formulae

It readily follows from RHP-$\boldsymbol{A}$ and $\bar\partial$P-$\boldsymbol{D}$ as well as Lemmas 3.3 and 3.5 that $\bar\partial$RHP-$\boldsymbol{X}$ is solved by

$$\boldsymbol{X}(z) = \boldsymbol{C}\boldsymbol{D}(z)\boldsymbol{Z}(z)\boldsymbol{M}(z).$$

Given a closed set $B \subset \overline{\mathbb{C}}\setminus[-1,1]$, we can choose r and R so that $\overline{E}_R \cap B = \varnothing$. Then it holds that $\boldsymbol{Y}(z) = \boldsymbol{X}(z)$ for $z \in B$ by (3.3). Write

$$\boldsymbol{D}(z)\boldsymbol{Z}(z) = \boldsymbol{I} + \begin{pmatrix} \upsilon_{n1}(z) & \upsilon_{n2}(z) \\ \upsilon_{n3}(z) & \upsilon_{n4}(z)\end{pmatrix}.$$

It follows from Lemmas 3.3 and 3.5 that $|\upsilon_{nj}(z)| = O(\varepsilon_n)$ uniformly in $\overline{\mathbb{C}}$ and that $\upsilon_{nj}(\infty) = 0$. Then we get from (3.2) and (3.4) that

$$P_n(z) = \left(1 + \upsilon_{n1}(z) + \frac{\upsilon_{n2}(z)}{2\varphi(z)}\right)\frac{(S_*S)(z)}{(S_*S)(\infty)}\left(\frac{\varphi(z)}{2}\right)^n, \quad z \in B.$$

Since $S_*(z)/S_*(\infty) = S_i(z)/S_i(\infty)$, the first claim of the theorem follows. Next, notice that the first column of $\boldsymbol{Y}(z)$ is entire and is equal to the first column of

$$\boldsymbol{X}_+(x)\begin{pmatrix} 1 & 0 \\ w_+(x)/(\rho(x)v_i(x)) & 1 \end{pmatrix}$$

for $x \in [-1,1]$ by (3.3) and Proposition 2.1. Since the functions $\upsilon_{ni}(z)$ are continuous across $[-1,1]$ and $S_{*\pm}(x)/S_*(\infty) = S_{i\pm}(x)/S_i(\infty)$, we deuce from (1.3), (1.6), (1.8), and (3.4) that

$$P_n(x) = (1+\upsilon_{n1}(x))\frac{(S_iS\varphi^n)_+(x) + (S_iS\varphi^n)_-(x)}{2^n(S_iS)(\infty)} + \upsilon_{n2}(x)\frac{(S_iS\varphi^{n-1})_+(x) + (S_iS\varphi^{n-1})_-(x)}{2^{n+1}(S_iS)(\infty)}$$

for any $x \in [-1,1]$. It now follows from (1.4), (1.6), and (1.8) that

$$(S_iS\varphi^k)_+(x) + (S_iS\varphi^k)_-(x) = \frac{2\cos\left(k\arccos(x) + \theta(x) + \theta_i(x)\right)}{\sqrt{\rho(x)|v_i(x)|}}, \quad x \in [-1,1].$$

The last two formulae now yield the second claim of the theorem. Finally, it is known, see [15, Equations (9.6) and (9.7)], that

$$\begin{cases} a_{n,i}^2 & = \lim_{z\to\infty} z^2[\boldsymbol{Y}(z)]_{12}[\boldsymbol{Y}(z)]_{21}, \\ b_{n,i} & = \lim_{z\to\infty} \left(z - P_{n+1,i}(z)[\boldsymbol{Y}(z)]_{22}\right), \end{cases}$$

where $\boldsymbol{Y}(z)$ corresponds to the index n. As in the first part of the proof, we get that

$$[\boldsymbol{Y}(z)]_{12} = [\boldsymbol{X}(z)]_{12} = \frac{1}{w(z)}\frac{1+\upsilon_{n1}(z) + \upsilon_{n2}(z)\varphi(z)/2}{2^n(S_*S)(\infty)(S_*S)(z)\varphi^n(z)}$$

and

$$[\boldsymbol{Y}(z)]_{21} = [\boldsymbol{X}(z)]_{21} = \left(\upsilon_{n3}(z) + \frac{1+\upsilon_{n4}(z)}{2\varphi(z)}\right)2^n(S_*S)(\infty)(S_*S)(z)\varphi^n(z)$$

for all z large. Since $\upsilon_{nj}(\infty) = 0$, it holds that

$$a_{n,i}^2 = \frac{1}{4} + \lim_{z\to\infty} z\upsilon_{n3}(z)(1+z\upsilon_{n2}(z)) = \frac{1}{4} + O(\varepsilon_n)$$

by the maximum modulus principle for holomorphic functions. Similarly, we have that

$$[\boldsymbol{Y}(z)]_{22} = [\boldsymbol{X}(z)]_{22} = \left(\upsilon_{n3}(z) + \frac{1}{2}(1+\upsilon_{n4}(z))\varphi(z)\right)\frac{1}{w(z)}\frac{2^n(S_*S)(\infty)}{(S_*S)(z)\varphi^n(z)}$$

for all z large. Hence,

$$P_{n+1,i}(z)[\boldsymbol{Y}(z)]_{22} = \frac{\varphi^2(z)}{4w(z)}\left(1 + \upsilon_{n+11}(z) + \frac{\upsilon_{n+12}(z)}{2\varphi(z)}\right)\left(1 + \upsilon_{n4}(z) + 2\frac{\upsilon_{n3}(z)}{\varphi(z)}\right)$$

in this case. It can be readily verified that

$$\frac{\varphi^2(z)}{4w(z)} = z + \frac{z}{2w(z)(z + w(z))} - \frac{1}{4w(z)} = z + O\left(\frac{1}{z}\right),$$

as $z \to \infty$. Therefore,

$$b_{n,i} = -\lim_{z\to\infty} z\big(\upsilon_{n+11}(z) + \upsilon_{n4}(z)\big) = O(\varepsilon_n)$$

again, by the maximum modulus principle for holomorphic functions. This finishes the proof of the theorem.

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