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SMOOTH NORMS IN ORLICZ SPACES

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ABSTRACT. Equivalent norms with best order of Frechet and uniformly Frechet differentiability in Orlicz spaces are constructed. Classes of Orlicz which admit infinitely many times Frechet differentiable equivalent norm are found.

1. **Introduction.** There are few examples of spaces for which the best order of Frechet differentiability of the usual or of some equivalent norm is known. Results in this field were obtained for the spaces ℓ_p, L_p, c_0 in [BF], [S]. Our aim is to investigate this problem for general Orlicz spaces as well as for the most common Orlicz spaces: ℓ_M, L_M when M satisfies the Δ_2-condition at 0, at ∞ respectively and the subspace of ℓ_M generated from the unit vector basis and the respective function space when M does not satisfy this condition.

The results contained in this paper were exposed in talks given by the second named author at the International Conference on Geometry of Banach Spaces and Related Topics, Mons 1987, Belgium and at the CMS Annual Seminar on Banach Spaces and Geometry of Convex Bodies, Banff 1988, Canada.

2. **Preliminaries.** We recall at first some definitions and results related to differentiability of functions on Banach spaces, to Orlicz spaces and give some preliminary results.

In the next X, Y are Banach spaces, $\mathbb{N}$ the naturals, $\mathbb{R}$ the reals. Everywhere differentiability is understood as Frechet differentiability.

We denote $B^j(X, Y)$ the space of all continuous symmetric j-linear forms $T: X \longrightarrow Y$ with the norm

$$\|T\| = \sup\{\, \|T(x_1, x_2, \ldots, x_j)\|;\ \|x_i\| = 1, i = 1, 2, \ldots, j\}.$$

It is well known that (see e.g. [SS], p. 10) this norm is equivalent to the norm

$$\|T\|_1 = \sup_{\|x\|=1} \|T(x, x, \ldots, x)\| = \sup_{\|x\|=1} \|T(x^{(j)})\|.$$

DEFINITION. The function $f: X \longrightarrow Y$ is said to be *k-times differentiable* at $x \in X$ if there exist $T_x^j \in B^j(X, Y)$, $1 \leq j \leq k$, such that

$$f(x + ty) = f(x) + \sum_{j=1}^{k} t^j T_x^j(y^{(j)}) + \sigma_x(|t|^k) \tag{1}$$

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uniformly on y from the unit sphere $S(X)$ of X. For an open $V \subset X, f \in F^k(V, Y)$ means f is k-times differentiable at every point of V. If (1) is fulfilled uniformly on x over a set $W \subset V$ we shall say that f is k-times uniformly differentiable over W and shall write $f \in UF^k(W, Y)$. $j!T_x^j$ is called j-th derivative of f at x and is denoted $D^j f(x; \cdot)$ or $f_x^{(j)}$. If $Y = \mathbb{R}$ the corresponding classes are denoted $F^k(V)^x, UF^k(W)$.

DEFINITION. We shall say that X is F^k-*smooth* if the norm in X is a function from $F^k(X \setminus \{0\})$ and UF^k-*smooth* if this norm belongs to $UF^k(S(X))$. The norm is said to be F^∞-*smooth* (UF^∞-*smooth*) if it is F^k-smooth (UF^k-smooth) for any $k \in \mathbb{N}$.

We recall that an even convex continuous function M, nondecreasing in $[0, \infty)$ is called an *Orlicz function* if $M(0) = 0$, $M(\infty) = \infty$. Let (S, Σ, μ) be a measure space. Consider the space of all classes equivalent μ-measurable functions x on S with $\tilde{M}(x/\rho) < \infty$ for some $\rho > 0$, where

$$\tilde{M}(x) = \int_S M(x(s))\, d\mu(s).$$

This space can be normed in different modes (see e.g. [KR], p. 37). We shall use the so called Luxemburg's norm, introduced by the formula

$$\|x\| = \inf\{\lambda > 0;\ \tilde{M}(x/\lambda) \leq 1\}.$$

The space we obtain is a Banach space denoted $L_M(S, \Sigma, \mu)$ and called the *Orlicz space generated by* M. The subspace of $L_M(S, \Sigma, \mu)$ consisting of all $x \in L_M(S, \Sigma, \mu)$ for which $\tilde{M}(x/\rho) < \infty$ for every $\rho > 0$, is denoted $H_M(S, \Sigma, \mu)$.

Usually three types of Orlicz spaces are considered with respect to the measure space (S, Σ, μ):

(A) $\mu(S) = \infty$, μ is purely atomic, $0 < \inf_\alpha \mu(\sigma_\alpha) \leq \sup_\alpha \mu(\sigma_\alpha) \leq \infty$, for the atoms $\{\sigma_\alpha\}$;

(B) $\mu(S) < \infty$, S is free of atoms;

(C) $\mu(S) = \infty$, S contains a set of positive measure free of atoms.

The most common examples of such Orlicz spaces are the sequence spaces h_M, ℓ_M for type (A) and the Orlicz function spaces $H_M(0,1), L_M(0,1)$ and $H_M(0,\infty), L_M(0,\infty)$ for types (B) and (C) respectively. Essential for $\ell_M, L_M(0,1)$ and $L_M(0,\infty)$ is the behaviour of M near $0, \infty$ and 0 and ∞ respectively. More precisely if two Orlicz functions M and N are equivalent ($M \sim N$) at 0 (∞, 0 and ∞), i.e.,

$$c^{-1}M(c^{-1}t) \leq N(t) \leq cM(ct),\ t \in [0,1]\ (t \in [1,\infty),\ t \in [0,\infty))$$

for some positive constants c, then ℓ_N ($L_N(0,1), L_N(0,\infty)$) is isomorphic to ℓ_M ($L_M(0,1)$, $L_M(0,\infty)$). This allows to introduce equivalent norms in ℓ_M, $L_M(0,1)$ and $L_M(0,\infty)$ through Orlicz functions, equivalent to M at 0, at ∞ or at 0 and ∞ respectively.

To every Orlicz function M the following numbers are associated (see [LT1, p. 143] and [LT2, p. 382]):

$$\alpha_M^0 = \sup\bigl\{p; \sup\{u^{-p}M(u,v)/M(v); u, v \in (0,1]\} < \infty\bigr\};$$
$$\alpha_M^\infty = \sup\bigl\{p; \sup\{u^{p}M(v)/M(u,v); u, v \in [1,\infty)\} < \infty\bigr\};$$
$$\alpha_M = \min(\alpha_M^0, \alpha_M^\infty).$$

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These numbers play a special role in the theory of isomorphic embedding of ℓ_p spaces into Orlicz spaces. A detailed study of this subject is contained in [LT1] and [LT2]. Here we mention only that always $\alpha_M \geq 1$.

Finally we recall that the Orlicz function M satisfies the Δ_2-condition at 0 (and ∞) if there is positive K such that

$$M(2t) \leq KM(t), \ t \in [0,1]\,(t \in [1,\infty)).$$

In this case $h_M = \ell_M\big(H_M(0,1) = L_M(0,1)\big)$. If M satisfies the Δ_2-condition at 0 and $\infty, H_M(0,\infty) = L_M(0,\infty)$.

For a fixed $k \in \mathbb{N}$ we shall investigate the class of all Orlicz functions M with the properties:

(i) $\alpha_M > k$;

(ii) $M^{(k)}$ is absolutely continuous in every finite interval;

(iii) $t^{k+1}|M^{(k+1)}(t)| \leq cM(ct)$ a.e. in $[0,\infty)$ for some $c > 0$.

This class is denoted AC^k while $AC^\infty = \bigcap_{k=1}^\infty AC^k$.

REMARK 1. We can assume without loss of generality that if $M \in AC^k$ then for arbitrary fixed $a \in (k, \alpha_M)$ the function M satisfies also the inequalities:

(iv) $M(\lambda t) \leq c\lambda^b M(t), \lambda \in [0,1], t \in (-\infty,+\infty)$ for every $0 < b \leq a$ and

(v) $t^i|M^{(i)}(t)| \leq cM(ct), t \in [0,\infty), i = 1,2,\ldots,k$ for some $c > 0$ that may depend on a.

Indeed, it is obvious that (i) implies for any fixed $a \in (k, \alpha_M)$

$$M(\lambda t) \leq c_a \lambda^a M(t), \ \lambda \in [0,1], t \in [0,\infty) \tag{2}$$

and therefore also (iv).

Using this inequality and the Taylor's formula at 0 one can prove inductively that

$$M^{(i)}(0) = 0, \ i = 1,2,\ldots,k.$$

Now from (iii) and (2) we get for $t \in [0,\infty)$

$$\begin{aligned} t^k|M^{(k)}(t)| &\leq t^k \int_0^t |M^{(k+1)}(u)|\,du \leq ct^k \int_0^t M(cu)u^{-k-1}\,du \\ &\leq ct^{k-a} \int_0^t (t/u)^a u^{a-k-1} M(cu)\,du \\ &\leq cc_a t^{k-a} M(ct) \int_0^t u^{a-k-1}\,du = cc_a M(ct)/(a-k), \end{aligned}$$

i.e. (v). Analogous inequalities hold also for the preceding derivatives.

We need for the sequel the following estimate which expresses the k-times differentiability of $M \in AC^k$.

LEMMA 2. *Let $M \in AC^k$. Then for any $u, v, t \in \mathbb{R}$*

$$|M(u+tv) - \sum_{j=0}^{k} \frac{1}{j!}(tv)^j M^{(j)}(u)| \leq (M(2cu) + M(v))\phi(t), \tag{3}$$

where ϕ depends only on a, k, c, i.e., on M and $\phi(t) = \sigma(|t|^k)$.

PROOF. Let $0 < \varepsilon < 1/2c$. We consider first the case $|u| < \varepsilon|v|$. The Taylor's formula, (iv) and (v) give for $|t| < \varepsilon$:

$$\begin{aligned}
\Big|M(u+tv) - \sum_{j=0}^{k} \frac{(tv)^j}{j!} M^{(j)}(u)\Big| &= \frac{|tv|^k}{k!}\Big|M^{(k)}(u+\theta tv) - M^{(k)}(u)\Big| \\
&\leq c\frac{|tv|^k}{k!}\Big(M\Big(2c\varepsilon v\frac{u+\theta tv}{2\varepsilon v}\Big)|u+\theta tv|^{-k} + M\Big(2c\varepsilon v\frac{u}{2\varepsilon v}\Big)|u|^{-k}\Big) \\
&\leq \frac{2c^2|t|^k}{k!(2\varepsilon)^k} M(2c\varepsilon v) \leq \frac{2^{a+1-k}}{k!} c^{a+3}\varepsilon^{a-k}|t|^k M(v).
\end{aligned}$$

Let now $|u| \geq \varepsilon|v|$. Using the integral form of the remainder in the Taylor's formula and (iii) we obtain for $|t| < (\varepsilon/2)^{k+2}$:

$$\begin{aligned}
\Big|M(u+tv) - \sum_{j=0}^{k} \frac{(tv)^j}{j!} M^{(j)}(u)\Big| &\leq \frac{1}{k!}\int_0^{|tv|} (|tv| - s)^k \Big|M^{k+1}(u+s)\Big|\, ds \\
&\leq \frac{c}{k!}\int_0^{|tv|} \frac{(|tv|-s)^k}{|u+s|^{k+1}} M(c(u+s))\, ds \\
&\leq \frac{c}{2(k!)}\Big(M(2cu) + M(2ctv)\Big) \cdot \int_0^{|tv|} \frac{(|tv|-s)^k}{|u+s|^{k+1}}\, ds \\
&\leq \frac{c}{2(k!)}\Big(M(2cu) + M(v)\Big)\left(\frac{2}{\varepsilon}\right)^{k+1} \int_0^{|tv|} \frac{(|tv|-s)^k}{|v|^{k+1}}\, ds \\
&= \frac{c}{4((k+1)!)}\Big(M(2cu) + M(v)\Big)\varepsilon|t|^k.
\end{aligned}$$

Thus Lemma 2 is proved.

To every function $M \in AC^k$ we associate in $X = H_M(S, \Sigma, \mu)$ the symmetric j-linear forms, $j = 1, 2, \ldots, k$:

$$\tilde{M}_j(x, y_1, y_2, \ldots, y_j) = \int_S M^{(j)}(x(s))y_1(s)y_2(s)\ldots y_j(s)\, d\mu(s)$$

LEMMA 3. *Let $M \in AC^k$. Then $\tilde{M}_j(x) \in B^j(X)$ for every $x \in X, j = 1, 2, \ldots, k$.*

PROOF. It is sufficient to show that

$$\sup\Big\{\Big|\tilde{M}_j(x; y^{(j)})\Big|; \|y\| \leq 1/c\Big\} < \infty.$$

Denote $S' = \{s \in S; 0 < |x(s)| \leq |y(s)|\}, S'' = S \setminus S'$.

Now using (iv) and (v) we obtain

$$\begin{aligned}
\left|\tilde{M}_j(x;y^{(j)})\right| &\le \int_S \left|M^{(j)}\big(x(s)\big)\right| |y(s)|^j \, d\mu(s) \\
&\le c\int_{S'} M\big(cx(s)\big) |y(s)/x(s)|^j \, d\mu(s) + c\int_{S''} M\big(cx(s)\big) d\mu(s) \\
&\le c^2 \int_{S'} M\big(cy(s)\big) d\mu(s) + c\int_{S''} M\big(cx(s)\big) d\mu(s) \\
&\le c^2\tilde{M}(cy) + \tilde{M}(cx) \le c\big(\tilde{M}(cx) + c\big).
\end{aligned}$$

The next lemma gives information about the differentiability of $\tilde{M}$ and $\tilde{M}_j$.

LEMMA 4. *Let $M \in AC^k$ and $B_r(X) = \{x \in X; \|x\| \le r\}$. Then $\tilde{M} \in UF^k\big(B_r(X)\big)$, $\tilde{M}_j \in UF^{k-j}\big(B_r(X), B_j(X)\big)$ for every $r \in (0, 1/2c)$ and*

$$D^j\tilde{M} = \tilde{M}_j,\ j = 1,2,\ldots,k, \quad D^i\tilde{M}_j = \tilde{M}_{i+j},\ i+j \le k. \tag{4}$$

PROOF. Let $y \in S_r(X)$, $0 < r \le 1/2c$. According to (3), Lemma 2 we obtain the estimate

$$\begin{aligned}
\left|\tilde{M}(x+ty) - \sum_{j=0}^{k} \frac{t^j}{j!}\tilde{M}_j\big(x;y^{(j)}\big)\right| &\le \big(\tilde{M}(2cx) + \tilde{M}(y)\big)\phi(t) \\
&\le 2\phi(t) = \sigma(|t|^k),
\end{aligned}$$

which ensures the k-times uniformly differentiability of $\tilde{M}$ over $S_r(X), 0 < r \le 1/2c$ and shows that $D^j\tilde{M} = \tilde{M}_j$.

The last equality implies $\tilde{M}_j \in UF^{k-j}\big(S_r(X), B^j(X)\big)$ and $D^i\tilde{M}^j = \tilde{M}_{i+j}$. Lemma 4 is proved.

REMARK 5. Obviously $\tilde{M} \in F^k(X\setminus\{0\}), \tilde{M}_j \in F^{k-j}\big(X\setminus\{0\}, B^j(X)\big)$.

3. **Main result.** We are ready to prove the following :

THEOREM 6. *Let $M \in AC^k$. If (S,Σ,μ) is a measure space then $H_M(S,\Sigma,\mu)$ is F^k-smooth. If, in addition, M satisfies the Δ_2-condition at 0 and ∞ then $L_M(S,\Sigma,\mu)$ is UF^k-smooth.*

PROOF. Let $n(x) = \|x\|, x \in X = H_M(S,\Sigma,\mu)$. From the differentiability of $\tilde{M}$ and the implicit function theorem applied to the equation

$$\tilde{M}\big(x/n(x)\big) - 1 = 0 \tag{5}$$

it follows that the norm in X is differentiable. After a differentiation in (5) we obtain for $x \ne 0$:

$$\tilde{M}'\Big(x/n(x); y/n(x) - \big(x/n^2(x)\big)n'_x(y)\Big) = 0$$

and from (4):

$$(6)\qquad n'_x = \frac{D\tilde{M}\big(x/n(x)\big)}{< D\tilde{M}\big(x/n(x)\big), x/n(x) >} = \frac{\tilde{M}_1\big(x/n(x)\big)}{\tilde{M}_1\big(x/n(x); x/n(x)\big)}.$$

Obviously $n \in F^j(X\setminus\{0\})$ for some j, $1 \leq j \leq k$, Remark 5 and (6) imply $n'_x \in F^j\big(X\setminus\{0\}, B(X)\big)$, i.e., $n \in F^{j+1}(X\setminus\{0\})$. In this way we obtain inductively that X is F^k-smooth.

If, in addition, M satisfies the Δ_2-condition at 0 and ∞ it can be shown as in [MT1] that $X = L_M(S,\Sigma,\mu)$ is uniformly smooth, i.e., $n \in UF\big(S(X)\big)$. But in this case $\tilde{M} \in UF^k\big(S(X)\big), \tilde{M}_j \in UF^{k-j}\big(S(X), B^j(X)\big)$ and the same reasoning as above given now $n \in UF^k\big(S(X)\big)$. Thus Theorem 6 is proved.

REMARK 7. If $M \in AC^\infty$ then $H_M(S,\Sigma,\mu)$ is F^∞-smooth.

From Theorem 6 directly follows for $M(t) = t^p$ the well-known result of Bonic and Frampton and Sundaresan:

COROLLARY 8 ([BF], [S]). The spaces $L_p(S,\Sigma,\mu), p > 1$, are $UF^{E(p)}$-smooth, where

$$E(p) = \begin{cases} p-1 & \text{if } p \text{ is integer} \\ [p] & \text{otherwise} \end{cases}$$

4. **Smooth renormings in Orlicz spaces.** It is clear that the usual norm in $X = H_M(S,\Sigma,\mu)$ is not differentiable over $X\setminus\{0\}$ even for measure spaces (S,Σ,μ) of type (A), (B) or (C) if the function M is arbitrary. So the problem of equivalent renorming arises naturally. The simplest way to treat this problem is to consider equivalent norms generated from suitably chosen Orlicz functions equivalent to M at 0 and ∞ (at 0 or at ∞). A first result in this direction is due to Akimovich [A] and can be formulated as follows: if M satisfies the Δ_2-condition at 0 (at ∞, at 0 and ∞) and $\alpha_M^0 > 1$ ($\alpha_M^\infty > 1, \alpha_M > 1$) then $L_M(S,\Sigma,\mu)$, (S,Σ,μ) of type (A), ((B) or (C)) admit an equivalent UF-smooth norm.

The crucial step to the construction of suitable Orlicz function generating equivalent smooth norm is the following

LEMMA 9. *Let M_1, M_2 be Orlicz functions. There exists an Orlicz function N which is infinitely many times differentiable and satisfies*

(i) $N \sim M_1$ at 0 and $N \sim M_2$ at ∞;

(ii) $t^k|N^{(k)}(t)| \leq c_k N(c_k t)$, $t \in [0,\infty)$, $c_k > 0$, $k = 1,2,\ldots$.

PROOF. Without loss of generality we may assume that $M_1(1) = M_2(1) = 1$. Consider the function

$$N_1(t) = \int_0^t N_0(u)\,du \text{ where } N_0(u) = \begin{cases} M_1(u)/u, & u \in (0,1] \\ M_2(u)/u, & u \in (1,\infty) \end{cases}$$

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N_1 is an Orlicz function equivalent to M_1 at 0 and to M_2 at ∞. This follows immediately if we observe that from the convexity of M_1 and M_2 it follows that $M_1(t)/t$ and $M_2(t)/t$ and therefore also N_0 are nondecreasing. Indeed, the inequalities

$$M_1(t/2) \le \int_{t/2}^{t} \big(M_1(u)/u\big)\, du \le \int_0^t M_1(u)\, du/u \le M_1(t),\ t \in [0,1]$$

show that $N_1 \sim M_1$ at 0. $N_1 \sim M_2$ at ∞ can be proved in the same manner.

Put now

$$N(t) = \int_0^t \frac{N_1(u)}{u} \exp\frac{u}{u-t}\, du = \int_0^1 N_1(vt) \exp\frac{v}{v-1}\, dv/v.$$

(i) is verified as above. Using the first representation it is easy to prove inductively

$$\begin{aligned} N^{(k)}(t) &= \int_0^t \frac{N_1(u)}{u} \frac{d^k}{dt^k} \exp\frac{u}{u-t}\, du \\ &= \int_0^t N_1(u) \sum_{j=0}^{k-1} c_j(k)(u-t)^{j-2k} u^{k-j-1} \exp\frac{u}{u-t}\, du, \end{aligned}$$

where $c_j(k) \in \mathbb{N}, j = 0, 1, \ldots, k-1$.

The substitution $u = vt$ in the last integral led us to the following estimate for $t \in [0, \infty)$:

$$t^k |N^{(k)}(t)| \le c(k) N_1(t),$$

where $c(k) = \int_0^1 \sum_{j=0}^{k-1} c_j(k)(1-v)^{j-2k} v^{k-j-1} \exp\frac{v}{v-1} dv > 0$.

Now to obtain (ii) it is sufficent to use $N_1 \sim N$ at 0 and ∞. Finally from the second representation of N it follows easily that N' is nondecreasing and therefore N is an Orlicz function. Lemma 9 is proved.

COROLLARY 10. *Let $1 \le k = E(\alpha_M^0)\big(E(\alpha_M^\infty), E(\alpha_M)\big)$. Then in h_M ($H_M(0,1)$, $H_M(0,\infty)$) there exists equivalent F^k-smooth norm. If M satisfies the Δ_2-condition at 0 (at ∞, at 0 and ∞) then in ℓ_M $\big(L_M(0,1), L_M(0,\infty)\big)$ there exists equivalent UF^k-smooth norm. Especially if $\alpha_M^0 = \infty$ ($\alpha_M^\infty = \infty, \alpha_M = \infty$) then in h_M $\big(H_M(0,1), H_M(0,\infty)\big)$ there exists equivalent F^∞-norm.*

PROOF. According to Theorem 6 it is sufficient to construct for the Orlicz function M we consider, Orlicz function $N \in AC^k$ equivalent to M at 0 (at ∞, at 0 and ∞) with $\alpha_N = \alpha_M^0$($\alpha_N = \alpha_M^\infty, \alpha_N = \alpha_M$). This is always possible using Lemma 9 for the functions $M_1 = M, M_2(t) = t^{\alpha_M^0}\big(M_1(t) = t^{\alpha_M^\infty}, M_2 = M$ or $M_1 = M_2 = M\big)$.

To treat the case $\alpha_M^0 = \infty$ ($\alpha_M^\infty = \infty, \alpha_M = \infty$) we have to use Lemma 9 for the functions $M_1 = M, M_2(t) = e^t - 1$ $\big(M_1(t) = te^{-1/t}, M_2 = M$ or $M_1 = M_2 = M\big)$ to find Orlicz function $N \in AC^\infty$, equivalent to M at 0 (at ∞, at 0 and ∞) with $\alpha_N = \infty$.

REMARK 11. Obviously analogous results hold true for Orlicz spaces $H_M(S, \Sigma, \mu)$, $L_M(S, \Sigma, \mu)$ over measure space (S, Σ, μ) of type (A), (B) or (C) and for general Orlicz spaces as well.

COROLLARY 12. *If* $1 \leq k < \alpha_M$ *then* $L_M(S, \Sigma, \mu)$ *admits* F^k*-smooth partition of unity.*

The proof follows immediately from Corollary 10 and [GTWZ].

REMARK 13. It is well-known (see e.g. [SS], p. 20) that $L_{2k} \in UF^\infty\big(S(L_{2k})\big), k \in \mathbb{N}$. It turns out that for Orlicz functions M which are not equivalent at 0 to $t^{2k}, k \in \mathbb{N}$ the equivalent norms for h_M and ℓ_M found in Corollary 10 are the best possible with respect to the order of smoothness. Moreover, the following stronger result is proved in [MT2]:

THEOREM 14. *Let* $k - 1 = E(\alpha_M^0)$. *For k odd or k even but* $M \not\sim t^k$ *at* 0 *in* h_M *there is no k-smooth real valued function with bounded support.*

Let us give some examples.

EXAMPLE 1. Let M be an Orlicz function such that near 0 and ∞ $M(t) = t^p|\log t|^q$, $p > 1$. Let M be $[p]$-times differentiable in $(0, \infty)$. Obviously M satisfies the Δ_2-condition at 0 and ∞ and $\alpha_M = p$. Therefore, according to Theorem 6 $L_M(0, 1) \in UF^{E(p)}$ and $E(p)$ is the best order of smoothness under equivalent renorming for any q, except the case p even, $q = 0$.

EXAMPLE 2. Let $M(t) = t\exp(-1/t)$. Now $M \in AC^\infty$ and therefore h_M is F^∞-smooth.

REMARK 15. Important results connecting the existence of C^∞-bump functions with the presence of subspaces isomorphic to $c_\circ$ or to ℓ_p for some pair p have been recently shown by R. Deville (see [D1] and [D2]). Related to this, Example 2 represents a space with symmetric basis, essentially different from $c_\circ$ and ℓ_p, which possesses C^∞-bump function.

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