THE QUOTIENT SEMIGROUP OF A SEMIGROUP THAT IS A SEMILATTICE OF GROUPS†

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1. Introduction. Let $Q(S)$ denote the maximal right quotient semigroup of the semigroup S as defined in [4]. In this paper, we initiate a study of $Q(S)$ when S is a semilattice of groups. A structure theorem for such semigroups is given by Theorem 4.11 of [2].

We prove that if S is a semilattice of groups, then so is $Q(S)$. In the process of showing this, we look at how right S-homomorphisms act on the groups making up S. In particular, a right S-homomorphism takes a group into a group with a lower index, and then maps this group one-to-one and onto itself.

If the set of idempotents of S forms a chain, then $Q(S)$ and S have exactly the same idempotents, and $Q(S)$ is just S union the group of units of $Q(S)$. If S is itself a chain, then $S = Q(S)$.

2. Preliminaries. Terminology throughout this note will be as found in [2] and [4].

DEFINITION 2.1. Let S be a subsemigroup of T. Then T is a *right quotient semigroup* of S if and only if, for any three elements $t_1, t_2, t \in T$ with $t_1 \neq t_2$, there exists an element $s \in S$ such that $t_1 s \neq t_2 s$ and $ts \in S$.

DEFINITION 2.2. If D is a right ideal of S, then D is said to be *dense* if and only if S is a right quotient semigroup of D. The set of all dense right ideals of S will be denoted by S^{Δ}.

Let us recall that $Q(S) = H_S/\equiv$, where $H_S = \bigcup\{\mathrm{Hom}_S(D, S): D \in S^{\Delta}\}$ and $\equiv$ is the congruence defined by $f_1 \equiv f_2$ if and only if f_1 agrees with f_2 on some dense right ideal contained in the intersection of their domains. We denote the domain of $f \in H_S$ by D_f, and the equivalence class containing f by $[f]$. Thus $[f] = [g]$ if and only if $f = g$ on some $D \in S^{\Delta}$ with $D \subseteq D_f \cap D_g$. S is considered as a subsemigroup of $Q(S)$ under the identification $x \to [x_l]$, where x_l is the left multiplication by x.

From now on, we shall let S be a semigroup with 0 and 1 that is a semilattice Y of groups $G_\alpha (\alpha \in Y)$, where Y is a semilattice order isomorphic to $E(S)$, the set of idempotents of S. Let e_α be the identity of the group G_α. The zero and identity of Y will also be denoted by 0 and 1. We recall that $S = \bigcup\{G_\alpha : \alpha \in Y\}$ with $G_\alpha \cap G_\beta = \emptyset$ if $\alpha \neq \beta$, and $G_\alpha G_\beta \subseteq G_{\alpha\beta}$.

By [2, exercise 2, p. 129], every one-sided ideal of S is two-sided. Thus $D \in S^{\Delta}$ if and only if, for any two elements $x_1, x_2 \in S$ with $x_1 \neq x_2$, there exists an element $d \in D$ such that $x_1 d \neq x_2 d$.

3. In this section we show that $Q(S)$ is also a semilattice of groups. We recall that a semigroup T is *regular* if and only if, for every element $x \in T$, there exists an element $y \in T$ such that $xyx = x$.

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PROPOSITION 3.1 ([**2**], pp. 128–129). *A semigroup T is regular with idempotents in the centre of T if and only if T is a semilattice of groups.*

We shall show that $Q(S)$ is regular and has central idempotents, but first we need the following lemmas.

LEMMA 3.2. *If D is an ideal of S, then D is a semilattice of groups.*

Proof. We assert that D is a semilattice X_D of groups G_β $(\beta \in X_D)$, where X_D is an ideal of Y. Let $d \in D$; then $d \in G_\beta$ for some $\beta \in Y$, and thus there exists an element $d^{-1} \in G_\beta$ such that $dd^{-1} = d^{-1}d = e_\beta$. Since D is an ideal, we have $e_\beta \in D$, and it follows that $G_\beta \subseteq D$. Set $X_D = \{\beta \in Y : e_\beta \in D\}$ and let $\beta \in X_D$, $\alpha \in Y$. Since $e_\beta \in D$, we have $e_\alpha e_\beta \in D$. Thus $e_\alpha e_\beta = e_{\alpha\beta}$ implies that $\alpha\beta \in X_D$. Hence X_D is an ideal of Y and D is a semilattice X_D of groups G_β $(\beta \in X_D)$.

We shall let $E(D) = \{e_\beta \in E(S) : e_\beta \in D\}$. Thus $E(D)$ is order isomorphic to X_D under the correspondence $\beta \to e_\beta$. If $f \in H_S$, let $E_f = \{e_\alpha \in E(S) : e_\alpha \in D_f\}$.

LEMMA 3.3. *Let $f \in H_S$. If J is an ideal of S such that $J \subseteq D_f$, then $f(J) \subseteq J$. In particular, $f(D_f) \subseteq D_f$.*

Proof. Let $x \in J$; then $x \in G_\alpha$ for some $\alpha \in X_J$, and $f(x) = f(xe_\alpha) = f(x)e_\alpha \in J$.

REMARK 3.4. If $e_\alpha \leqq e_\beta$ $(\alpha \leqq \beta)$, then $e_\beta x = x$ for all $x \in G_\alpha$ (for $e_\beta x = e_\beta(e_\alpha x) = (e_\beta e_\alpha)x = e_\alpha x = x$).

LEMMA 3.5. *Let $f \in H_S$; then for all $e_\beta \in E_f$, there exists a unique $e_\gamma \in E_f$, with $e_\gamma \leqq e_\beta$, such that $f(G_\beta) \subseteq G_\gamma$. Also, f restricted to G_γ is a one-to-one mapping of G_γ onto G_γ.*

Proof. Let $e_\beta \in E_f$, and consider the element $f(e_\beta)$. From 3.2 and 3.3, we have that $f(e_\beta) \in G_\gamma$ for some $e_\gamma \in E_f$. Now $f(e_\beta) = f(e_\beta e_\beta) = f(e_\beta)e_\beta \in G_\gamma G_\beta \subseteq G_{\gamma\beta}$. Hence $f(e_\beta) \in G_{\gamma\beta} \cap G_\gamma$ and thus $G_{\gamma\beta} = G_\gamma$, which implies that $\gamma\beta = \gamma$. Therefore $\gamma \leqq \beta$ $(e_\gamma \leqq e_\beta)$. Now let $x \in G_\beta$; then $f(x) = f(e_\beta x) = f(e_\beta)x \in G_\gamma G_\beta \subseteq G_{\gamma\beta} = G_\gamma$. Thus we have $f(G_\beta) \subseteq G_\gamma$. It is clear that e_γ is unique since S is the disjoint union of the groups G_α $(\alpha \in Y)$.

If $y \in G_\gamma$, then we have $e_\beta y = y$, by 3.4. Thus we have $f(y) = f(e_\beta y) = f(e_\beta)y \in G_\gamma G_\gamma \subseteq G_\gamma$. Hence $f(G_\gamma) \subseteq G_\gamma$. Finally it remains to show that f takes G_γ one-to-one and onto itself. Assume that $y, z \in G_\gamma$, with $f(y) = f(z)$; then $f(y) = f(e_\gamma y) = f(e_\gamma)y = f(e_\gamma)z = f(e_\gamma z) = f(z)$. Cancelling $f(e_\gamma)$, we have that $y = z$. Now let $w \in G_\gamma$; then there exists an element $u \in G_\gamma$ such that $f(e_\gamma)u = w$. But $f(e_\gamma)u = f(u)$, and this completes the proof.

REMARK 3.6. Suppose that $f \in H_S$ and $e_\beta \in E_f$. Let e_γ be as given in 3.5. Then ff is also a one-to-one mapping of G_γ onto G_γ.

THEOREM 3.7. *$Q(S)$ is a regular semigroup.*

Proof. Let $[f] \in Q(S)$. We shall define a mapping $g \in H_S$ such that $[f][g][f] = [f]$. Let $x \in D_f$, so that $x \in G_\beta$ for some $e_\beta \in E_f$. Let e_γ be as in 3.5. Then from 3.6, we see that there exists a unique $y \in G_\gamma$ such that $ff(y) = f(x)$. Define the mapping $g : D_f \to S$ by $g(x) = y$. We assert that g is a right S-homomorphism. Assume that $x \in D_f$ with $x \in G_\beta$, and $s \in S$ with $s \in G_\alpha$. Let $y \in G_\gamma$ be as chosen above. Set $z = g(xs)$. Since $f(xs) = f(x)s \in G_\gamma G_\alpha \subseteq G_{\gamma\alpha}$, it follows that $z \in G_{\gamma\alpha}$ with $ff(z) = f(xs)$. Now $ff(z) = f(xs) = f(x)s = (ff(y))s = ff(ys)$. Since $z, ys \in G_{\gamma\alpha}$ and ff is one-to-one on $G_{\gamma\alpha}$, we have $z = ys$; that is, $g(xs) = g(x)s$.

We show that $[f][g][f] = [f]$ by proving that fgf agrees with f on D_f. Again let $x \in D_f$ with $x \in G_\beta$, and G_γ be as above. Now $fgf(x) = f(u)$, where $u = g(f(x)) \in G_\gamma$, and $ff(u) = f(f(x))$. Since f is one-to-one on G and $f(u), f(x) \in G_\gamma$, we have $f(u) = f(x)$; that is, $fgf(x) = f(x)$.

We recall from 3.1 that every idempotent of S is in the center of S. This fact will be used throughout the proofs of the following lemmas.

LEMMA 3.8. *Let $f \in H_S$. If $ff = f$ on some ideal J with $J \subseteq D_f$, then $f(e) \in E(J)$ for all $e \in E(J)$.*

Proof. Let $e \in E(J)$; then $f(e) = ff(e) = ff(ee) = f(f(e)e) = f(ef(e)) = f(e)f(e)$.

LEMMA 3.9. *Let $f \in H_S$; then $ff = f$ on an ideal J with $J \subseteq D_f$ if and only if $f(xy) = f(x)f(y)$ for all $x, y \in J$.*

Proof. Assume that $ff = f$ on $J \subseteq D_f$, and let $x, y \in J$ with $x \in G_\alpha$ and $y \in G_\beta$. Applying 3.8, we have

$$\begin{aligned} f(xy) &= ff(xy) = ff(e_\alpha x e_\beta y) = ff(e_\alpha e_\beta xy) \\ &= (ff(e_\alpha e_\beta))xy = (f(f(e_\alpha)e_\beta))xy = (f(e_\alpha f(e_\beta)))xy \\ &= f(e_\alpha)f(e_\beta)xy = f(e_\alpha)xf(e_\beta)y = f(x)f(y). \end{aligned}$$

For the converse, let $z \in J$ with $z \in G_\gamma$. Then

$$f(z) = f(e_\gamma z) = f(e_\gamma)f(z) = f(e_\gamma f(z)) = f(f(z)e_\gamma) = f(f(z)) = ff(z).$$

PROPOSITION 3.10 (2.33 of [4]). *If T is a right quotient semigroup of S, then an element of T commutes with every element of S if and only if it is in the centre of T.*

PROPOSITION 3.11. *The idempotents of $Q(S)$ are in the center of $Q(S)$.*

Proof. We need only show that if $[f]$ is an idempotent of $Q(S)$, then $[f]x = x[f]$ for all $x \in S$. That is we must show that the mappings $x_l f$ and fx_l agree on some dense ideal of S. Assume that $ff = f$ on $D \in S^\Delta$, with $D \subseteq D_f$. Set $D^* = D \cap D_{fx_l}$ and let $d \in D^*$ with $d \in G_\alpha$. Applying 3.8 and 3.9, we have

$$\begin{aligned} (fx_l)(d) &= f(xd) = f(xe_\alpha d) = f(xe_\alpha)f(d) = f(e_\alpha)xf(d) \\ &= xf(e_\alpha)f(d) = xf(e_\alpha d) = xf(d) = (x_l f)(d). \end{aligned}$$

Hence $x_l f = fx_l$ on $D^* \in S^\Delta$.

THEOREM 3.12. *$Q(S)$ is a semilattice of groups.*

Proof. From 3.7 and 3.11, $Q(S)$ is a regular semigroup with central idempotents. Hence, by 3.1, $Q(S)$ is a semilattice of groups.

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From 3.1, a commutative semigroup is regular if and only if it is a semilattice of groups. The following example is a commutative example in which $Q(T)$ is regular but T is not. Hence the converse to 3.12 is not necessarily true.

EXAMPLE 3.13. Let T be the infinite cyclic semigroup generated by the element a, with 0 and 1 adjoined; that is, $T = \{a, a^2, a^3, \ldots\} \cup 0 \cup 1$. Thus T is a commutative semigroup that is not regular. Every ideal of T is of the form $\{a^k, a^{k+1}, \ldots\} \cup 0$ where $k \geqq 1$. It can be shown that every ideal of T is dense, and every $f \in H_T$ is one-to-one. Let f' be the inverse mapping of f. Hence f' is a right S-homomorphism from $f(D_f) \in T^\Delta$ into T such that $ff'f = f$ on D_f. Therefore $[f][f'][f] = [f]$, which implies that $Q(T)$ is a regular semigroup. $Q(T)$ is commutative, by 2.35 of [**4**].

4. Throughout this section, we shall assume that $E(S)$ is a chain.

PROPOSITION 4.1. *Let G_1 denote the group of units of S. If $D \in S^\Delta$, then $D = S$ or $D = S - G_1$, where $S - G_1 = \{x \in S : x \notin G_1\}$.*

Proof. Assume that $D \in S^\Delta$ with $D \nsubseteq S$. From 3.2, D is a semilattice X_D of groups, where X_D is isomorphic to $E(D)$. Thus we need only show that $E(D) = E(S) - \{1\}$. Let $e_\beta \in E(S) - E(D)$. It is easy to verify that $e_\alpha \leqq e_\beta$ for all $e_\alpha \in E(D)$. Hence, from 3.4, $1d = d = e_\beta d$ for all $d \in D$. Since $D \in S^\Delta$, this implies that $1 = e_\beta$.

LEMMA 4.2. *Let $[f] \in Q(S)$ and $D = S - G_1$. If $ff = f$ on D and $e_\alpha, e_\beta \in E(D) - f(D)$, then $f(e_\alpha) = f(e_\beta)$.*

Proof. By 4.1, we have $D \subseteq D_f$. Also $f(e_\alpha), f(e_\beta) \in E(D)$, from 3.8. Assume that $e_\alpha \leqq e_\beta$. We assert that $f(e_\beta) < e_\alpha$. If $e_\alpha \leqq f(e_\beta)$, then $e_\alpha = f(e_\beta)e_\alpha = f(e_\beta e_\alpha) = f(e_\alpha)$, which contradicts the fact that $e_\alpha \notin f(D)$. Hence $f(e_\beta) = f(e_\beta)e_\alpha = f(e_\beta e_\alpha) = f(e_\alpha)$.

THEOREM 4.3. *The idempotents of S and $Q(S)$ are identical.*

Proof. Let $E(Q)$ denote the set of idempotents of $Q(S)$. There are two cases: $S - G_1 \in S^\Delta$ or $S - G_1 \notin S^\Delta$.

Assume that $S - G_1 \notin S^\Delta$ and let $[f] \in Q(S)$. Then $f \in \mathrm{Hom}_S(S, S)$ and hence $[f] = [(f(1))_l] = f(1) \in S$. Therefore $S = Q(S)$.

Now let $D = S - G_1$ and suppose that $D \in S^\Delta$. If $[f] \in E(Q)$, then $ff = f$ on D. From 3.3, $f(D) \subseteq D$. Assume that $f(D) = D$. We claim that $f = 1_D$, where 1_D is the identity map on D. If $d \in D$, then there exists an element $x \in D$ such that $f(x) = d$. Thus $f(d) = ff(x) = f(x) = d$. In [**4**] it was shown that $[1_D] = 1$. Therefore $[f] = [1_D] = 1 \in E(S)$.

Let $f(D) \nsubseteq D$. Since $f(D)$ is an ideal of S, 3.2 implies that there exists an element $e_\alpha \in E(D) - f(D)$. Set $e = f(e_\alpha)$; then $e \in E(S)$. Let $d \in D$ with $d \in G_\beta$. If $e_\beta \leqq e_\alpha$, then $f(d) = f(e_\alpha d) = f(e_\alpha)d = ed$. If $e_\alpha < e_\beta$, then $e_\beta \in E(D) - f(D)$ and we have $f(d) = f(e_\beta d) = f(e_\beta)d = f(e_\alpha)d = ed$, by 4.2. Hence $[f] = [e_l] = e \in E(S)$.

Theorem 16 of [**1**] states that, if S is a semilattice ($G_\gamma = \{e_\gamma\}$ for all $\gamma \in Y$), then so is $Q(S)$. The following corollary then follows.

COROLLARY 4.4. *If S is a chain, then $S = Q(S)$.*

On page 45 of [3], it is shown that if R is a Boolean ring ($aa = a$ for all $a \in R$), then its Dedekind–MacNeille completion is isomorphic over R to the maximal right quotient ring of R. An analogous theorem is not true for semilattices: that is, if S is a non-complete chain, then $S = Q(S)$, which is properly contained in its completion.

If T is a semigroup, then $E(T)$ is *dually well-ordered* if every non-empty subset of $E(T)$ has a greatest element in the set.

THEOREM 4.5. *If T is a regular semigroup such that $E(T)$ is dually well-ordered, then $T = Q(T)$.*

Proof. We first show that every right ideal is generated by an idempotent. Let R be a right ideal of T. Since T is regular, we have $R \cap E(T) \neq \emptyset$. Let e be the greatest idempotent of T contained in R. Clearly $eT \subseteq R$. If $x \in R$, then there exists an element $x' \in T$ such that $xx'x = x$ and $xx' \in E(T)$. Now $xx' \in R \cap E(T)$, so that $xx' \leqq e$. Thus $x = (xx')x = e(xx')x \in eT$. Hence $eT = R$.

Now let $f \in H_T$; then $D_f = iT$, where $i \in E(T)$. We have $f(iy) = f(iiy) = f(i)iy$ for all $iy \in iT$. By 2.31 of [4], $T = Q(T)$.

We shall now write $Q(S)$ as the semilattice I of groups $H_\alpha (\alpha \in I)$, where I is isomorphic to $E(Q)$. Note that we may assume that $Y \subseteq I$ and $G_\alpha \subseteq H_\alpha$ for all $\alpha \in Y$.

LEMMA 4.6. *If $\alpha \in Y$ with $\alpha \neq 1$, then $G_\alpha = H_\alpha$.*

Proof. Let $[f] \in H_\alpha$, where $\alpha \in Y$ with $\alpha \neq 1$. Thus $e_\alpha \neq 1$. Set $e = e_\alpha$; then $[f]e = [f]$, which implies that $fe_l = f$ on some $D \in S^\Delta$, with $D \subseteq D_f$. Since $D = S$ or $D = S - G_1$, we have $e \in D$. Hence $(fe_l)(d) = f(ed) = f(e)d$. Therefore $[f] = [f]e = [(f(e))_l] \in S$, and thus $[f] \in G_\alpha$.

THEOREM 4.7. $Q(S) = (\bigcup_{\alpha \neq 1} G_\alpha) \cup H_1$.

Proof. By 4.3, $Y = I$ and hence the result follows from 4.6.

5. For the remainder of this paper, let T be a semigroup with 0 and 1. A right ideal R of T is said to be *minimal* if $R \neq 0$ and if K is a right ideal of T with $0 \neq K \subseteq R$, then $K = R$. T is said to satisfy the *minimum condition* on right ideals if every non-empty set of right ideals of T has a minimal member.

PROPOSITION 5.1. *If T has a minimal dense right ideal, then it is unique.*

Proof. This follows from the fact that the intersection of two dense ideals is a dense ideal.

Assume that T has a minimal dense right ideal D, and let $f, g \in \mathrm{Hom}_T(D, T)$. Let fg be the composition map with domain $g^{-1}D = \{x \in D: g(x) \in D\}$. By 2.14 of [4], $g^{-1}D \in T^\Delta$, which implies that $g^{-1}D = D$ since D is minimal. Thus $\mathrm{Hom}_T(D, T)$ is a semigroup under this operation.

THEOREM 5.2. *If T has a minimal dense right ideal D, then $Q(T)$ is isomorphic to $\mathrm{Hom}_T(D, T)$.*

Proof. Define the mapping $\mu: Q(T) \to \mathrm{Hom}_T(D, T)$ by $\mu([f]) = f|_D$, where $f|_D$ is the restriction of f to D. μ is an isomorphism.

COROLLARY 5.3. *Let T satisfy the minimum condition on right ideals, and let D be the unique minimal dense right ideal of T. Then $Q(T)$ is isomorphic to* $\mathrm{Hom}_T(D, T)$.

COROLLARY 5.4. *Assume that S is a semilattice of groups and $E(S)$ is a finite set. Let D^* be the intersection of all the dense ideals of S. Then $Q(S)$ is isomorphic to* $\mathrm{Hom}_S(D^*, S)$.

REFERENCES

1. P. Berthiaume, The injective envelope of S-sets, *Canad. Math. Bull.* **10** (1967), 261–273.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1, Math. Surveys of the Amer. Math. Soc., 7 (Providence, R.I., 1961).

3. J. Lambek, *Lectures on rings and modules* (Blaisdell, 1966).

4. F. R. McMorris, On quotient semigroups; submitted.

5. Y. Utumi, On quotient rings, *Osaka Math. J.* **8** (1956), 1–18.

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