On Positive Definiteness Over Locally Compact Quantum Groups

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Abstract. The notion of positive-definite functions over locally compact quantum groups was recently introduced and studied by Daws and Salmi. Based on this work, we generalize various well-known results about positive-definite functions over groups to the quantum framework. Among these are theorems on "square roots" of positive-definite functions, comparison of various topologies, positive-definite measures and characterizations of amenability, and the separation property with respect to compact quantum subgroups.

Introduction

Positive-definite functions over locally compact groups, introduced by Godement in [17], play a central role in abstract harmonic analysis. If G is a locally compact group, a continuous function $f: G \to \mathbb{C}$ is called *positive definite* if for every $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in G$, the matrix $(f(s_i^{-1}s_j))_{1 \le i, j \le n}$ is positive (we always take continuity as part of the definition). Positive-definite functions are tightly connected with various aspects of the group, such as representations, group properties (amenability and other approximation properties, property (T), etc.), the Banach algebras associated to the group and many more, as exemplified by the numerous papers dedicated to them. It is thus natural to extend this theory to a framework more general than locally compact groups. This was done in the context of Kac algebras by Enock and Schwartz [13, Section 1.3]. Recently, Daws [6] and Daws and Salmi [8] generalized this work to the much wider context of locally compact quantum groups in the sense of Kustermans and Vaes [30, 31]. They introduced several notions of positive definiteness, corresponding to the classical ones, and established the precise relations between them.

These foundations having been laid, the next step should be generalizing well-known useful results from abstract harmonic analysis about positive-definite functions to locally compact quantum groups. This is the purpose of the present paper, which is organized as follows.

In Section 2 we generalize a result of Godement, essentially saying that a positive-definite function has a "square root" if and only if it is square integrable.

A theorem of Raïkov [40] and Yoshizawa [57] says that on the set of positive-definite functions of norm 1, the w^*-topology induced by L^1 coincides with the topology of uniform convergence on compact subsets. This result was improved by several

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authors, and eventually Granirer and Leinert [18] generalized it to treat the different topologies on the unit sphere of the Fourier–Stieltjes algebra. Hu, Neufang, and Ruan asked in [22] whether this result extends to locally compact quantum groups. We give an affirmative answer to their question in Section 4. Generalizing other results from [18] as well, we require the theory of non-commutative L^p-spaces of locally compact quantum groups. The background on this subject appears in Section 3.

Another notion due to Godement is that of positive-definite measures. He established an important connection between these and amenability of the group in question. In Section 5 we extend this result to locally compact quantum groups.

The separation property of locally compact groups with respect to closed subgroups was introduced by Lau and Losert [34] and Kaniuth and Lau [26], and was subsequently studied by several authors. A fundamental result is that the separation property is always satisfied with respect to compact subgroups. Section 6 is devoted to generalizing this to locally compact quantum groups. We introduce the separation property with respect to closed quantum subgroups, find a condition under which the separation property is satisfied with respect to a given compact quantum subgroup, and show that it is indeed satisfied in many examples, including $\mathbb{T}$ as a closed quantum subgroup of quantum $E(2)$.

We remark that most sections are independent of each other, but results from Section 4 are needed in other sections.

1 Preliminaries

We begin with fixing some conventions. Given a Hilbert space $\mathcal{H}$ and vectors $\zeta, \eta \in \mathcal{H}$, we denote by $\omega_{\zeta,\eta}$ the functional that takes $x \in B(\mathcal{H})$ to $\langle x\zeta, \eta \rangle$, and let $\omega_\zeta := \omega_{\zeta,\zeta}$. The identity map on a C^*-algebra A is denoted by id, and its unit, if it exists, by $\mathbb{1}$. For a functional $\omega \in A^*$, we define $\overline{\omega} \in A^*$ by $\overline{\omega}(x) := \overline{\omega(x^*)}$, $x \in A$. When no confusion is caused, we also write ω for its unique extension to the multiplier algebra $M(A)$ that is strictly continuous on the closed unit ball of $M(A)$ [33, Corollary 5.7].

Let A, B be C^*-algebras. A $*$-homomorphism from A to B or, more generally, to $M(B)$, that is nondegenerate (namely, span $\Phi(A)B$ is dense in B) has a unique extension to a (unital) $*$-homomorphism from $M(A)$ to $M(B)$ [33, Proposition 2.1]. We use the same notation for this extension.

For an n.s.f. (normal, semi-finite, faithful) weight φ on a von Neumann algebra M [46, Chapter VII], we denote $\mathcal{N}_\varphi := \{x \in M : \varphi(x^*x) < \infty\}$.

The symbol σ stands for the flip operator $x \otimes y \mapsto y \otimes x$, for x, y in some C^*-algebras. We use the symbols $\otimes, \overline{\otimes}, \otimes_{\min}$ for the Hilbert space, normal spatial and minimal tensor products, respectively.

The basics of positive-definite functions on locally compact groups are presented in the book by Dixmier [10]. From time to time we will refer to the Banach algebras associated with a locally compact group G, such as the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$; see Eymard [14]. For the Tomita–Takesaki theory, see the books by Strătilă [43] and Takesaki [46], or Takesaki's original monograph [44]. We recommend Bédos, Murphy, and Tuset [1, Section 2] for statements and proofs of folklore facts about the slice maps at the C^*-algebraic level.

1.1 Locally Compact Quantum Groups

The following axiomatization of locally compact quantum groups is due to Kustermans and Vaes [30, 31] (see also Van Daele [55]). It describes the same objects as that of Masuda, Nakagami and Woronowicz [35]. Unless stated otherwise, the material in this subsection is taken from [30, 31].

Definition 1.1 A *locally compact quantum group* (henceforth abbreviated to LCQG) is a pair $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta)$ with the following properties:

(i) $L^\infty(\mathbb{G})$ is a von Neumann algebra;
(ii) $\Delta \colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$ is a co-multiplication, that is, a faithful, normal, unital $*$-homomorphism which is co-associative: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;
(iii) there exist n.s.f. weights φ, ψ on $L^\infty(\mathbb{G})$, called the Haar weights, satisfying

$$\varphi((\omega \otimes \mathrm{id})\Delta(x)) = \omega(\mathbb{1})\varphi(x)$$

for all $\omega \in L^\infty(\mathbb{G})_*^+, x \in L^\infty(\mathbb{G})^+$ such that $\varphi(x) < \infty$ (left invariance), and

$$\psi((\mathrm{id} \otimes \omega)\Delta(x)) = \omega(\mathbb{1})\psi(x)$$

for all $\omega \in L^\infty(\mathbb{G})_*^+, x \in L^\infty(\mathbb{G})^+$ such that $\psi(x) < \infty$ (right invariance).

Let $\mathbb{G}$ be a LCQG. The left and right Haar weights, only whose existence is assumed, are unique up to scaling. The predual of $L^\infty(\mathbb{G})$ is denoted by $L^1(\mathbb{G})$. We define a convolution $*$ on $L^1(\mathbb{G})$ by $(\omega_1 * \omega_2)(x) := (\omega_1 \otimes \omega_2)\Delta(x)$ $(\omega_1, \omega_2 \in L^1(\mathbb{G})$, $x \in L^\infty(\mathbb{G}))$, making the pair $(L^1(\mathbb{G}), *)$ into a Banach algebra. We write $L^2(\mathbb{G})$ for the Hilbert space of the GNS construction for $(L^\infty(\mathbb{G}), \varphi)$, and let $\Lambda \colon \mathcal{N}_\varphi \to L^2(\mathbb{G})$ stand for the canonical injection. A fundamental feature of the theory is that of duality: $\mathbb{G}$ has a *dual* LCQG $\widehat{\mathbb{G}} = (L^\infty(\widehat{\mathbb{G}}), \widehat{\Delta})$. Objects pertaining to $\widehat{\mathbb{G}}$ will be denoted by adding a hat, *e.g.*, $\widehat{\varphi}, \widehat{\psi}$. The GNS construction for $(L^\infty(\widehat{\mathbb{G}}), \widehat{\varphi})$ yields the same Hilbert space $L^2(\mathbb{G})$, and henceforth we will consider both $L^\infty(\mathbb{G})$ and $L^\infty(\widehat{\mathbb{G}})$ as acting (standardly) on $L^2(\mathbb{G})$. We write $J, \widehat{J}$ for the modular conjugations relative to $L^\infty(\mathbb{G}), L^\infty(\widehat{\mathbb{G}})$, respectively, both acting on $L^2(\mathbb{G})$.

Example 1.2 Every locally compact group G induces two LCQGs as follows. First, the LCQG that is identified with G is $(L^\infty(G), \Delta)$, where $(\Delta(f))(t, s) := f(ts)$ for $f \in L^\infty(G)$ and $t, s \in G$ using the identification $L^\infty(G) \overline{\otimes} L^\infty(G) \cong L^\infty(G \times G)$, and φ and ψ are integration against the left and right Haar measures of G, respectively. All LCQGs whose $L^\infty(\mathbb{G})$ is commutative have this form. Second, the dual of the above, which is the LCQG $(\mathrm{VN}(G), \Delta)$, where $\mathrm{VN}(G)$ is the left von Neumann algebra of G, Δ is the unique normal $*$-homomorphism $\mathrm{VN}(G) \to \mathrm{VN}(G) \overline{\otimes} \mathrm{VN}(G)$ mapping the translation $\lambda_t, t \in G$, to $\lambda_t \otimes \lambda_t$, and φ and ψ are the Plancherel weight on $\mathrm{VN}(G)$. The LCQGs that are co-commutative, namely whose $L^1(\mathbb{G})$ is commutative, are precisely the ones of this form. The L^2-Hilbert space of both LCQGs is $L^2(G)$.

The *left regular co-representation* of $\mathbb{G}$ is a unitary $W \in L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$ satisfying $\Delta(x) = W^*(\mathbb{1} \otimes x)W$ for every $x \in L^\infty(\mathbb{G})$ and $(\Delta \otimes \mathrm{id})(W) = W_{13}W_{23}$ (using leg

numbering). The left regular co-representation of $\widehat{\mathbb{G}}$ is $\widehat{W} = \sigma(W^*)$. The set

$$C_0(\mathbb{G}) := \overline{\{(\mathrm{id} \otimes \widehat{\omega})(W) : \widehat{\omega} \in L^1(\widehat{\mathbb{G}})\}}^{\|\cdot\|}$$

is a weakly dense C^*-subalgebra of $L^\infty(\mathbb{G})$, satisfying $\Delta(C_0(\mathbb{G})) \subseteq M(C_0(\mathbb{G}) \otimes_{\min} C_0(\mathbb{G}))$. This allows us to define a convolution $\star$ on $C_0(\mathbb{G})^*$, which becomes a Banach algebra. Viewing $L^1(\mathbb{G})$ as a subspace of $C_0(\mathbb{G})^*$ by restriction, the former is a (closed, two-sided) ideal in the latter. We define a map $\lambda: L^1(\mathbb{G}) \to C_0(\widehat{\mathbb{G}})$ by $\lambda(\omega) := (\omega \otimes \mathrm{id})(W)$. It is easily checked that λ is a contractive homomorphism.

We review the construction of the left-invariant weight $\widehat{\varphi}$ of $\widehat{\mathbb{G}}$. Let $\mathcal{I}$ stand for all "square-integrable elements of $L^1(\mathbb{G})$", namely all $\omega \in L^1(\mathbb{G})$ such that there is $M < \infty$ with $|\omega(x^*)| \le M\|\Lambda(x)\|$ for every $x \in \mathcal{N}_\varphi$; equivalently, there is $\xi = \xi(\omega) \in L^2(\mathbb{G})$ such that $\omega(x^*) = \langle \xi, \Lambda(x) \rangle$ for every $x \in \mathcal{N}_\varphi$. Then $\widehat{\varphi}$ is the unique n.s.f. weight on $L^\infty(\widehat{\mathbb{G}})$ whose GNS construction $(L^2(\mathbb{G}), \widehat{\Lambda})$ satisfies $\widehat{\Lambda}(\lambda(\omega)) = \xi(\omega)$ for all $\omega \in \mathcal{I}$ and that $\lambda(\mathcal{I})$ is a $*$-ultrastrong–norm core for $\widehat{\Lambda}$.

A fundamental object for $\mathbb{G}$ is its *antipode* S, which is a $*$-ultrastrongly closed, densely defined, generally unbounded linear operator on $L^\infty(\mathbb{G})$. It has the "polar decomposition" $S = R \circ \tau_{-i/2}$, where R stands for the *unitary antipode* and $(\tau_t)_{t\in\mathbb{R}}$ for the *scaling group*. We will not discuss here the definitions of these maps. The subspace

$$L^1_*(\mathbb{G}) := \left\{\omega \in L^1_*(\mathbb{G}) : (\exists \rho \in L^1(\mathbb{G}) \, \forall x \in D(S)) \quad \rho(x) = \overline{\omega}(S(x))\right\}$$

is a dense subalgebra of $L^1(\mathbb{G})$. For $\omega \in L^1_*(\mathbb{G})$, let ω^* be the unique element $\rho \in L^1(\mathbb{G})$ such that $\rho(x) = \overline{\omega}(S(x))$ for each $x \in D(S)$. Then $\omega \mapsto \omega^*$ is an involution on $L^1_*(\mathbb{G})$, and $\lambda|_{L^1_*(\mathbb{G})}$ is a $*$-homomorphism. Moreover, $L^1_*(\mathbb{G})$ is an involutive Banach algebra when equipped with the new norm $\|\omega\|_* := \max(\|\omega\|, \|\omega^*\|)$.

A useful construction is the opposite LCQG $\mathbb{G}^{\mathrm{op}}$ [31, Section 4], which has $L^\infty(\mathbb{G}^{\mathrm{op}}) := L^\infty(\mathbb{G})$ and co-multiplication given by $\Delta^{\mathrm{op}} := \sigma \circ \Delta$.

The *universal* setting of $\mathbb{G}$ was defined by Kustermans [29] as follows. Let $C_0^{\mathrm{u}}(\mathbb{G})$ be the enveloping C^*-algebra of $L^1_*(\widehat{\mathbb{G}})$. The canonical embedding of $L^1_*(\widehat{\mathbb{G}})$ in $C_0^{\mathrm{u}}(\mathbb{G})$ is denoted by $\widehat{\lambda}_{\mathrm{u}}$. By universality, there exists a surjective $*$-homomorphism $\pi_{\mathrm{u}}: C_0^{\mathrm{u}}(\mathbb{G}) \to C_0(\mathbb{G})$ satisfying $\pi_{\mathrm{u}}(\widehat{\lambda}_{\mathrm{u}}(\omega)) = \widehat{\lambda}(\omega)$ for every $\omega \in L^1_*(\widehat{\mathbb{G}})$. There exists a co-multiplication $\Delta_{\mathrm{u}}: C_0^{\mathrm{u}}(\mathbb{G}) \to M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} C_0^{\mathrm{u}}(\mathbb{G}))$ satisfying $(\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})\Delta_{\mathrm{u}} = \Delta\pi_{\mathrm{u}}$, inducing a convolution in $C_0^{\mathrm{u}}(\mathbb{G})^*$, making it an involutive Banach algebra. Using the isometry $\pi_{\mathrm{u}}^*: C_0(\mathbb{G})^* \to C_0^{\mathrm{u}}(\mathbb{G})^*$, one can see $C_0(\mathbb{G})^*$ as a subset of $C_0^{\mathrm{u}}(\mathbb{G})^*$, which is a (closed, two-sided) ideal. Furthermore, $L^1(\mathbb{G})$ is also a (closed, two-sided) ideal in $C_0^{\mathrm{u}}(\mathbb{G})^*$ [5, Proposition 8.3].

The left regular co-representation of $\mathbb{G}$ has a universal version. It is a unitary $\mathbb{W} \in M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$ satisfying $(\Delta_{\mathrm{u}} \otimes \mathrm{id})(\mathbb{W}) = \mathbb{W}_{13}\mathbb{W}_{23}$ and $(\pi_{\mathrm{u}} \otimes \widehat{\pi}_{\mathrm{u}})(\mathbb{W}) = W$. Its dual object is $\widehat{\mathbb{W}} = \sigma(\mathbb{W}^*)$. Letting $\mathbb{W} := (\mathrm{id} \otimes \widehat{\pi}_{\mathrm{u}})(\mathbb{W})$ and $\mathbb{W} := (\pi_{\mathrm{u}} \otimes \mathrm{id})(\mathbb{W})$, we have $\mathbb{W} \in M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} C_0(\widehat{\mathbb{G}}))$, $\mathbb{W} \in M(C_0(\mathbb{G}) \otimes_{\min} C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$ and $(\mathrm{id} \otimes \pi_{\mathrm{u}})\Delta_{\mathrm{u}}(x) = \mathbb{W}^*(\mathbb{1} \otimes \pi_{\mathrm{u}}(x))\mathbb{W}$ for every $x \in C_0^{\mathrm{u}}(\mathbb{G})$. Moreover, representing $C_0^{\mathrm{u}}(\mathbb{G})$ faithfully on a Hilbert space $\mathcal{H}_{\mathrm{u}}$ and viewing the operator $\mathbb{W} \in M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} C_0(\widehat{\mathbb{G}}))$ as an element of $B(\mathcal{H}_{\mathrm{u}} \otimes L^2(\mathbb{G}))$, we have $\mathbb{W} \in M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} \mathcal{K}(L^2(\mathbb{G})))$. Also $\lambda_{\mathrm{u}}(\omega) = (\omega \otimes \mathrm{id})(\mathbb{W})$ for every $\omega \in L^1_*(\mathbb{G})$,

and the map $\lambda^{\mathrm{u}} \colon C_0^{\mathrm{u}}(\mathbb{G})^* \to M(C_0(\widehat{\mathbb{G}}))$, $\omega \mapsto (\omega \otimes \mathrm{id})(\mathbb{W})$ for $\omega \in C_0^{\mathrm{u}}(\mathbb{G})^*$, is a $*$-homomorphism.

The universality property of $C_0^{\mathrm{u}}(\mathbb{G})$ implies the existence of the *co-unit*, which is the unique $*$-homomorphism $\epsilon \in C_0^{\mathrm{u}}(\mathbb{G})_+^*$ such that $(\epsilon \otimes \mathrm{id}) \circ \Delta_{\mathrm{u}} = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta_{\mathrm{u}}$. It satisfies $(\epsilon \otimes \mathrm{id})(\mathbb{W}) = \mathbb{1}_{M(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))}$.

For a Banach algebra A, the canonical module action of A on its dual A^* is denoted by juxtaposition, that is,

$$(\mu a)(b) = \mu(ab) \quad \text{and} \quad (a\mu)(b) = \mu(ba) \qquad (\forall \mu \in A^*, a, b \in A).$$

This notation will be used for the actions of $L^\infty(\mathbb{G})$, $C_0(\mathbb{G})$ and $C_0^{\mathrm{u}}(\mathbb{G})$ on their duals.

The canonical module actions of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$ will be denoted by '$\cdot$', so we have

$$\omega \cdot a = (\mathrm{id} \otimes \omega)\Delta(a) \quad \text{and} \quad a \cdot \omega = (\omega \otimes \mathrm{id})\Delta(a) \qquad (\forall \omega \in L^1(\mathbb{G}), a \in L^\infty(\mathbb{G})).$$

Each of $\{\omega \cdot a : \omega \in L^1(\mathbb{G}), a \in C_0(\mathbb{G})\}$ and $\{a \cdot \omega : \omega \in L^1(\mathbb{G}), a \in C_0(\mathbb{G})\}$ spans a norm dense subset of $C_0(\mathbb{G})$.

More generally, every $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ acts on $L^\infty(\mathbb{G})$ as follows: for $a \in L^\infty(\mathbb{G})$, $\mu \cdot a$ and $a \cdot \mu$ are defined to be the unique elements of $L^\infty(\mathbb{G})$ satisfying

$$\omega(\mu \cdot a) = (\omega * \mu)(a), \quad \omega(a \cdot \mu) = (\mu * \omega)(a) \qquad (\forall \omega \in L^1(\mathbb{G})).$$

Note that if $\mu_1, \mu_2 \in C_0^{\mathrm{u}}(\mathbb{G})^*$ and $a \in L^\infty(\mathbb{G})$, then

$$\omega[\mu_1 \cdot (\mu_2 \cdot a)] = (\omega * \mu_1)(\mu_2 \cdot a) = (\omega * \mu_1 * \mu_2)(a) = \omega[(\mu_1 * \mu_2) \cdot a],$$

thus $\mu_1 \cdot (\mu_2 \cdot a) = (\mu_1 * \mu_2) \cdot a$. Similarly, $(a \cdot \mu_1) \cdot \mu_2 = a \cdot (\mu_1 * \mu_2)$.

Lemma 1.3 *If $a \in C_0(\mathbb{G})$ and $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$, then $\mu \cdot a, a \cdot \mu \in C_0(\mathbb{G})$.*

Proof Fix $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$. If $\omega \in L^1(\mathbb{G})$ and $b \in C_0(\mathbb{G})$, then $\mu \cdot (\omega \cdot b) = (\mu * \omega) \cdot b \in C_0(\mathbb{G})$ as $\mu * \omega \in L^1(\mathbb{G})$. By density, $\mu \cdot a \in C_0(\mathbb{G})$ for all $a \in C_0(\mathbb{G})$. The proof for $a \cdot \mu$ is similar. ∎

1.2 Types of LCQGs

Compact quantum groups were introduced by Woronowicz [56], and discrete quantum groups by Effros and Ruan [12] and by Van Daele [54]. We will not present their original definitions, but define them through the Kustermans–Vaes axiomatization. Complete proofs of the equivalence of various characterizations of compact and discrete quantum groups can be found in [41].

A LCQG $\mathbb{G}$ is *compact* if its left Haar weight φ is finite. This is equivalent to $C_0(\mathbb{G})$ being unital. In this case, we denote $C_0(\mathbb{G})$ by $C(\mathbb{G})$. Moreover, the right Haar weight ψ is also finite, and assuming, as customary, that both φ and ψ are states, they are equal.

A LCQG $\mathbb{G}$ is *discrete* if it is the dual of a compact quantum group. This is equivalent to $(L^1(\mathbb{G}), *)$ admitting a unit ϵ. In this case, we denote $C_0(\mathbb{G}), L^\infty(\mathbb{G})$ by $c_0(\mathbb{G}), \ell^\infty(\mathbb{G})$, respectively, and have

$$c_0(\mathbb{G}) \cong c_0 - \bigoplus_{\alpha \in \mathrm{Irred}(\widehat{\mathbb{G}})} M_{n(\alpha)} \quad \text{and} \quad \ell^\infty(\mathbb{G}) \cong \ell^\infty - \bigoplus_{\alpha \in \mathrm{Irred}(\widehat{\mathbb{G}})} M_{n(\alpha)},$$

where Irred($\mathbb{G}$) is the set of equivalence classes of (necessarily finite-dimensional) irreducible unitary co-representations of $\widehat{\mathbb{G}}$, and for every $\alpha \in \text{Irred}(\widehat{\mathbb{G}})$, $n(\alpha) \in \mathbb{N}$ denotes the dimension of the representation. Particularly, the summand corresponding to the trivial co-representation of $\widehat{\mathbb{G}}$ gives a central minimal projection p in $\ell^\infty(\mathbb{G})$, satisfying $ap = \epsilon(a)p = pa$ for every $a \in \ell^\infty(\mathbb{G})$.

A LCQG $\mathbb{G}$ is called *co-amenable* (see Bédos and Tuset [2] or Desmedt, Quaegebeur and Vaes [9], who use a different terminology) if $L^1(\mathbb{G})$ admits a bounded approximate identity. This is equivalent to the Banach algebra $(C_0(\mathbb{G})^*, *)$ having a unit [2, Theorem 3.1], which is called the *co-unit* of $\mathbb{G}$ and denoted by ϵ. It is also equivalent to the surjection $\pi_u : C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$ being an isomorphism, in which case we simply identify $C_0^u(\mathbb{G})$ with $C_0(\mathbb{G})$.

Every locally compact group G is co-amenable as a (commutative) quantum group, while its co-commutative dual $\widehat{G}$ is co-amenable if and only if G is amenable as a group. Discrete quantum groups are trivially co-amenable.

1.3 Positive-definite Functions Over LCQGs

Let $\mathbb{G}$ be a LCQG. In [6,8], Daws and Salmi introduced four notions of positive definiteness for elements of $L^\infty(\mathbb{G})$. Here we will need only two of them, namely (1) and (2) of [8]. Note that we use different notation: $\overline{\omega}, \omega^*$ are denoted by $\omega^*, \omega^\sharp$ in [6,8].

Definition 1.4 Let $\mathbb{G}$ be a LCQG.

(i)   A *positive-definite function* is $x \in L^\infty(\mathbb{G})$ satisfying $(\omega^* * \omega)(x^*) \geq 0$ for every $\omega \in L^1_*(\mathbb{G})$.

(ii) A *Fourier–Stieltjes transform of a positive measure* is an element x of the form $(\text{id} \otimes \widehat{\mu})(W^*) = \hat{\lambda}^u(\widehat{\mu})$ for some $\widehat{\mu} \in C_0^u(\widehat{\mathbb{G}})^*_+$. Note that $x \in M(C_0(\mathbb{G}))$ in this case.

Theorem 1.5 ([8, Lemma 1 and Theorem 15]) *For $x \in L^\infty(\mathbb{G})$, we have* (ii) $\Rightarrow$ (i), *and the converse holds when $\mathbb{G}$ is co-amenable.*

For co-amenable $\mathbb{G}$, we will therefore just use the adjective "positive definite" for these elements.

Remark 1.6 Let $\mathbb{G}$ be a co-amenable LCQG with co-unit $\epsilon \in C_0(\mathbb{G})^*$. Write ϵ also for its strictly continuous extension to $M(C_0(\mathbb{G}))$. If $x \in L^\infty(\mathbb{G})$ is positive definite, then $\|x\| = \epsilon(x)$, for writing $x = (\text{id} \otimes \widehat{\mu})(W^*)$ with $\widehat{\mu} \in C_0^u(\widehat{\mathbb{G}})^*_+$, we have

$$\|x\| \geq \epsilon(x) = \epsilon((\text{id} \otimes \widehat{\mu})(W^*)) = \widehat{\mu}((\epsilon \otimes \text{id})(W^*)) = \widehat{\mu}(\mathbb{1}) = \|\widehat{\mu}\| \geq \|x\|$$

(see [1, Corollary 2.2] and [2, Theorem 3.1]).

2 Square-integrable Positive-definite Functions Over Locally Compact Quantum Groups

This section is dedicated to proving a generalization of Godement's theorem on square-integrable positive-definite functions. It can be established directly along the

lines of [10, Section 13.8], but we feel that it is more correct to do it through the generalization of this result to left Hilbert algebras given by Phillips [39]. We start with some background. Let $\mathcal{A}$ be a full (that is, achieved) left Hilbert algebra [44, 46] and $\mathcal{H}$ be the completion of $\mathcal{A}$. We denote by $\pi(\xi)$ (resp. $\pi'(\xi)$) the operator corresponding to a left-bounded (resp. right-bounded) vector $\xi \in \mathcal{H}$.

Definition 2.1 (Perdrizet [38], Haagerup [19])   Let

$$\mathcal{P}^\flat := \left\{ \eta \in \mathcal{H} : \langle \eta, \xi^\sharp \xi \rangle \geq 0 \text{ for every } \xi \in \mathcal{A} \right\}.$$

This set is evidently a cone in $\mathcal{H}$.

Remark 2.2   Let $\eta \in \mathcal{H}$. Takesaki [46, Theorem VI.1.26 (ii)] implies that $\eta \in \mathcal{P}^\flat$ if and only if $\langle \eta, \pi(\xi)^* \xi \rangle \geq 0$ for every left-bounded vector $\xi \in \mathcal{H}$.

Definition 2.3 ([39])   Let $\eta \in \mathcal{P}^\flat$.

(i)   Say that η is *integrable* if $\sup\{ \langle \eta, \xi \rangle : \xi \text{ is a selfadjoint idempotent in } \mathcal{A} \} < \infty$.
(ii)   Say that $\zeta \in \mathcal{P}^\flat$ is a *square root* of η if $\langle \xi, \eta \rangle = \langle \pi(\xi)\zeta, \zeta \rangle$ for every $\xi \in \mathcal{A}$.

We denote the set of all integrable elements of $\mathcal{P}^\flat$ by $\mathcal{P}^\flat_{\text{int}}$.

Theorem 2.4 ([39, Theorem 1.10])   *Let $\eta \in \mathcal{P}^\flat$. Then η is integrable if and only if it has a square root $\zeta \in \mathcal{P}^\flat$. If $\eta \in \mathcal{A}'$, then also $\zeta \in \mathcal{A}'$, and $\zeta\zeta = \eta$.*

Moreover, the span of $\mathcal{P}^\flat_{\text{int}}$ can be endowed with a natural norm making it isometrically isomorphic to a dense subspace of the predual of the (left) von Neumann algebra $\mathcal{R}_\ell(\mathcal{A})$ of $\mathcal{A}$ [39, Theorem 2.9]. In particular, $\eta \in \mathcal{P}^\flat_{\text{int}}$ with square root $\zeta \in \mathcal{P}^\flat$ induces the element $\omega_\zeta|_{\mathcal{R}_\ell(\mathcal{A})}$ of $\mathcal{R}_\ell(\mathcal{A})_*$.

Let $\mathbb{G}$ be a LCQG, and set $\mathcal{J} := \mathcal{I} \cap L^1_*(\mathbb{G})$.

Lemma 2.5   *Let $x, y \in L^\infty(\mathbb{G})$. If $(\omega_1^* * \omega_2)^*(y) = (\overline{\omega_1^*} * \overline{\omega_2})(x)$ for every $\omega_1, \omega_2 \in \mathcal{J}$. Then $y \in D(S)$ and $S(y) = x$.*

Proof   The assertion follows by repeating the argument of [8, proof of Lemma 5] with $L^1_*(\mathbb{G})$ being replaced by $\mathcal{J}$. This is possible as $\mathcal{I}$, and hence $\mathcal{J}$, are invariant under the scaling group adjoint $(\tau_t^*)_{t \in \mathbb{R}}$, and $\mathcal{J}, \mathcal{J}^*$ are norm dense in $L^1(\mathbb{G})$ [31, Lemma 2.5 and its proof]. ∎

We need a slight strengthening of [8, Theorem 6] and part of [31, Proposition 2.6].

Lemma 2.6   *The set $\{ \omega_1^* * \omega_2 : \omega_1, \omega_2 \in \mathcal{J} \}$ is total in $(L^1_*(\mathbb{G}), \|\cdot\|_*)$. Thus the subspace $\mathcal{J} \cap \mathcal{J}^*$ is dense in $(L^1_*(\mathbb{G}), \|\cdot\|_*)$.*

Proof   Since $\mathcal{I}$ is a left ideal [55, Lemma 4.8], $\{ \omega_1^* * \omega_2 : \omega_1, \omega_2 \in \mathcal{J} \}$ is contained in $\mathcal{J} \cap \mathcal{J}^*$. Adapting the argument of [8, proof of Theorem 6], if $\{ \omega_1^* * \omega_2 : \omega_1, \omega_2 \in \mathcal{J} \}$ were not total in $(L^1_*(\mathbb{G}), \|\cdot\|_*)$, then there would be $x, y \in L^\infty(\mathbb{G})$ such that

$$0 = (\omega_1^* * \omega_2)(x) + \overline{(\omega_1^* * \omega_2)^*(y)},$$

that is, $(\omega_1^* * \omega_2)^*(y) = (\overline{\omega_1^*} * \overline{\omega_2})(-x^*)$, for every $\omega_1, \omega_2 \in \mathcal{J}$. Lemma 2.5 gives that $y \in D(S)$ and $S(y) = -x^*$, and hence the element of $(L_*^1(\mathbb{G}), \|\cdot\|_*)^*$ corresponding to $(x, \overline{y})$ is zero. ∎

Considering the full left Hilbert algebra $\mathcal{A}_{\widehat{\varphi}}$ associated with the left-invariant weight $\widehat{\varphi}$ of $\widehat{\mathbb{G}}$, we let $\mathcal{P}_{\widehat{\varphi}}^\flat$ stand for the corresponding cone.

Lemma 2.7 *Let $x \in L^\infty(\mathbb{G})$. If $x \in \mathcal{N}_\varphi$, then x is a positive-definite function if and only if $\Lambda(x) \in \mathcal{P}_{\widehat{\varphi}}^\flat$.*

Proof By definition, x is positive definite if and only if $(\omega^* * \omega)(x^*) \geq 0$ for every $\omega \in L_*^1(\mathbb{G})$. From Lemma 2.6, it suffices to check this for $\omega \in \mathcal{J}$. But if $\omega \in \mathcal{J}$, then also $\omega^* * \omega \in \mathcal{J}$ and for $\widehat{y} := \lambda(\omega)$ we have $\widehat{y}^* \widehat{y} = \lambda(\omega^* * \omega)$ and

$$(\omega^* * \omega)(x^*) = \langle \widehat{\Lambda}(\lambda(\omega^* * \omega)), \Lambda(x) \rangle = \langle \widehat{y}^* \widehat{\Lambda}(\widehat{y}), \Lambda(x) \rangle.$$

By Remark 2.2, $\Lambda(x) \in \mathcal{P}_{\widehat{\varphi}}^\flat$ if and only if $\langle \Lambda(x), \widehat{y}^* \widehat{\Lambda}(\widehat{y}) \rangle \geq 0$ for every $\widehat{y} \in \mathcal{N}_{\widehat{\varphi}}$. Using [31, Lemma 2.5], that is equivalent to $\langle \Lambda(x), \widehat{y}^* \widehat{\Lambda}(\widehat{y}) \rangle \geq 0$ for every $\widehat{y} \in \lambda(\mathcal{J})$. This completes the proof. ∎

Proposition 2.8 *Let $\mathbb{G}$ be a co-amenable LCQG. There exists a contractive approximate identity for $(L_*^1(\mathbb{G}), \|\cdot\|_*)$ in $\mathcal{J} \cap \mathcal{J}^*$.*

Proof By [8, Theorem 13], $(L_*^1(\mathbb{G}), \|\cdot\|_*)$ has a contractive approximate identity. Combining this with Lemma 2.6, the assertion is proved. ∎

The following result generalizes [39, Theorem 1.6], saying that if G is a locally compact group and $f \in L^2(G)$ is positive definite and essentially bounded on a neighborhood of the identity, then it belongs to $A(G)$.

Corollary 2.9 *Let $\mathbb{G}$ be a co-amenable LCQG. If $x \in \mathcal{N}_\varphi$ and x is positive definite, then $\Lambda(x)$ is integrable with respect to $\mathcal{A}_{\widehat{\varphi}}$ (see Definition 2.3).*

Proof Let (ϵ_i) be a contractive approximate identity for $(L_*^1(\mathbb{G}), \|\cdot\|_*)$ in $\mathcal{J} \cap \mathcal{J}^*$. Then letting $\xi_i := \widehat{\Lambda}(\lambda(\epsilon_i))$, we get a net (ξ_i) in the left Hilbert algebra $\mathcal{A}_{\widehat{\varphi}}$. Since $x \in \mathcal{N}_\varphi$, we have for every i,

$$\langle \Lambda(x), \xi_i^\sharp \xi_i \rangle = \langle \Lambda(x), \widehat{\Lambda}(\lambda(\epsilon_i^* * \epsilon_i)) \rangle = \overline{\langle \widehat{\Lambda}(\lambda(\epsilon_i^* * \epsilon_i)), \Lambda(x) \rangle} = \overline{(\epsilon_i^* * \epsilon_i)(x^*)},$$

and so $\langle \Lambda(x), \xi_i^\sharp \xi_i \rangle \leq \|\epsilon_i^*\| \|\epsilon_i\| \|x\| \leq \|x\|$. Since $\Lambda(x) \in \mathcal{P}_{\widehat{\varphi}}^\flat$ by Lemma 2.7 and $(\lambda(\epsilon_i))$ converges strongly to $\mathbb{1}$ (for $L_*^1(\mathbb{G})$ is dense in $L^1(\mathbb{G})$), [39, Proposition 1.5] applies, and yields that $\Lambda(x)$ is integrable with respect to $\mathcal{A}_{\widehat{\varphi}}$. ∎

We now prove the main result of this section, generalizing a theorem of Godement [17, Théorème 17].

Theorem 2.10 *Let $\mathbb{G}$ be a co-amenable LCQG. If $x \in \mathcal{N}_\varphi$ and x is positive definite, then $\Lambda(x)$ has a square root in $\mathcal{P}_{\widehat{\varphi}}^\flat$ (Definition 2.3); equivalently, there exists $\zeta \in \mathcal{P}_{\widehat{\varphi}}^\flat$ such that $x = \widehat{\lambda}(\widehat{\omega}_\zeta)$. If, additionally, $\Lambda(x) \in \mathcal{A}_{\widehat{\varphi}}'$, then also $\zeta \in \mathcal{A}_{\widehat{\varphi}}'$, in which case*

$\Lambda(x) = \widehat{\pi}'(\zeta)\zeta$. That is, if $\widehat{w} \in \mathcal{N}_{\widehat{\varphi}}$ is positive and $\widehat{J\Lambda}(\widehat{w}) \in \Lambda(\mathcal{N}_{\varphi})$, then the (positive) square root of $\widehat{w}$ also belongs to $\mathcal{N}_{\widehat{\varphi}}$.

Proof The first part of the first assertion, as well as the second assertion, follow from Theorem 2.4 by using Corollary 2.9. For the part after "equivalently", $\Lambda(x)$ having a square root in $\mathcal{P}^{\flat}_{\widehat{\varphi}}$ means, by definition, that there exists $\zeta \in \mathcal{P}^{\flat}_{\widehat{\varphi}}$ such that $\langle \widehat{\Lambda}(\widehat{y}), \Lambda(x) \rangle = \widehat{\omega}_{\zeta}(\widehat{y})$ for every $\widehat{y} \in \mathcal{N}_{\widehat{\varphi}} \cap \mathcal{N}^*_{\widehat{\varphi}}$. Thus for every $\widehat{y} \in \mathcal{N}_{\widehat{\varphi}}$ [46, Theorem VI.1.26 (ii)]. In particular, for every $\omega \in \mathcal{I}$,

$$\omega(x^*) = \langle \widehat{\Lambda}(\lambda(\omega)), \Lambda(x) \rangle = \widehat{\omega}_{\zeta}(\lambda(\omega)) = \omega[(\mathrm{id} \otimes \widehat{\omega}_{\zeta})(W)].$$

The density of $\mathcal{I}$ in $L^1(\mathbb{G})$ entails that $x = (\mathrm{id} \otimes \widehat{\omega}_{\zeta})(W)^* = (\mathrm{id} \otimes \widehat{\omega}_{\zeta})(W^*) = \widehat{\lambda}(\widehat{\omega}_{\zeta})$. The converse is proved similarly.

For the last sentence, note that $\mathcal{P}^{\flat}_{\widehat{\varphi}} \cap \mathcal{A}'_{\widehat{\varphi}} = \{\widehat{J\Lambda}(\widehat{w}) : \widehat{w} \in \mathcal{N}_{\widehat{\varphi}} \text{ and } \widehat{w} \geq 0\}$ (see the right version of [38, Proposition 2.5]). If an element there is in $\Lambda(\mathcal{N}_{\varphi})$, then its square root in $\mathcal{P}^{\flat}_{\widehat{\varphi}}$ has the form $\widehat{J\Lambda}(\widehat{z})$ for $\widehat{z} \in \mathcal{N}_{\widehat{\varphi}}$ with $\widehat{z} \geq 0$, and the equality $\widehat{J\Lambda}(\widehat{w}) = \widehat{J\Lambda}(\widehat{z}^2)$ implies that $\widehat{w} = \widehat{z}^2$. ∎

Remark 2.11 In the situation of Theorem 2.10 we have $\|x\| = \|\widehat{\omega}_{\zeta}\| = \|\zeta\|^2$ by Remark 1.6.

3 Convolution in $L^p(\mathbb{G})$

This section contains the preliminaries on non-commutative L^p-spaces of LCQGs needed in the next section. The theory of non-commutative L^p-spaces of von Neumann algebras was developed in three approaches, which turned out to be equivalent: the "abstract" one by Haagerup [20], the "spatial" one by Connes and Hilsum [21], and the one using interpolation theory, whose final form is by Izumi [23] (see also Terp [48, 49]). Here we rely on the work of Caspers [4], who introduced and studied non-commutative L^p-spaces of LCQGs based on Izumi's approach with interpolation parameter $\alpha = -\frac{1}{2}$. This has two clear virtues. The first, which is intrinsic in interpolation theory, is the fact that as vector spaces, all non-commutative L^p-spaces are realized as subspaces of a larger space, allowing consideration of their intersections. Caspers proved that when $\alpha = -\frac{1}{2}$, some of these intersections take a particularly natural form. The second is simplicity: the statement (but not the proof) of the construction's basic ingredients does not require modular theory.

We offer now a succinct account of the theory. A pair of Banach spaces (A_0, A_1) is called *compatible* in the sense of interpolation theory (see Bergh and Löfström [3, Section 2.3]) if they are continuously embedded in a Hausdorff topological vector space. For $0 < \theta < 1$, the Calderón *complex interpolation method* [3, Chapter 4] gives the interpolation Banach space $C_{\theta}(A_0, A_1)$. As a vector space it satisfies $A_0 \cap A_1 \subseteq C_{\theta}(A_0, A_1) \subseteq A_0 + A_1$, and these inclusions are contractive when $A_0 \cap A_1$ and $A_0 + A_1$ are given the norms $\|a\|_{A_0 \cap A_1} := \max(\|a\|_{A_0}, \|a\|_{A_1})$, $a \in A_0 \cap A_1$, and $\|a\|_{A_0+A_1} := \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1, a = a_0 + a_1\}$, $a \in A_0 + A_1$. Moreover, $A_0 \cap A_1$ is dense in $C_{\theta}(A_0, A_1)$ [3, Theorem 4.2.2]. The functor C_{θ} is an *exact interpolation functor of exponent* θ in the following sense. Given another compatible pair (B_0, B_1),

two bounded maps $T_i: A_i \to B_i$, $i = 0, 1$, are called *compatible* if they agree on $A_0 \cap A_1$. Then the induced linear map $T: A_0 + A_1 \to B_0 + B_1$ satisfies $TC_\theta(A_0, A_1) \subseteq C_\theta(B_0, B_1)$, and the restriction $T: C_\theta(A_0, A_1) \to C_\theta(B_0, B_1)$ has norm at most $\|T_0\|^{1-\theta}\|T_1\|^\theta$.

Let M be a von Neumann algebra, and let φ be an n.s.f. weight on M. Define

$$L := \left\{ x \in \mathcal{N}_\varphi : (\exists_x \varphi \in M_* \, \forall y \in \mathcal{N}_\varphi) \, _x\varphi(y^*) = \varphi(y^*x) \right\},$$

$$R := \left\{ x \in \mathcal{N}_\varphi^* : (\exists \varphi_x \in M_* \, \forall y \in \mathcal{N}_\varphi) \, \varphi_x(y) = \varphi(xy) \right\}.$$

The spaces L, R are precisely $L_{(-1/2)}, L_{(1/2)}$ in Izumi's notation [4, Proposition 2.14]. Endow L, R with norms by putting $\|x\|_L := \max(\|x\|_M, \|_x\varphi\|_{M_*})$ for $x \in L$ and $\|x\|_R := \max(\|x\|_M, \|\varphi_x\|_{M_*})$ for $x \in R$. Define linear mappings $l^1: L \to M_*$, $l^\infty: L \to M$, $r^1: R \to M_*$, and $r^\infty :: R \to M$ by $l^1(x) := {}_x\varphi$ and $l^\infty(x) := x$ for $x \in L$, and similarly $r^1(x) := \varphi_x$ and $r^\infty(x) := x$ for $x \in R$. These maps are contractive and injective. Furthermore, the adjoints $(l^1)^*: M \to L^*$, $(l^\infty)^*: M_* \to L^*$, $(r^1)^*: M \to R^*$, and $(r^\infty)^*: M_* \to R^*$ are also injective (in the second and the fourth we restricted the usual adjoint from M^* to M_*). By [23, Theorem 2.5], the diagram on the left-hand side is commutative:

(3.1)

In addition, by [23, Corollary 2.13],

$$(3.2) \qquad ((r^1)^* \circ l^\infty)(L) = (r^1)^*(M) \cap (r^\infty)^*(M_*) = ((r^\infty)^* \circ l^1)(L),$$

allowing us to regard L as the "intersection of M and M_* in R^*".

Viewing M, M_* as embedded as vector spaces in R^* via $(r^1)^*, (r^\infty)^*$, the pair (M, M_*) is thus compatible. For $1 < p < \infty$, we define $(L^p(M)_{\text{left}}, \|\cdot\|_p)$ to be the interpolation Banach space $C_{1/p}(M, M_*)$. As above, we have $(r^1)^*(M) \cap (r^\infty)^*(M_*) \subseteq L^p(M)_{\text{left}} \subseteq (r^1)^*(M) + (r^\infty)^*(M_*)$ (all inside R^*) with contractive inclusions and $(r^1)^*(M) \cap (r^\infty)^*(M_*)$ is dense in $(L^p(M)_{\text{left}}, \|\cdot\|_p)$. From (3.2) we get a contractive injection $l^p: L \to L^p(M)_{\text{left}}$ with dense range, and the diagram on the right-hand side of (3.1) is commutative.

Denote by $(\mathcal{H}, \Lambda)$ the GNS construction for (M, φ). The map $l^2(x) \mapsto \Lambda(x)$, $x \in L$, extends to a unitary U_l from $L^2(M)_{\text{left}}$ to $\mathcal{H}$, allowing us to identify these spaces. We have the useful identity $\langle U_l^* \xi, y \rangle_{R^*, R} = \langle \xi, \Lambda(y^*) \rangle_{\mathcal{H}}$ for all $\xi \in \mathcal{H}$ and $y \in R$ [4, Propositions 2.21, 2.22]. In the sequel we put $L^\infty(M)_{\text{left}} := M$ and $L^1(M)_{\text{left}} := M_*$, and view M, M_* and $\mathcal{H}$ as linear subspaces of R^* by eliminating the usage of $(r^1)^*$, $(r^\infty)^*$ and U_l^*.

Define $\mathcal{I} := \left\{ \omega \in M_* : (\exists \xi(\omega) \in \mathcal{H} \, \forall x \in \mathcal{N}_\varphi) \, \omega(x^*) = \langle \xi(\omega), \Lambda(x) \rangle \right\}$, and note that this is precisely $\mathcal{I}$ defined for $L^\infty(\mathbb{G})$ in Section 1. By [4, Theorem 3.3], we have $\mathcal{I} = \mathcal{H} \cap M_*$ in R^*, with $\omega \in \mathcal{I}$ being equal to $\xi(\omega)$. Moreover, the pair $(\mathcal{H}, M_*)$ is evidently also compatible. It was proved in [4, Theorem 3.7] using the reiteration

theorem that for $1 < p < 2$, we have $C_{\frac{2}{p}-1}(\mathcal{H}, M_*) = L^p(M)_{\text{left}}$ in the simplest sense that they are equal as vector subspaces of R^* and have the same norm.

Definition 3.1   Let $\mathbb{G}$ be a LCQG. For $1 \leq p \leq \infty$, we define $L^p(\mathbb{G})_{\text{left}}$ to be $L^p(L^\infty(\mathbb{G}))_{\text{left}}$, calculated with respect to the left Haar weight φ. We identify $L^p(\mathbb{G})_{\text{left}}$ with $L^p(\mathbb{G})$ for $p = 1, 2, \infty$.

The following generalization of [4, Theorem 6.4 (i)–(iii)] is proved in the same way, with obvious modifications. For completeness, we give full details. Handling the last part of the theorem, relating convolutions and the Fourier transform on noncommutative L^p-spaces, requires too much background and is not needed in this paper. It is thus left to the reader. A special case of this construction was developed by Forrest, Lee, and Samei [16, Subsection 6.2].

Theorem 3.2   *Let $\mathbb{G}$ be a LCQG, $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ and $1 < p < 2$. Consider the maps $\mu *^1 \in B(L^1(\mathbb{G}))$, $L^1(\mathbb{G}) \ni \omega \mapsto \mu * \omega$, and $\mu *^2 := \lambda^{\mathrm{u}}(\mu) \in B(L^2(\mathbb{G}))$. Then these maps are compatible, and the resulting induced operator $\mu *^p \in B(L^p(\mathbb{G})_{\text{left}})$ satisfies $\|\mu *^p\| \leq \|\mu\|$.*

Proof   Fix $\omega \in \mathcal{I}$. For $\widehat{\omega} \in \widehat{\mathcal{I}}$, write $y := \widehat{\lambda}(\widehat{\omega}) \in C_0(\mathbb{G}) \cap \mathcal{N}_\varphi$, and calculate

$$(\mu * \omega)(y^*) = (\mu \otimes \omega)(\mathbb{W}^*(\mathbb{1} \otimes y^*)\mathbb{W}) = (\mu \otimes \omega)(\mathbb{W}^*(\mathbb{1} \otimes (\mathrm{id} \otimes \overline{\omega})(W))\mathbb{W})$$
$$= (\mu \otimes \omega \otimes \overline{\omega})(\mathbb{W}_{12}^* \mathbb{W}_{23} \mathbb{W}_{12}) = (\mu \otimes \omega \otimes \overline{\omega})(\mathbb{W}_{13}\mathbb{W}_{23})$$
$$= \overline{\omega}[(\mu \otimes \mathrm{id})(\mathbb{W}) \cdot (\omega \otimes \mathrm{id})(\mathbb{W})] = \overline{\omega}[(\lambda^{\mathrm{u}}(\mu)\lambda(\omega))^*]$$
$$= \langle \widehat{\Lambda}(\lambda^{\mathrm{u}}(\mu)\lambda(\omega)), \Lambda(y)\rangle = \langle \lambda^{\mathrm{u}}(\mu)\widehat{\Lambda}(\lambda(\omega)), \Lambda(y)\rangle.$$

As $\widehat{\lambda}(\mathcal{I})$ is a core for Λ, we deduce that $\mu * \omega \in \mathcal{I}$ and $\xi(\mu * \omega) = \lambda^{\mathrm{u}}(\mu)\xi(\omega)$ (a slight generalization of [55, Lemma 4.8]). This means precisely that $\mu *^1$ and $\mu *^2$ are compatible.

Since $C_{\frac{2}{p}-1}(L^2(\mathbb{G}), L^1(\mathbb{G})) = L^p(\mathbb{G})_{\text{left}}$ and since C_θ is an exact interpolation functor of exponent θ, we have the existence of $\mu *^p$, and

$$\|\mu *^p\| \leq \|\mu *^1\|^{1-((2/p)-1)}\|\mu *^2\|^{(2/p)-1} \leq \|\mu\|^{1-((2/p)-1)}\|\mu\|^{(2/p)-1} = \|\mu\|. \qquad \blacksquare$$

Remark 3.3   For $p > 2$ it may be generally impossible to give a proper meaning to $\mu *^p \omega$ when $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ and $\omega \in L^p(\mathbb{G})_{\text{left}}$.

3.1 Duality

For $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, Izumi, generalizing the classical duality of L^p-spaces, proved that $L^p(\mathbb{G})_{\text{left}}^* \cong L^q(\mathbb{G})_{\text{left}}$ via a natural sesquilinear form $(\cdot|\cdot)_p$ over $L^p(\mathbb{G})_{\text{left}} \times L^q(\mathbb{G})_{\text{left}}$ ([24, Theorem 6.1]; as usual, we assume $\alpha = -\frac{1}{2}$ throughout). For $x, y \in L$, we have $(l^p(x)|l^q(y))_p = {}_x\varphi(y^*) = \varphi(y^*x)$ [24, Theorem 2.5].

If $1 < p \leq 2$, $\omega \in \mathcal{I} = L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ and $y \in L$, then $(\omega|l^q(y))_p = \omega(y^*)$. Indeed, endow $\mathcal{I}$ with the natural norm $\|\omega\|_{\mathcal{I}} := \max(\|\omega\|_{L^1(\mathbb{G})}, \|\xi(\omega)\|_{L^2(\mathbb{G})})$, $\omega \in \mathcal{I}$. The embedding $L \hookrightarrow \mathcal{I}$, $L \ni x \mapsto {}_x\varphi$, is contractive with dense range [4, Proposition 3.4].

If (x_n) is a sequence in L such that $_{x_n}\varphi \to \omega$ in $\mathcal{I}$, then $l^p(x_n) \to \omega$ in $L^p(\mathbb{G})_{\text{left}}$, and so $(\omega | l^q(y))_p \leftarrow (l^p(x_n) | l^q(y))_p = {_{x_n}}\varphi(y^*) \to \omega(y^*)$.

4 Comparison of Topologies on the Unit Sphere of $C_0^{\mathrm{u}}(\mathbb{G})^*$

In this section we generalize the main results of Granirer and Leinert [18], and in particular obtain a result (Theorem 4.8) about positive-definite functions over LCQGs extending [40, 57].

Definition 4.1 Let $\mathbb{G}$ be a LCQG. We define several topologies on $C_0^{\mathrm{u}}(\mathbb{G})^*$ as follows.

(i) The *strict topology* is the one induced by the semi-norms $\mu \mapsto \|\omega * \mu\|_{L^1(\mathbb{G})}$ and $\mu \mapsto \|\mu * \omega\|_{L^1(\mathbb{G})}$, $\omega \in L^1(\mathbb{G})$.

(ii) For $p \in [1, 2]$, the *p-strict topology* is the one induced by the semi-norms $\mu \mapsto \|\mu *^p \omega\|_p$, $\omega \in L^p(\mathbb{G})_{\text{left}}$.

(iii) For $p \in [1, 2]$, a net (μ_β) in $C_0^{\mathrm{u}}(\mathbb{G})^*$ converges to $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ in the *weak p-strict topology* if $\mu_\beta *^p \omega \to \mu *^p \omega$ in the *w*-topology $\sigma(L^p(\mathbb{G})_{\text{left}}, L^p(\mathbb{G})^*_{\text{left}})$ for every $\omega \in L^p(\mathbb{G})_{\text{left}}$.

(iv) A net (μ_β) in $C_0^{\mathrm{u}}(\mathbb{G})^*$ converges to $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ in τ_{nw^*} if $\mu_\beta \xrightarrow{w^*} \mu$ and $\|\mu_\beta\| \to \|\mu\|$.

(v) A net (μ_β) in $C_0^{\mathrm{u}}(\mathbb{G})^*$ converges to $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ in τ_{bw^*} if $\mu_\beta \xrightarrow{w^*} \mu$ and (μ_β) is bounded.

We now generalize [18, Theorem A], answering affirmatively a question raised by Hu, Neufang, and Ruan [22, p. 140].

Theorem 4.2 *Let $\mathbb{G}$ be a LCQG. On $C_0^{\mathrm{u}}(\mathbb{G})^*$, the strict topology is weaker than τ_{nw^*}.*

Lemma 4.3 *Let A be a C^*-algebra and (e_α) be an approximate identity for A. Let (μ_β) be a net in A^* and $\mu \in A^*$ be such that $\mu_\beta \xrightarrow{w^*} \mu$ and $\|\mu_\beta\| \to \|\mu\|$. Then for every $\varepsilon > 0$ there are α_0, β_0 such that $\|e_{\alpha_0}\mu_\beta - \mu_\beta\| < \varepsilon$ (resp., $\|\mu_\beta e_{\alpha_0} - \mu_\beta\| < \varepsilon$) for every $\beta \geq \beta_0$ and $\|e_{\alpha_0}\mu - \mu\| < \varepsilon$ (resp., $\|\mu - \mu e_{\alpha_0}\| < \varepsilon$).*

Proof If M is a von Neumann algebra (e.g., A^{**}), recall that the "absolute value" of $v \in M_*$ can be defined in two ways, as the unique $|v| \in M_*^+$ with $\||v|\| = \|v\|$ satisfying either $|v(x)|^2 \leq \|v\| \cdot |v|(x^*x)$ or $|v(x)|^2 \leq \|v\| \cdot |v|(xx^*)$ for all $x \in M$. We will use the first way to establish half of the lemma's assertion, the other half being established similarly using the second way.

For every $v \in A^*$ and $a \in A$ we have, writing $\mathbb{1}$ for $\mathbb{1}_{M(A)}$,

$$|(v - e_\alpha v)(a)|^2 = |v(a(\mathbb{1} - e_\alpha))|^2 \leq \|v\| |v|[(\mathbb{1} - e_\alpha)a^*a(\mathbb{1} - e_\alpha)]$$
$$\leq \|v\| \|a\|^2 |v|((\mathbb{1} - e_\alpha)^2) \leq \|v\| \|a\|^2 |v|(\mathbb{1} - e_\alpha).$$

Hence $\|v - e_\alpha v\|^2 \leq \|v\| |v|(\mathbb{1} - e_\alpha)$. Since (e_α) is an approximate identity for A, we have $|v|(\mathbb{1} - e_\alpha) \to 0$ by strict continuity. Let α_0 be such that $\|\mu\| |\mu|(\mathbb{1} - e_{\alpha_0}) < \varepsilon^2$.

Since $\mu_\beta \xrightarrow{w^*} \mu$ and $\|\mu_\beta\| \to \|\mu\|$, we have $|\mu_\beta| \xrightarrow{w^*} |\mu|$ (see Effros [11, Lemma 3.5] or [45, Proposition III.4.11]). Therefore,

$$\|\mu_\beta - e_{\alpha_0}\mu_\beta\|^2 \le \|\mu_\beta\| |\mu_\beta|(\mathbb{1} - e_{\alpha_0}) \xrightarrow[\beta]{} \|\mu\| |\mu|(\mathbb{1} - e_{\alpha_0}) < \varepsilon^2,$$

so we can choose β_0 as asserted. ∎

Lemma 4.4   *Let $a, b \in C_0^{\mathrm{u}}(\mathbb{G})$. The map*

$$(C_0^{\mathrm{u}}(\mathbb{G})^*, \tau_{bw^*}) \to (C_0^{\mathrm{u}}(\mathbb{G})^*, \text{strict topology})$$

given by $\mu \mapsto a\mu b$ is continuous.

Proof   Let (μ_β) be a bounded net in $C_0^{\mathrm{u}}(\mathbb{G})^*$ and $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ be such that $\mu_\beta \xrightarrow{w^*} \mu$. Representing $C_0^{\mathrm{u}}(\mathbb{G})$ faithfully on a Hilbert space $\mathcal{H}_{\mathrm{u}}$, we view the operator $\mathbb{W} \in M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} C_0(\widehat{\mathbb{G}}))$ as an element of $B(\mathcal{H}_{\mathrm{u}} \otimes L^2(\mathbb{G}))$. Recall [29, Proposition 8.3 and its proof] that for every $\nu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ and $\omega \in C_0(\mathbb{G})^*$, the functional $\nu * \omega \in C_0^{\mathrm{u}}(\mathbb{G})^*$ corresponds to the element of $C_0(\mathbb{G})^*$ given by

$$C_0(\mathbb{G}) \ni x \mapsto (\nu \otimes \omega)(\mathbb{W}^*(\mathbb{1} \otimes x)\mathbb{W}),$$

which makes sense because $\mathbb{W}^*(1 \otimes x)\mathbb{W} \in M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} C_0(\mathbb{G}))$.

Fix $\omega \in L^1(\mathbb{G})$, write $\omega = \omega_{\zeta,\eta}$ for $\zeta, \eta \in L^2(\mathbb{G})$ (this is possible as $L^\infty(\mathbb{G})$ is in standard form on $L^2(\mathbb{G})$), and let $e_\zeta, e_\eta \in \mathcal{K}(L^2(\mathbb{G}))$ be the projections of $L^2(\mathbb{G})$ onto $\mathbb{C}\zeta, \mathbb{C}\eta$, respectively. Then for $\nu \in C_0^{\mathrm{u}}(\mathbb{G})^*$, the functional $(a\nu b) * \omega$ corresponds to

$$C_0(\mathbb{G}) \ni x \mapsto (\nu \otimes \omega_{\zeta,\eta})\big((b \otimes \mathbb{1})\mathbb{W}^*(\mathbb{1} \otimes x)\mathbb{W}(a \otimes \mathbb{1})\big)$$
$$= (\nu \otimes \omega_{\zeta,\eta})\big((b \otimes e_\eta)\mathbb{W}^*(\mathbb{1} \otimes x)\mathbb{W}(a \otimes e_\zeta)\big).$$

Since $\mathbb{W} \in M\big(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} \mathcal{K}(L^2(\mathbb{G}))\big)$, both $\mathbb{W}(a \otimes e_\zeta)$ and $(b \otimes e_\eta)\mathbb{W}^*$ belong to $C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} \mathcal{K}(L^2(\mathbb{G}))$. As a result, approximating them in norm by elements of the corresponding algebraic tensor product, we see that (μ_β) being bounded and the fact that $\mu_\beta \xrightarrow{w^*} \mu$ imply that $(a\mu_\beta b) * \omega \xrightarrow{\|\cdot\|} (a\mu b) * \omega$. By using the universal version of the unitary antipode $R_{\mathrm{u}} \colon C_0^{\mathrm{u}}(\mathbb{G}) \to C_0^{\mathrm{u}}(\mathbb{G})$ and its properties [29, Proposition 7.2], we conclude that also $\omega * (a\mu_\beta b) \xrightarrow{\|\cdot\|} \omega * (a\mu b)$. ∎

Proof of Theorem 4.2   Let (μ_β) be a net in $C_0^{\mathrm{u}}(\mathbb{G})^*$ and $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ be such that $\mu_\beta \xrightarrow{nw^*} \mu$, and let $\omega \in L^1(\mathbb{G})$ and $\varepsilon > 0$. Fix an approximate identity (e_α) for $C_0^{\mathrm{u}}(\mathbb{G})$. By invoking Lemma 4.3 twice, we find $\alpha_1, \alpha_2, \beta_1$ such that $\|e_{\alpha_1}\mu_\beta e_{\alpha_2} - \mu_\beta\| < \varepsilon$ for every $\beta \ge \beta_1$ and $\|e_{\alpha_1}\mu e_{\alpha_2} - \mu\| < \varepsilon$. From Lemma 4.4, there is β_2 such that

$$\big\|(e_{\alpha_1}\mu_\beta e_{\alpha_2}) * \omega - (e_{\alpha_1}\mu e_{\alpha_2}) * \omega\big\|, \big\|\omega * (e_{\alpha_1}\mu_\beta e_{\alpha_2}) - \omega * (e_{\alpha_1}\mu e_{\alpha_2})\big\| < \varepsilon$$

for every $\beta \ge \beta_2$. We conclude that the strict topology is weaker than τ_{nw^*}. ∎

We now generalize most of [18, Theorem D] for $1 \le p \le 2$.

Corollary 4.5   *Let $1 \le p \le 2$. On $C_0^{\mathrm{u}}(\mathbb{G})^*$ the p-strict topology is weaker than τ_{nw^*}, and on bounded sets, the w^*-topology is weaker than the weak p-strict topology.*

Proof Let (μ_β) be a net in $C_0^{\mathrm{u}}(\mathbb{G})^*$ and $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$. We use Theorem 3.2 and its notation. Let $\mu_\beta \xrightarrow{nw^*} \mu$. Let $\omega \in \mathcal{I}$ and $\xi := \xi(\omega)$ (so $\omega = \xi$ in R^*). By Theorem 4.2, $(\mu_\beta - \mu) *^1 \omega \to 0$ in $L^1(\mathbb{G})$. Moreover, $(\mu_\beta - \mu) *^2 \xi = \lambda^{\mathrm{u}}(\mu_\beta - \mu)\xi \to 0$ in $L^2(\mathbb{G})$ (see Theorem 4.6, (vii) $\Rightarrow$ (ii) below). Since the canonical embedding $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \hookrightarrow (L^p(\mathbb{G})_{\mathrm{left}}, \|\cdot\|_p)$ is contractive, we infer that $(\mu_\beta - \mu) *^p \omega \to 0$ in $L^p(\mathbb{G})_{\mathrm{left}}$. That embedding has dense range and $((\mu_\beta - \mu)*^p)_\beta$ is bounded in $B(L^p(\mathbb{G})_{\mathrm{left}})$; hence $(\mu_\beta - \mu) *^p \omega \to 0$ for all $\omega \in L^p(\mathbb{G})_{\mathrm{left}}$.

For the second statement, suppose that (μ_β) is bounded and that $\mu_\beta \to \mu$ in the weak p-strict topology. We claim that $(\mu_\beta - \mu) * \omega \to 0$ in the w^*-topology for every $\omega \in L^1(\mathbb{G})$. Assume for the moment that $p > 1$ and let $q \in [2, \infty)$ be the conjugate of p. Let $\omega \in \mathcal{I}$ and $y \in L$. If $(\mu_\beta - \mu) *^p \omega \to 0$ weakly, then by Subsection 3.1, we have

$$(4.1) \qquad \big((\mu_\beta - \mu) * \omega\big)(y^*) = \big((\mu_\beta - \mu) *^p \omega | l^q(y)\big)_p \to 0.$$

Denoting by $\mathcal{T}_\varphi$ the Tomita algebra of φ, the set $\{ ab : a, b \in \mathcal{T}_\varphi \}$ is contained in L by [23, Proposition 2.3]. As $\varphi|_{C_0(\mathbb{G})_+}$ is a C^*-algebraic KMS weight on $C_0(\mathbb{G})$ whose modular automorphism group is the restriction of that of φ to $C_0(\mathbb{G})$ [31, Proposition 1.6 and its proof], $\mathcal{T}_\varphi \cap C_0(\mathbb{G})$ is norm dense in $C_0(\mathbb{G})$. Hence $L \cap C_0(\mathbb{G})$ is norm dense in $C_0(\mathbb{G})$, and $\pi_{\mathrm{u}}^{-1}(L \cap C_0(\mathbb{G}))$ is norm dense in $C_0^{\mathrm{u}}(\mathbb{G})$. Consequently, (4.1) implies that as elements of $C_0^{\mathrm{u}}(\mathbb{G})^*$, $(\mu_\beta - \mu) * \omega \to 0$ pointwise on a norm dense subset of $C_0^{\mathrm{u}}(\mathbb{G})$, which by the boundedness of (μ_β) implies that $(\mu_\beta - \mu) * \omega \to 0$ in the w^*-topology. By density of $\mathcal{I}$ in $L^1(\mathbb{G})$ and boundedness again, this holds for every $\omega \in L^1(\mathbb{G})$ as claimed. In the case that $p = 1$ we have the same result, since the assumption that $(\mu_\beta - \mu) * \omega \to 0$ in the w-topology $\sigma(L^1(\mathbb{G}), L^\infty(\mathbb{G}))$ is formally stronger.

Since $\{ (\mathrm{id} \otimes \omega)(\mathrm{W}^*(\mathbb{1} \otimes b)\mathrm{W}) : \omega \in L^1(\mathbb{G}), b \in C_0(\mathbb{G}) \}$ is dense in $C_0^{\mathrm{u}}(\mathbb{G})$ and (μ_β) is bounded, we infer from the claim that $\mu_\beta \to \mu$ in the w^*-topology. $\blacksquare$

Let G be a locally compact group. If (g_β) is a bounded net in $B(G)$ and $g \in B(G)$, then $g_\beta \to g$ uniformly on the compact subsets of G if and only if $fg_\beta \to fg$ in the $C_0(G)$ norm for every $f \in C_0(G)$. Indeed, one direction is trivial, and for the other, notice that (g_β) is bounded in $C_b(G)$ since $\|\cdot\|_{C_b(G)} \leq \|\cdot\|_{B(G)}$. Hence, the following result generalizes [18, Theorem B_2].

Theorem 4.6   *Let $\mathbb{G}$ be a LCQG and let S denote the unit sphere of $C_0^{\mathrm{u}}(\mathbb{G})^*$. If (μ_β) is a net in S and $\mu \in S$, then the following are equivalent:*

(i) $\mu_\beta \to \mu$ *in the w^*-topology,*

(ii) $\lambda^{\mathrm{u}}(\mu_\beta) \to \lambda^{\mathrm{u}}(\mu)$ *in the strict topology on $M(C_0(\widehat{\mathbb{G}}))$,*

(iii) $\mu_\beta \cdot a \to \mu \cdot a$ *and* $a \cdot \mu_\beta \to a \cdot \mu$ *in $C_0(\mathbb{G})$ for every $a \in C_0(\mathbb{G})$ (see Lemma 1.3),*

(iv) $\mu_\beta \cdot a \to \mu \cdot a$ *and* $a \cdot \mu_\beta \to a \cdot \mu$ *in the w^*-topology $\sigma(L^\infty(\mathbb{G}), L^1(\mathbb{G}))$ for every $a \in L^\infty(\mathbb{G})$, that is: $\mu_\beta * \omega \to \mu * \omega$ and $\omega * \mu_\beta \to \omega * \mu$ in the w-topology $\sigma(L^1(\mathbb{G}), L^\infty(\mathbb{G}))$ for every $\omega \in L^1(\mathbb{G})$,*

(v) $(\mu_\beta * \omega)a \to (\mu * \omega)a$ *and* $a(\mu_\beta * \omega) \to a(\mu * \omega)$ *in $L^1(\mathbb{G})$ for every $a \in C_0(\mathbb{G})$, $\omega \in L^1(\mathbb{G})$,*

(vi) *for every $a \in C_0(\mathbb{G})$ and $\omega \in L^1(\mathbb{G})$, $(\mu_\beta * \omega)a \to (\mu * \omega)a$ and $a(\mu_\beta * \omega) \to a(\mu * \omega)$ in the w-topology $\sigma(L^1(\mathbb{G}), L^\infty(\mathbb{G}))$,*

(vii) $\mu_\beta \to \mu$ *in the strict topology,*
(viii) *for some* $1 \le p \le 2$, $\mu_\beta \to \mu$ *in the p-strict topology,*
(ix) *for some* $1 \le p \le 2$, $\mu_\beta \to \mu$ *in the weak p-strict topology.*

Proof From Theorem 4.2 and Corollary 4.5, conditions (i), (iv), (vii), (viii), and (ix) are equivalent. It is clear that ((vii) $\Rightarrow$ (v) $\Rightarrow$ (vi). (vii) $\Rightarrow$ (ii). Since λ^u is a homomorphism, we have $\lambda^u(\mu_\beta)\lambda(\omega) \to \lambda^u(\mu)\lambda(\omega)$ and $\lambda(\omega)\lambda^u(\mu_\beta) \to \lambda(\omega)\lambda^u(\mu)$ for every $\omega \in L^1(\mathbb{G})$. As $\{\lambda(\omega) : \omega \in L^1(\mathbb{G})\}$ is norm dense in $C_0(\widehat{\mathbb{G}})$ and $(\lambda^u(\mu_\beta))$ is bounded, we conclude that $\lambda^u(\mu_\beta) \to \lambda^u(\mu)$ in the strict topology on $M(C_0(\widehat{\mathbb{G}}))$.

(ii) $\Rightarrow$ (i). Since $\lambda^u(\mu_\beta) \to \lambda^u(\mu)$ in the strict topology on $M(C_0(\widehat{\mathbb{G}}))$ and $(\lambda^u(\mu_\beta))$ is bounded, this convergence holds in the ultraweak topology as well. So for all $\widehat{\omega} \in L^1(\widehat{\mathbb{G}})$, we have $(\mu_\beta - \mu)((\mathrm{id} \otimes \widehat{\omega})(\mathbb{W})) = \widehat{\omega}(\lambda^u(\mu_\beta - \mu)) \to 0$. As $\{(\mathrm{id} \otimes \widehat{\omega})(\mathbb{W}) : \widehat{\omega} \in L^1(\widehat{\mathbb{G}})\}$ is dense in $C_0^u(\mathbb{G})$ and (μ_β) is bounded, we infer that $\mu_\beta \to \mu$ in the w^*-topology.

(vii) $\Rightarrow$ (iii). We may assume that $a = \omega \cdot b$ for some $\omega \in L^1(\mathbb{G})$ and $b \in C_0(\mathbb{G})$, because the set of these elements spans a dense subset of $C_0(\mathbb{G})$. Hence

$$(\mu_\beta - \mu) \cdot a = (\mu_\beta - \mu) \cdot (\omega \cdot b) = ((\mu_\beta - \mu) * \omega) \cdot b \to 0,$$

and similarly $a \cdot (\mu_\beta - \mu) \to 0$.

The proofs of (iii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i) are left to the reader (see the proof of Corollary 4.5, and use that $C_0(\mathbb{G})^2 = C_0(\mathbb{G})$). ∎

Remark 4.7 In view of Theorem 4.6, the following is noteworthy. Let $\mathbb{G}$ be a compact quantum group. Generalizing a classical result about discrete groups, Kyed [32, Theorem 3.1] proved that the discrete dual $\widehat{\mathbb{G}}$ has property (T) if and only if every net of states of $C^u(\mathbb{G})$, converging in the w^*-topology to the co-unit, converges in norm.

A classical result [40, 57] says that if G is a locally compact group, then on the set of positive-definite functions of $L^\infty(G)$-norm 1, the w^*-topology $\sigma(L^\infty(G), L^1(G))$ and the topology of uniform convergence on compact subsets coincide. The following generalizes this to LCQGs.

Theorem 4.8 *Assume that* $\mathbb{G}$ *is a co-amenable LCQG. On the subset S of* $M(C_0(\mathbb{G}))$ *consisting of all positive-definite elements of norm* 1*, the strict topology induced by* $C_0(\mathbb{G})$ *coincides with the weak and the strong operator topologies on* $L^2(\mathbb{G})$.

Proof By co-amenability, the map $\widehat{\mu} \mapsto \widehat{\lambda}^u(\widehat{\mu})$ is an isometric isomorphism between the unit sphere of $C_0^u(\widehat{\mathbb{G}})_+^*$ and S (Theorem 1.5 and Remark 1.6). Now apply Theorem 4.6, (i) if and only if (ii), to $\widehat{\mathbb{G}}$ in place of $\mathbb{G}$, and notice that for a bounded net in $C_0^u(\widehat{\mathbb{G}})^*$, w^*-convergence is equivalent to convergence in the weak operator topology of its image under $\widehat{\lambda}^u$. Moreover, on bounded sets the strict topology on $M(C_0(\mathbb{G}))$ is finer than the strong operator topology. ∎

Remark 4.9 Attempting to prove Theorem 4.8 by generalizing the proof of [10, Theorem 13.5.2] was only partially successful: we were able to establish that on S, the weak operator topology coincides with the topology on $M(C_0(\mathbb{G}))$ in which a net

(x_β) converges to x if and only if $y x_\beta z \to y x z$ for every $y, z \in C_0(\mathbb{G})$. This topology evidently coincides with the strict one when $\mathbb{G}$ is commutative, but not generally.

However, it is worth noting that taking this approach, one encounters a straightforward generalization of a very useful inequality, namely that if φ is a (continuous) positive definite-function on a locally compact group G, then $|\varphi(s) - \varphi(t)|^2 \le 2\varphi(e)(\varphi(e) - \operatorname{Re}\varphi(s^{-1}t))$ for every $s, t \in G$ [10, Proposition 13.4.7]. As $\varphi(s^{-1}) = \overline{\varphi(s)}$, this is equivalent to $|\varphi(st) - \varphi(t)|^2 \le 2\varphi(e)(\varphi(e) - \operatorname{Re}\varphi(s))$ for every $s, t \in G$. If $\mathbb{G}$ is a co-amenable LCQG and y is positive definite over $\mathbb{G}$, write $y = (\operatorname{id} \otimes \widehat{\mu})(W^*)$ for a suitable $\widehat{\mu} \in C_0^{\mathrm{u}}(\widehat{\mathbb{G}})_+^*$. Now $\Delta(y) - \mathbb{1} \otimes y = (\operatorname{id} \otimes \operatorname{id} \otimes \widehat{\mu})(W_{23}^*(W_{13}^* - \mathbb{1}))$, and as $\operatorname{id} \otimes \operatorname{id} \otimes \widehat{\mu}$ is a completely positive map of cb-norm $\|\widehat{\mu}\| = \|y\|$, the Kadison–Schwarz inequality implies that

$$(4.2) \quad [\Delta(y) - \mathbb{1} \otimes y]^*[\Delta(y) - \mathbb{1} \otimes y]$$
$$\le \|y\|(\operatorname{id} \otimes \operatorname{id} \otimes \widehat{\mu})((W_{13} - \mathbb{1})W_{23}W_{23}^*(W_{13}^* - \mathbb{1}))$$
$$= \|y\|(\operatorname{id} \otimes \operatorname{id} \otimes \widehat{\mu})(2\mathbb{1} - W_{13} - W_{13}^*)$$
$$= \|y\|[2\|y\|\mathbb{1} - (y^* + y)] \otimes \mathbb{1}.$$

5 A Characterization of Co-amenability of the Dual

Related to the notion of a positive-definite function is the notion of a (generally unbounded) positive-definite measure ([17], [10, Section 13.7]). The purpose of this section is to generalize a classical result of Godement connecting amenability to positive definiteness ([10, Proposition 18.3.6], originally [17, pp. 76–77], see also Valette [53]).

Definition 5.1 An element $\mu \in C_0(\mathbb{G})^*$ is called a *bounded positive-definite measure on $\mathbb{G}$* if $\lambda(\mu)$ is positive in $M(C_0(\widehat{\mathbb{G}}))$.

Theorem 5.2 *Let $\mathbb{G}$ be a co-amenable LCQG. The following conditions are equivalent.*

(i) $\widehat{\mathbb{G}}$ *is co-amenable.*
(ii) *Every positive-definite function on $\mathbb{G}$ is the strict limit in $M(C_0(\mathbb{G}))$ of a bounded net of positive-definite functions in $\widehat{\lambda}(L^1(\widehat{\mathbb{G}})_+) \cap \mathcal{N}_\varphi$.*
(iii) *Every positive-definite function on $\mathbb{G}$ is the strict limit in $M(C_0(\mathbb{G}))$ of a bounded net of positive-definite functions in $\widehat{\lambda}(L^1(\widehat{\mathbb{G}})_+)$.*
(iv) $\mu(x^*) \ge 0$ *for every bounded positive-definite measure μ on $\mathbb{G}$ and every positive-definite function x on $\mathbb{G}$.*
(v) $\mu(\mathbb{1}_{M(C_0(\mathbb{G}))}) \ge 0$ *for every bounded positive-definite measure μ on $\mathbb{G}$.*

Lemma 5.3 *Let $\mathbb{G}$ be a co-amenable LCQG. Then the cone $Q := \widehat{\lambda}(C_0(\widehat{\mathbb{G}})_+^*)$ is ultraweakly closed in $L^\infty(\mathbb{G})$.*

Proof By the Krein–Šmulian theorem, it suffices to prove that Q_1, the intersection of Q with the closed unit ball of $L^\infty(\mathbb{G})$, is ultraweakly closed. Let (x_α) be a net in Q_1 converging ultraweakly to some $x \in L^\infty(\mathbb{G})$. Write $x_\alpha = \widehat{\lambda}(\widehat{\mu}_\alpha)$, $\widehat{\mu}_\alpha \in C_0(\widehat{\mathbb{G}})_+^*$, for every α. By Remark 1.6, $(\widehat{\mu}_\alpha)$ is bounded by one, and so it has a subnet converging in the w^*-topology to some $\widehat{\mu} \in C_0(\widehat{\mathbb{G}})_+^*$. Hence $x = \widehat{\lambda}^{\mathrm{u}}(\widehat{\mu}) \in Q_1$. ∎

Proof of Theorem 5.2 (i) $\Rightarrow$ (ii): every positive-definite function has the form $\widehat{\lambda}(\widehat{\nu})$ for some $\widehat{\nu} \in C_0^{\mathrm{u}}(\widehat{\mathbb{G}})_+^* = C_0(\widehat{\mathbb{G}})_+^*$ by co-amenability of $\widehat{\mathbb{G}}$ (Theorem 1.5). Now $\widehat{\nu}$ is the w^*-limit of a bounded net $(\widehat{\omega}_\beta)$ in $L^1(\widehat{\mathbb{G}})_+$. Since each element of $L^1(\widehat{\mathbb{G}})_+$ can be approximated in norm by elements of $\widehat{\mathcal{I}}_+$ of the same norm [55, Lemma 4.7], we may assume that $\widehat{\omega}_\beta \in \widehat{\mathcal{I}}$, and hence $\widehat{\lambda}(\widehat{\omega}_\beta) \in \mathcal{N}_\varphi$, for every β. From Theorem 4.6 applied to $\widehat{\mathbb{G}}$, we infer that $\widehat{\lambda}(\widehat{\omega}_\beta) \to \widehat{\lambda}(\widehat{\nu})$ strictly in $M(C_0(\mathbb{G}))$.

(ii) $\Rightarrow$ (iii): clear.

(iii) $\Rightarrow$ (iv): let μ be a bounded positive-definite measure on $\mathbb{G}$. For every $\widehat{\omega} \in L^1(\widehat{\mathbb{G}})_+$,

$$\overline{\mu}\big(\widehat{\lambda}(\widehat{\omega})\big) = (\overline{\mu} \otimes \widehat{\omega})(W^*) = \widehat{\omega}(\lambda(\mu)^*) \geq 0.$$

If x is a positive-definite function on $\mathbb{G}$ and $(\widehat{\omega}_\beta)$ is a net in $L^1(\widehat{\mathbb{G}})_+$ such that $\widehat{\lambda}(\widehat{\omega}_\beta) \to x$ strictly in $M(C_0(\mathbb{G}))$, then $\overline{\mu}\big(\widehat{\lambda}(\widehat{\omega}_\beta)\big) \to \overline{\mu}(x) = \overline{\mu(x^*)}$. Hence $\overline{\mu(x^*)}$, or equivalently $\mu(x^*)$, is non-negative.

(iv) $\Rightarrow$ (v): trivial, as $\mathbb{1} := \mathbb{1}_{M(C_0(\mathbb{G}))} = \widehat{\lambda}^{\mathrm{u}}(\widehat{\epsilon})$ is positive definite.

(v) $\Rightarrow$ (i): as $\widehat{\lambda}^{\mathrm{u}}$ is injective, we should establish that $\mathbb{1}$ belongs to Q. By Lemma 5.3, Q is an ultraweakly closed cone, so it is enough to show that $\mathbb{1}$ belongs to the bipolar of Q. Here we are using the version of the bipolar theorem in which the pre-polar of Q is given by $Q_\circ := \big\{ \omega \in L^1(\mathbb{G}) : (\forall x \in Q) \quad 0 \leq \mathrm{Re}\, \omega(x) \big\}$, and its polar is defined similarly. Note that Q is invariant under the scaling group, as $\tau_t\big(\widehat{\lambda}(\widehat{\mu})\big) = \widehat{\lambda}(\widehat{\mu} \circ \widehat{\tau}_{-t})$ for every $\widehat{\mu} \in C_0(\widehat{\mathbb{G}})^*$, $t \in \mathbb{R}$ [30, Propositions 8.23 and 8.25]. Consequently,

$$V := Q_\circ \cap D\big((\tau_*)_{-i/2}\big)$$

is norm dense in $Q_\circ$ by a standard smearing argument (see [30, proof of Proposition 5.26]). So picking $\omega_0 \in V$, we should show that $0 \leq \mathrm{Re}\, \omega_0(\mathbb{1})$. For every $\widehat{\nu} \in C_0(\widehat{\mathbb{G}})_+^*$ we have

$$0 \leq \mathrm{Re}\, \omega_0\big(\widehat{\lambda}(\widehat{\nu})\big) = \mathrm{Re}\, \overline{\omega_0\big(\widehat{\lambda}(\widehat{\nu})\big)} = \mathrm{Re}\big(\overline{\omega_0} \otimes \widehat{\nu}\big)(W) = \mathrm{Re}\, \widehat{\nu}\big(\lambda(\overline{\omega_0})\big).$$

Thus $0 \leq \lambda(\overline{\omega_0}) + \lambda(\overline{\omega_0})^* = \lambda(\overline{\omega_0} + \overline{\omega_0}^*)$, that is, $\overline{\omega_0} + \overline{\omega_0}^*$ as an element of $L^1(\mathbb{G}) \hookrightarrow C_0(\mathbb{G})^*$, is a bounded positive-definite measure. By assumption, $0 \leq (\overline{\omega_0} + \overline{\omega_0}^*)(\mathbb{1}) = (\overline{\omega_0} + \omega_0)(\mathbb{1}) = 2\,\mathrm{Re}\, \omega_0(\mathbb{1})$ as $\mathbb{1} \in D(S)$ and $S(\mathbb{1}) = \mathbb{1}$. In conclusion, $\mathbb{1}$ belongs to the bipolar of Q. ∎

6 The Separation Property

6.1 Preliminaries

Definition 6.1 (Lau and Losert [34], Kaniuth and Lau [26]) Let G be a locally compact group and H be a closed subgroup of G. We say that G has the *H-separation property* if for every $g \in G \backslash H$ there exists a positive-definite function φ on G with $\varphi|_H \equiv 1$ but $\varphi(g) \neq 1$.

It was first observed in [34] that G has the H-separation property if H is either normal, compact or open. Generalizing a result of Forrest [15], it was proved that

G has the H-separation property provided that G has small H-invariant neighborhoods [26, Proposition 2.2]. The property was subsequently explored further in several papers, including [27, 28]. It is somewhat related to another property connecting positive-definite functions and closed subgroups, namely the extension property.

In this section we introduce the separation property for LCQGs and obtain a first result about it. To this end, we continue with some background on closed quantum subgroups of LCQGs. To simplify the notation a little, throughout this section we will use π for the surjection $\pi_{\mathrm{u}} \colon C_0^{\mathrm{u}}(\mathbb{G}) \to C_0(\mathbb{G})$, $\mathbb{G}$ being a LCQG.

Definition 6.2 (Meyer, Roy, and Woronowicz [36])   Let $\mathbb{G}, \mathbb{H}$ be LCQGs. A *strong quantum homomorphism from $\mathbb{H}$ to $\mathbb{G}$* is a nondegenerate $*$-homomorphism $\Phi \colon C_0^{\mathrm{u}}(\mathbb{G}) \to M(C_0^{\mathrm{u}}(\mathbb{H}))$ such that $(\Phi \otimes \Phi) \circ \Delta_{\mathbb{G}}^{\mathrm{u}} = \Delta_{\mathbb{H}}^{\mathrm{u}} \circ \Phi$.

Every such Φ has a dual object [36, Proposition 3.9 and Theorem 4.8], which is the (unique) strong quantum homomorphism $\widehat{\Phi}$ from $\widehat{\mathbb{G}}$ to $\widehat{\mathbb{H}}$ that satisfies

$$(6.1) \qquad\qquad (\Phi \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) = (\mathrm{id} \otimes \widehat{\Phi})(\mathbb{W}_{\mathbb{H}}).$$

(Here and in the sequel we use the left version of this theory, in contrast to [7, 36], which use the right one.) As is customary, we will also write Φ for its unique extension to a $*$-homomorphism $M(C_0^{\mathrm{u}}(\mathbb{G})) \to M(C_0^{\mathrm{u}}(\mathbb{H}))$.

Definition 6.3 (Daws, Kasprzak, Skalski, and Sołtan [7, Definitions 3.1, 3.2 and Theorems 3.3, 3.6])   Let $\mathbb{G}, \mathbb{H}$ be LCQGs.

(i)   We say that $\mathbb{H}$ is a *closed quantum subgroup of $\mathbb{G}$ in the sense of Vaes* if there exists a faithful normal $*$-homomorphism $\gamma \colon L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}})$ such that $(\gamma \otimes \gamma) \circ \Delta_{\widehat{\mathbb{H}}} = \Delta_{\widehat{\mathbb{G}}} \circ \gamma$.

(ii)   We say that $\mathbb{H}$ is a *closed quantum subgroup of $\mathbb{G}$ in the sense of Woronowicz* if there exists a strong quantum homomorphism Φ from $\mathbb{H}$ to $\mathbb{G}$ such that $\Phi(C_0^{\mathrm{u}}(\mathbb{G})) = C_0^{\mathrm{u}}(\mathbb{H})$.

A fundamental result [7, Theorem 3.5] is that if $\mathbb{H}$ is a closed quantum subgroup of $\mathbb{G}$ in the sense of Vaes, then it is also a closed quantum subgroup of $\mathbb{G}$ in the sense of Woronowicz. In this case, the maps γ and Φ are related by the identity $\gamma|_{C_0(\widehat{\mathbb{H}})} \circ \pi_{\widehat{\mathbb{H}}} = \pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi}$. The converse is true if $\mathbb{G}$ is either commutative, co-commutative or discrete, or if $\mathbb{H}$ is compact [7, Sections 4–6].

6.2   The Separation Property for LCQGs

Definition 6.4   Let $\mathbb{G}$ be a LCQG and $\mathbb{H}$ be a closed quantum subgroup of $\mathbb{G}$ in the sense of Woronowicz via a strong quantum homomorphism $\Phi \colon C_0^{\mathrm{u}}(\mathbb{G}) \to C_0^{\mathrm{u}}(\mathbb{H})$. We say that $\mathbb{G}$ has the $\mathbb{H}$-*separation property* if whenever $\mu \in C_0^{\mathrm{u}}(\mathbb{G})_+^*$ is a state such that $(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \notin \widehat{\Phi}(M(C_0^{\mathrm{u}}(\widehat{\mathbb{H}})))$, there is $\widehat{\omega} \in C_0^{\mathrm{u}}(\widehat{\mathbb{G}})_+^*$ so that $\Phi((\mathrm{id} \otimes \widehat{\omega})(\mathbb{W}_{\mathbb{G}})) = \mathbb{1}_{M(C_0^{\mathrm{u}}(\mathbb{H}))}$ but $\mu((\mathrm{id} \otimes \widehat{\omega})(\mathbb{W}_{\mathbb{G}})) \neq 1$.

If $\mathbb{G}$ (thus $\mathbb{H}$) is commutative, this definition reduces to the classical one. Generally, for $\widehat{\omega} \in C_0^{\mathrm{u}}(\widehat{\mathbb{G}})_+^*$, note that $\Phi((\mathrm{id} \otimes \widehat{\omega})(\mathbb{W}_{\mathbb{G}})) = (\mathrm{id} \otimes (\widehat{\omega} \circ \widehat{\Phi}))(\mathbb{W}_{\mathbb{H}})$ by (6.1) and

$(\mathrm{id} \otimes \widehat{\epsilon}_{\mathbb{H}})(\mathbb{W}_{\mathbb{H}}) = \mathbb{1}_{M(C_0^u(\mathbb{H}))}$, hence the equality $\Phi((\mathrm{id} \otimes \widehat{\omega})(\mathbb{W}_{\mathbb{G}})) = \mathbb{1}_{M(C_0^u(\mathbb{H}))}$ is equivalent to $\widehat{\omega} \circ \widehat{\Phi} = \widehat{\epsilon}_{\mathbb{H}}$.

Theorem 6.5 *Let $\mathbb{G}$ be a LCQG and $\mathbb{H}$ a compact quantum subgroup of $\mathbb{G}$. Let $\widehat{p}$ be the central minimal projection in $\ell^{\infty}(\widehat{\mathbb{H}})$ with $\widehat{a}\widehat{p} = \widehat{\epsilon}_{\mathbb{H}}(\widehat{a})\widehat{p} = \widehat{p}\widehat{a}$ for every $\widehat{a} \in \ell^{\infty}(\widehat{\mathbb{H}})$, and assume that the following condition holds:*

(6.2)   *for every $\widehat{z} \in M(C_0(\widehat{\mathbb{G}}))$, if $\widehat{\Delta}_{\mathbb{G}}(\widehat{z})(\gamma(\widehat{p}) \otimes \mathbb{1}) = \gamma(\widehat{p}) \otimes \widehat{z}$ then $\widehat{z} \in \mathrm{Im}\,\gamma$.*

Then $\mathbb{G}$ has the $\mathbb{H}$-separation property.

It will be clear from the proof of Theorem 6.5 that a condition weaker than (6.2) is enough. However, (6.2) is often easier to check.

Before proving the theorem, observe that each $\widehat{z} \in \mathrm{Im}\,\gamma$ indeed satisfies

$$\widehat{\Delta}_{\mathbb{G}}(\widehat{z})(\gamma(\widehat{p}) \otimes \mathbb{1}) = \gamma(\widehat{p}) \otimes \widehat{z}$$

(see [54, Proposition 3.1]). Also, if $\widehat{z} \in L^{\infty}(\widehat{\mathbb{G}})$ satisfies this identity, then taking $\widehat{\omega} \in L^1(\widehat{\mathbb{G}})$ with $\widehat{\omega}(\gamma(\widehat{p})) = 1$, we get $(\gamma(\widehat{p})\widehat{\omega} \otimes \mathrm{id})\widehat{\Delta}_{\mathbb{G}}(\widehat{z}) = \widehat{z}$, so in the terminology of [42], we have $\widehat{z} \in \mathrm{LUC}(\widehat{\mathbb{G}})$, thus $\widehat{z} \in M(C_0(\widehat{\mathbb{G}}))$ [42, Theorem 2.4]. Furthermore, if $\mathbb{G}$ is commutative or co-commutative, then (6.2) holds automatically by [7, Sections 4, 5]; we prove the former case below, and the second one, in which $\mathbb{G} = \widehat{G}$ for some locally compact group G and $\mathbb{H} = \widehat{G/A}$ for an open normal subgroup A of G, is a simple observation. At the moment it is unclear whether (6.2) always holds, but we will show in Subsection 6.3 that it holds in an abundance of examples in which closed quantum subgroups appear naturally, namely via the bicrossed product construction, and in Subsection 6.4 that it holds for $\mathbb{T}$ as a closed quantum subgroup of quantum $E(2)$.

Proposition 6.6 *Condition (6.2) holds when $\mathbb{G}$ is commutative.*

Proof Let G be a locally compact group and H a compact subgroup of G. The embedding $\gamma \colon \mathrm{VN}(H) \to M(C_r^*(G)) \subseteq \mathrm{VN}(G)$ is the natural one, mapping $\lambda_h \in \mathrm{VN}(H)$, $h \in H$, to λ_h in $\mathrm{VN}(G)$. Also $\gamma(\widehat{p}) = \int_H \lambda_h \, \mathrm{d}h$. Replacing $\widehat{z}$ by its adjoint in (6.2), suppose that $\widehat{z} \in \mathrm{VN}(G)$ and $(\gamma(\widehat{p}) \otimes \mathbb{1})\widehat{\Delta}(\widehat{z}) = \gamma(\widehat{p}) \otimes \widehat{z}$. Denote by ℓ_t, $t \in G$, the left shift operators over $A(G)$. For all $\omega_1, \omega_2 \in A(G)$ and $t \in G$, one calculates that $(\omega_1 \otimes \omega_2)[(\lambda_t \otimes \mathbb{1})\widehat{\Delta}(\widehat{z})] = (\ell_{t^{-1}}(\omega_1) \cdot \omega_2)(\widehat{z})$, and thus

$$(\omega_1 \otimes \omega_2)[(\gamma(\widehat{p}) \otimes \mathbb{1})\widehat{\Delta}(\widehat{z})] = \int_H (\ell_{h^{-1}}(\omega_1) \cdot \omega_2)(\widehat{z}) \, \mathrm{d}h = \left(\left(\int_H \ell_{h^{-1}}(\omega_1) \, \mathrm{d}h \right) \cdot \omega_2 \right)(\widehat{z})$$

(the second integral is in the norm of $A(G)$), and by assumption it is equal to

$$(\omega_1 \otimes \omega_2)(\gamma(\widehat{p}) \otimes \widehat{z}) = \int_H \omega_1(h) \, \mathrm{d}h \cdot \omega_2(\widehat{z}).$$

Fix a closed set C with $C \cap H = \varnothing$ and $\omega_2 \in A(G)$ that is supported by C. Noticing that $HC \cap H = \varnothing$, let $\omega_1 \in A(G)$ be such that $\omega_1|_H \equiv 1$ and $\omega_1|_{HC} \equiv 0$ [14, Lemme 3.2]. We have

$$0 = \left(\left(\int_H \ell_{h^{-1}}(\omega_1) \, \mathrm{d}h \right) \cdot \omega_2 \right)(\widehat{z}) = \int_H \omega_1(h) \, \mathrm{d}h \cdot \omega_2(\widehat{z}) = \omega_2(\widehat{z}).$$

Consequently, the support of $\widehat{z}$ (see [14, Définition 4.5 and Proposition 4.8]) is contained in H. Consequently, by Takesaki and Tatsuuma [47], $\widehat{z}$ belongs to $\gamma(\mathrm{VN}(H))$, as desired. ∎

Proof of Theorem 6.5 Let $\mu \in C_0^{\mathrm{u}}(\mathbb{G})_+^*$ be a state such that

$$(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \notin \widehat{\Phi}(M(c_0(\widehat{\mathbb{H}}))).$$

We should prove that there exists $\widehat{\omega} \in C_0^{\mathrm{u}}(\widehat{\mathbb{G}})_+^*$ so that $\widehat{\omega} \circ \widehat{\Phi} = \widehat{\epsilon}_{\mathbb{H}}$ but $\mu((\mathrm{id} \otimes \widehat{\omega})(\mathbb{W}_{\mathbb{G}})) \neq 1$. Assume by contradiction that $\mu((\mathrm{id} \otimes \widehat{\omega})(\mathbb{W}_{\mathbb{G}})) = 1$ for every $\widehat{\omega} \in C_0^{\mathrm{u}}(\widehat{\mathbb{G}})_+^*$ such that $\widehat{\omega} \circ \widehat{\Phi} = \widehat{\epsilon}_{\mathbb{H}}$. Representing $M(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$ faithfully on some Hilbert space, every unit vector $\zeta \in \mathrm{Im}\,\widehat{\Phi}(\widehat{p})$ satisfies $\widehat{\omega}_\zeta \circ \widehat{\Phi} = \widehat{\epsilon}_{\mathbb{H}}$. Hence

$$\widehat{\omega}_\zeta\big[(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}})\big] = 1$$

for every such vector, and as $\|(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}})\| = 1$, we obtain

$$(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}})\widehat{\Phi}(\widehat{p}) = \widehat{\Phi}(\widehat{p}) = \widehat{\Phi}(\widehat{p})(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}).$$

Denote $\widehat{y} := (\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \in M(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$. Since μ is a state, a variant of (4.2) implies that

$$\big[\widehat{\Delta}_{\mathbb{G}}^{\mathrm{u}}(\widehat{y}) - \mathbb{1} \otimes \widehat{y}\big]^*\big[\widehat{\Delta}_{\mathbb{G}}^{\mathrm{u}}(\widehat{y}) - \mathbb{1} \otimes \widehat{y}\big] \leq \big[2\mathbb{1} - (\widehat{y}^* + \widehat{y})\big] \otimes \mathbb{1}.$$

Multiplying by $\widehat{\Phi}(\widehat{p}) \otimes \mathbb{1}$ on both sides we get $\big[\widehat{\Delta}_{\mathbb{G}}^{\mathrm{u}}(\widehat{y}) - \mathbb{1} \otimes \widehat{y}\big](\widehat{\Phi}(\widehat{p}) \otimes \mathbb{1}) = 0$, that is, $\widehat{\Delta}_{\mathbb{G}}^{\mathrm{u}}(\widehat{y})(\widehat{\Phi}(\widehat{p}) \otimes \mathbb{1}) = \widehat{\Phi}(\widehat{p}) \otimes \widehat{y}$. Applying $\pi_{\widehat{\mathbb{G}}} \otimes \pi_{\widehat{\mathbb{G}}}$ to both sides and using that $\pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi} = \gamma$, we get $\widehat{\Delta}_{\mathbb{G}}(\pi_{\widehat{\mathbb{G}}}(\widehat{y}))(\gamma(\widehat{p}) \otimes \mathbb{1}) = \gamma(\widehat{p}) \otimes \pi_{\widehat{\mathbb{G}}}(\widehat{y})$. By (6.2),

$$(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) = \pi_{\widehat{\mathbb{G}}}(\widehat{y}) \in (\pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi})(M(c_0(\widehat{\mathbb{H}}))).$$

From Lemma 6.7 we obtain $(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \in \widehat{\Phi}(M(c_0(\widehat{\mathbb{H}})))$, a contradiction. ∎

Lemma 6.7   *Let $\mathbb{G}$ be a LCQG and $\mathbb{H}$ be a compact quantum subgroup of $\mathbb{G}$. If $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ is such that $\widehat{x} := (\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}})$ satisfies $\pi_{\widehat{\mathbb{G}}}(\widehat{x}) \in (\pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi})(M(c_0(\widehat{\mathbb{H}})))$, then $\widehat{x} \in \widehat{\Phi}(M(c_0(\widehat{\mathbb{H}})))$.*

Proof Recall that up to isomorphism, $c_0(\widehat{\mathbb{H}})$ decomposes as $c_0 - \bigoplus_{\alpha \in \mathrm{Irred}(\mathbb{H})} M_{n(\alpha)}$. For each $\alpha \in \mathrm{Irred}(\mathbb{H})$, write $\widehat{p}_\alpha \in c_0(\widehat{\mathbb{H}})$ for the identity of $M_{n(\alpha)}$, and let $\omega_\alpha \in C^{\mathrm{u}}(\mathbb{H})^*$ be such that $\widehat{p}_\alpha = (\omega_\alpha \otimes \mathrm{id})(\mathbb{W}_{\mathbb{H}})$ (which exists by the Peter–Weyl theory for compact quantum groups [56]). Then $(\pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi})(\widehat{p}_\alpha) = ((\omega_\alpha \circ \Phi) \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}})$ by (6.1), and

$$(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \cdot (\pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi})(\widehat{p}_\alpha) \in (\pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi})\big(M(c_0(\widehat{\mathbb{H}}))\big).$$

If $\widehat{y}_\alpha \in M_{n(\alpha)}$ is such that $(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \cdot (\pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi})(\widehat{p}_\alpha) = (\pi_{\widehat{\mathbb{G}}} \circ \widehat{\Phi})(\widehat{y}_\alpha)$, there exists $\rho_\alpha \in C^{\mathrm{u}}(\mathbb{H})^*$ with $\widehat{y}_\alpha = (\rho_\alpha \otimes \mathrm{id})(\mathbb{W}_{\mathbb{H}})$. Thus

$$\big((\mu * (\omega_\alpha \circ \Phi)) \otimes \mathrm{id}\big)(\mathbb{W}_{\mathbb{G}}) = ((\rho_\alpha \circ \Phi) \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}).$$

Hence $\mu * (\omega_\alpha \circ \Phi) = \rho_\alpha \circ \Phi$ as $\lambda_{\mathbb{G}}^{\mathrm{u}}$ is injective, and we can replace $\mathbb{W}_{\mathbb{G}}$ by $\mathbb{W}_{\mathbb{G}}$ to obtain $(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \cdot \widehat{\Phi}(\widehat{p}_\alpha) \in \widehat{\Phi}(M(c_0(\widehat{\mathbb{H}})))$. But $\sum_{\alpha \in \mathrm{Irred}(\mathbb{H})} \widehat{\Phi}(\widehat{p}_\alpha) = \mathbb{1}$ strictly in $M(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$ since $\sum_{\alpha \in \mathrm{Irred}(\mathbb{H})} \widehat{p}_\alpha = \mathbb{1}$ strictly in $M(c_0(\widehat{\mathbb{H}}))$ and $\widehat{\Phi}$ is nondegenerate, so we conclude that

$$(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \in \widehat{\Phi}(M(c_0(\widehat{\mathbb{H}}))).$$

∎

Remark 6.8 For $\mu \in C_0^u(\mathbb{G})_+^*$, the condition $(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) \notin \widehat{\Phi}(M(C_0^u(\widehat{\mathbb{H}})))$ from Definition 6.4 implies that $\mu \notin \Phi^*(C_0^u(\mathbb{H})^*)$, because if $\mu = \nu \circ \Phi$ for some $\nu \in C_0^u(\mathbb{H})_+^*$, then $(\mu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{G}}) = \widehat{\Phi}((\nu \otimes \mathrm{id})(\mathbb{W}_{\mathbb{H}})) \in \widehat{\Phi}(M(C_0^u(\widehat{\mathbb{H}})))$ by (6.1). Moreover, if $\mathbb{G}$ is commutative, the two conditions are equivalent. We do not know whether Theorem 6.5 holds with this weaker condition as well.

6.3 Examples Arising From the Bicrossed Product Construction

A natural way to construct a closed quantum subgroup of a LCQG is the bicrossed product [50]. Let $\mathbb{G}_1, \mathbb{G}_2$ be LCQGs. We say that $(\mathbb{G}_1, \mathbb{G}_2)$ is a *matched pair* [50, Definition 2.1] if it admits a cocycle matching $(\tau, \mathscr{U}, \mathscr{V})$, which means that

$$\tau : L^\infty(\mathbb{G}_1) \overline{\otimes} L^\infty(\mathbb{G}_2) \longrightarrow L^\infty(\mathbb{G}_1) \overline{\otimes} L^\infty(\mathbb{G}_2)$$

is a faithful, normal, unital $*$-homomorphism and $\mathscr{U} \in L^\infty(\mathbb{G}_1) \overline{\otimes} L^\infty(\mathbb{G}_1) \overline{\otimes} L^\infty(\mathbb{G}_2)$, $\mathscr{V} \in L^\infty(\mathbb{G}_1) \overline{\otimes} L^\infty(\mathbb{G}_2) \overline{\otimes} L^\infty(\mathbb{G}_2)$ are unitaries such that the $*$-homomorphisms

$$\alpha : L^\infty(\mathbb{G}_2) \to L^\infty(\mathbb{G}_1) \overline{\otimes} L^\infty(\mathbb{G}_2), \qquad \beta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_1) \overline{\otimes} L^\infty(\mathbb{G}_2)$$

given by $\alpha(y) := \tau(\mathbb{1} \otimes y)$, $y \in L^\infty(\mathbb{G}_2)$, and $\beta(x) := \tau(x \otimes \mathbb{1})$, $x \in L^\infty(\mathbb{G}_1)$, satisfy the following conditions:

(i) $(\alpha, \mathscr{U})$ is a left cocycle action of $\mathbb{G}_1$ on $L^\infty(\mathbb{G}_2)$, *i.e.*,

$$(\mathrm{id} \otimes \alpha)(\alpha(y)) = \mathscr{U}(\Delta_1 \otimes \mathrm{id})(\alpha(y))\mathscr{U}^* \qquad (\forall y \in L^\infty(\mathbb{G}_2)),$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \alpha)(\mathscr{U})(\Delta_1 \otimes \mathrm{id} \otimes \mathrm{id})(\mathscr{U}) = (\mathbb{1} \otimes \mathscr{U})(\mathrm{id} \otimes \Delta_1 \otimes \mathrm{id})(\mathscr{U});$$

(ii) $(\sigma\beta, \mathscr{V}_{321})$ is a left cocycle action of $\mathbb{G}_2$ on $L^\infty(\mathbb{G}_1)$, *i.e.*,

$$(\beta \otimes \mathrm{id})(\beta(x)) = \mathscr{V}(\mathrm{id} \otimes \Delta_2^{\mathrm{op}})(\beta(x))\mathscr{V}^* \qquad (\forall x \in L^\infty(\mathbb{G}_1)),$$

$$(\beta \otimes \mathrm{id} \otimes \mathrm{id})(\mathscr{V})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_2^{\mathrm{op}})(\mathscr{V}) = (\mathscr{V} \otimes \mathbb{1})(\mathrm{id} \otimes \Delta_2^{\mathrm{op}} \otimes \mathrm{id})(\mathscr{V});$$

(iii) $(\alpha, \mathscr{U})$ and $(\beta, \mathscr{V})$ are matched, *i.e.*,

$$\tau_{13}(\alpha \otimes \mathrm{id})(\Delta_2(y)) = \mathscr{V}_{132}(\mathrm{id} \otimes \Delta_2)(\alpha(y))\mathscr{V}_{132}^* \qquad (\forall y \in L^\infty(\mathbb{G}_2)),$$

$$\tau_{23}\sigma_{23}(\beta \otimes \mathrm{id})(\Delta_1(x)) = \mathscr{U}(\Delta_1 \otimes \mathrm{id})(\beta(x))\mathscr{U}^* \qquad (\forall x \in L^\infty(\mathbb{G}_1)),$$

(6.3) $(\Delta_1 \otimes \mathrm{id} \otimes \mathrm{id})(\mathscr{V})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_2^{\mathrm{op}})(\mathscr{U}^*)$

$$= (\mathscr{U}^* \otimes \mathbb{1})(\mathrm{id} \otimes \tau\sigma \otimes \mathrm{id})\big[(\beta \otimes \mathrm{id} \otimes \mathrm{id})(\mathscr{U}^*)(\mathrm{id} \otimes \mathrm{id} \otimes \alpha)(\mathscr{V})\big](\mathbb{1} \otimes \mathscr{V}).$$

Suppose that such a matched pair is given. For convenience, write $\mathcal{H}_i := L^2(\mathbb{G}_i)$, $i = 1, 2$, and let $\tilde{W} := (W_1 \otimes \mathbb{1})\mathscr{U}^* \in L^\infty(\mathbb{G}_1) \overline{\otimes} B(\mathcal{H}_1) \overline{\otimes} L^\infty(\mathbb{G}_2)$. Recall that the cocycle crossed product $\mathbb{G}_{1\ \alpha,\mathscr{U}} \ltimes L^\infty(\mathbb{G}_2)$ is the von Neumann subalgebra of $B(\mathcal{H}_1) \overline{\otimes} L^\infty(\mathbb{G}_2)$ generated by $\alpha(L^\infty(\mathbb{G}_2))$ and $\{(\omega \otimes \mathrm{id} \otimes \mathrm{id})(\tilde{W}) : \omega \in L^1(\mathbb{G}_1)\}$. Letting $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2$, define unitaries $W, \widehat{W} \in B(\mathcal{H} \otimes \mathcal{H})$ by

$$\widehat{W} := (\beta \otimes \mathrm{id} \otimes \mathrm{id})\big[(W_1 \otimes \mathbb{1})\mathscr{U}^*\big](\mathrm{id} \otimes \mathrm{id} \otimes \alpha)\big[\mathscr{V}(\mathbb{1} \otimes \widehat{W}_2)\big], \quad W := \sigma(\widehat{W}^*).$$

By [50, Theorem 2.13], there is a LCQG $\mathbb{G}$ with $L^\infty(\mathbb{G}) = \mathbb{G}_{1\ \alpha,\mathscr{U}} \ltimes L^\infty(\mathbb{G}_2)$, $L^2(\mathbb{G}) = \mathcal{H}$, and W being its left regular co-representation. Defining $\tilde{\tau} := \sigma\tau\sigma$, $\tilde{\mathscr{U}} := \mathscr{V}_{321}$ and $\tilde{\mathscr{V}} := \mathscr{U}_{321}$, one checks that $(\tilde{\tau}, \tilde{\mathscr{U}}, \tilde{\mathscr{V}})$ is a cocycle matching making $(\mathbb{G}_2, \mathbb{G}_1)$ into a matched pair. Its ambient LCQG is, up to flipping from $\mathcal{H}_2 \otimes \mathcal{H}_1$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$, precisely the dual $\widehat{\mathbb{G}}$. In what follows we use a subscript to indicate that a symbol relates to $\mathbb{G}_i$,

$i = 1, 2$, and a lack of subscript if it relates to $\mathbb{G}$. For instance, J_1, J_2, and J are the modular conjugations of $L^\infty(\mathbb{G}_1)$, $L^\infty(\mathbb{G}_2)$, and $L^\infty(\mathbb{G})$, respectively.

Since $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_2$ [50, Proposition 2.4], we see that $\widehat{\mathbb{G}}_2$ is a closed quantum subgroup of $\widehat{\mathbb{G}}$ in the sense of Vaes, thus also in the sense of Woronowicz.

It is proved in [50, Section 3] that there is a bijection between (cocycle) bicrossed products and *cleft extensions* of LCQGs. To elaborate, consider the unitary

$$Z_2 := (J_1 \otimes \widehat{J})(\mathrm{id} \otimes \beta)(\widehat{W}_1^*)(J_1 \otimes \widehat{J}).$$

Then the formula $\theta(z) := Z_2(\mathbb{1} \otimes z)Z_2^*$ defines a map $\theta \colon L^\infty(\mathbb{G}) \to L^\infty(\widehat{\mathbb{G}}_1)\overline{\otimes}L^\infty(\mathbb{G})$, which is an action of $\widehat{\mathbb{G}}_1^{\mathrm{op}}$ on $\mathbb{G}$. The exactness of the sequence at $\mathbb{G}$ is manifested by the following characterization of the fixed-point algebra of θ:

$$(6.4) \qquad\qquad L^\infty(\mathbb{G})^\theta = \alpha(L^\infty(\mathbb{G}_2)).$$

The proof of this is by no means technical: it strongly relies on the structure of $\mathbb{G}$ and its dual.

Example 6.9   Assume henceforth that $\mathbb{G}_2$ is discrete and, denoting by p the central minimal projection in $L^\infty(\mathbb{G}_2)$ with $yp = \epsilon_2(y)p = py$ for every $y \in L^\infty(\mathbb{G}_2)$, that

$$(6.5) \qquad\qquad\qquad \alpha(p) = \mathbb{1}_{L^\infty(\mathbb{G}_1)} \otimes p,$$

$$(6.6) \qquad\qquad\qquad (\mathrm{id} \otimes \epsilon_2)\beta = \mathrm{id},$$

$$(6.7) \qquad (\mathrm{id} \otimes \mathrm{id} \otimes \epsilon_2)(\mathscr{V}) = \mathbb{1}_{L^\infty(\mathbb{G}_1)} \otimes \mathbb{1}_{L^\infty(\mathbb{G}_2)} = (\mathrm{id} \otimes \epsilon_2 \otimes \mathrm{id})(\mathscr{V}).$$

Condition (6.5) means essentially that $\mathbb{G}_1$ is "connected", while (6.6) and (6.7) are natural as $\mathbb{G}_2$ is discrete (see Vaes and Vergnioux [52, Definition 1.24] and Packer and Raeburn [37, Definition 2.1]).

For starters, notice that

$$(6.8) \qquad\qquad (\mathrm{id} \otimes \mathrm{id} \otimes \epsilon_2)(\mathscr{U}) = \mathbb{1}_{L^\infty(\mathbb{G}_1)} \otimes \mathbb{1}_{L^\infty(\mathbb{G}_1)}.$$

Indeed, denote the left-hand side by U. Applying the $*$-homomorphism $\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\epsilon_2$ to (6.3) and using (6.7), we obtain

$$\mathscr{U}^* = \mathscr{U}^*(\mathrm{id} \otimes \tau\sigma)\big[(\beta \otimes \mathrm{id})(U^*)(\mathrm{id} \otimes \mathrm{id} \otimes (\mathrm{id} \otimes \epsilon_2)\alpha)(\mathscr{V})\big],$$

and since $\mathrm{id} \otimes \tau\sigma$ is faithful, $(\beta \otimes \mathrm{id})(U) = (\mathrm{id} \otimes \mathrm{id} \otimes (\mathrm{id} \otimes \epsilon_2)\alpha)(\mathscr{V})$. Applying $\mathrm{id}\otimes\epsilon_2\otimes\mathrm{id}$ and using (6.6) and (6.7), we get $U = (\mathrm{id} \otimes (\mathrm{id}\otimes\epsilon_2)\alpha)(\mathbb{1}) = \mathbb{1}$, as desired.

We claim that for every $z \in L^\infty(\mathbb{G})$,

$$\Delta(z)(\alpha(p) \otimes \mathbb{1}_{L^\infty(\mathbb{G})}) = \alpha(p) \otimes z \quad \implies \quad z \in \alpha(L^\infty(\mathbb{G}_2)).$$

Indeed, suppose that the assumption is met. Then $\Delta^{\mathrm{op}}(z)(\mathbb{1}_{L^\infty(\mathbb{G})} \otimes \mathbb{1}_{B(\mathcal{H}_1)} \otimes p) = z \otimes \mathbb{1}_{B(\mathcal{H}_1)} \otimes p$ by (6.5). From [50, Lemma 2.3] we get

$$
\begin{aligned}
\Delta^{\mathrm{op}}&(z)(\mathbb{1}_{L^\infty(\mathbb{G})} \otimes \mathbb{1}_{B(\mathcal{H}_1)} \otimes p) \\
&= (\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W})[(\mathrm{id} \otimes \mathrm{id} \otimes \alpha)(\mathcal{V}(\mathrm{id} \otimes \Delta_2^{\mathrm{op}})(z)\mathcal{V}^*)] \\
&\quad \times (\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W}^*)(\mathbb{1}_{L^\infty(\mathbb{G})} \otimes \mathbb{1}_{B(\mathcal{H}_1)} \otimes p) \\
&= (\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W})[(\mathrm{id} \otimes \mathrm{id} \otimes \alpha)(\mathcal{V}(\mathrm{id} \otimes \Delta_2^{\mathrm{op}})(z)(\mathbb{1}_{L^\infty(\mathbb{G})} \otimes p)\mathcal{V}^*)] \\
&\quad \times (\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W}^*) \\
&= (\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W})[(\mathrm{id} \otimes \mathrm{id} \otimes \alpha)(\mathcal{V}(z \otimes p)\mathcal{V}^*)](\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W}^*).
\end{aligned}
$$

By (6.7) and (6.5) we thus have

$$
\begin{aligned}
\Delta^{\mathrm{op}}(z)&(\mathbb{1}_{L^\infty(\mathbb{G})} \otimes \mathbb{1}_{B(\mathcal{H}_1)} \otimes p) \\
&= (\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W})[(\mathrm{id} \otimes \mathrm{id} \otimes \alpha)(z \otimes p)](\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W}^*) \\
&= (\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W})(z \otimes \mathbb{1}_{B(\mathcal{H}_1)} \otimes p)(\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W}^*).
\end{aligned}
$$

The assumption hence implies that

$$
(\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W})(z \otimes \mathbb{1}_{B(\mathcal{H}_1)} \otimes p)(\beta \otimes \mathrm{id} \otimes \mathrm{id})(\widetilde{W}^*) = z \otimes \mathbb{1}_{B(\mathcal{H}_1)} \otimes p.
$$

Applying $\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \epsilon_2$ to both sides, we deduce from (6.8) that

$$
(\beta \otimes \mathrm{id})(W_1)(z \otimes \mathbb{1}_{B(\mathcal{H}_1)})(\beta \otimes \mathrm{id})(W_1^*) = z \otimes \mathbb{1}_{B(\mathcal{H}_1)}.
$$

Writing $w := \widehat{J}z\widehat{J}$ and recalling that $w = R(z^*) \in L^\infty(\mathbb{G})$ where R is the unitary antipode of $\mathbb{G}$, the last equation is equivalent to $\theta(w) = \mathbb{1} \otimes w$, that is, $w \in L^\infty(\mathbb{G})^\theta$. By (6.4), $R(z^*)$ belongs to the image of α. By the von Neumann algebraic version of [30, Corollary 5.46], we have $R \circ \alpha = \alpha \circ R_2$. Therefore z belongs to the image of α, and the proof is complete.

Remark 6.10 The LCQG $\mathbb{G}$ constructed in Example 6.9 and its dual are neither necessarily amenable nor necessarily co-amenable [9, Theorems 13 and 15].

Remark 6.11 The last part of the reasoning in Example 6.9 uses in an essential way the *exactness* of the short exact sequence of LCQGs. As mentioned above, bi-crossed products are characterized as *cleft* extensions. By [50, Propositions 1.22 and 1.24], this amounts to the structure of $L^\infty(\mathbb{G})$ as a cocycle crossed product. Examining the argument in Example 6.9, this structure is used mainly in the simplification of $\Delta^{\mathrm{op}}(z)(\mathbb{1} \otimes \mathbb{1} \otimes p)$. It is not clear at the moment whether this argument generalizes further, thus leaving open the general case of compact (or, even more generally, closed) *normal* quantum subgroups (see Vaes and Vainerman [51]).

6.4 Example: Quantum $E(2)$ Group

We prove that the complex unit circle $\mathbb{T}$, as a closed quantum subgroup of $E(2)$, has the separation property. Considering the quantum groups $E(2)$ and $\widehat{E}(2)$, we essentially follow the notation of Jacobs [25] although it does not always agree with

ours; further details can be found there. Fix $0 < \mu < 1$. Set $\mathbb{R}^\mu := \{\mu^k : k \in \mathbb{Z}\}$, $\overline{\mathbb{R}}^\mu := \mathbb{R}^\mu \cup \{0\}$, $\mathbb{R}(\mu^{1/2}) := \{\mu^{k/2} : k \in \mathbb{Z}\}$ and $\overline{\mathbb{R}}(\mu^{1/2}) := \mathbb{R}(\mu^{1/2}) \cup \{0\}$.

The following is taken from [25, Section 2.3]. Let $(e_k)_{k \in \mathbb{Z}}$ be an orthonormal basis of $\ell_2(\mathbb{Z})$. Denote by s the unitary operator over $\ell_2(\mathbb{Z})$ which is the shift given by $se_k := e_{k+1}$, $k \in \mathbb{Z}$. Denote by m the strictly positive (unbounded) operator over $\ell_2(\mathbb{Z})$ that acts on its core $\mathrm{span}\{e_k : k \in \mathbb{Z}\}$ by $me_k := \mu^k e_k$, $k \in \mathbb{Z}$.

Set $\mathcal{H} := \ell_2(\mathbb{Z}) \otimes \ell_2(\mathbb{Z})$ and $e_{k,l} := e_k \otimes e_l$ for $k, l \in \mathbb{Z}$. Consider the unbounded operators over $\mathcal{H}$ defined by $a := m^{-1/2} \otimes m$ and $b := m^{1/2} \otimes s$. Then a is strictly positive, b has polar decomposition $b = u|b|$ with $u := \mathbb{1} \otimes s$ and $|b| = m^{1/2} \otimes \mathbb{1}$, and $\sigma(a) = \overline{\mathbb{R}}(\mu^{1/2}) = \sigma(b)$. Since $a, |b|$ commute, they have a joint Borel functional calculus. As observed in [25, Remark 2.5.20], the joint continuous functional calculus of $a, |b|$ is determined by the values of the functions on

$$E := \{(p, q) \in \mathbb{R}(\mu^{1/2}) \times \overline{\mathbb{R}}(\mu^{1/2}) : pq \in \overline{\mathbb{R}}^\mu\}.$$

Similarly, as $|b|$, just like a, is injective, the joint Borel functional calculus of $a, |b|$ is determined by $F := \{(p, q) \in \mathbb{R}(\mu^{1/2}) \times \mathbb{R}(\mu^{1/2}) : pq \in \mathbb{R}^\mu\}$. Writing $\mathbb{B}(F)$ for the algebra of all bounded complex-valued functions over F, we get an injection $\mathbb{B}(F) \ni g \mapsto g(a, |b|) \in B(\mathcal{H})$.

The operator $W \in B(\mathcal{H} \otimes \mathcal{H})$ is the unitary that satisfies

$$(6.9) \quad ((\omega_{e_{k,l}, e_{p,q}} \otimes \mathrm{id})(W))e_{m,n} = B(q - l, k - l - n + 1)\delta_{k,p}e_{m-k+2q, n-k+l+q}$$
$$(\forall k, l, p, q, m, n \in \mathbb{Z}),$$

where $(B(k, n))_{k,n \in \mathbb{Z}}$ are special scalars in the complex unit disc.

The right, resp. left, leg of W norm-spans a C^*-algebra A, resp. $\widehat{A}$, which is the reduced C^*-algebra underlying the LCQG $E(2)$, resp. $\widehat{E}(2)$, and $W \in M(\widehat{A} \otimes_{\min} A)$ [25, Sections 2.4, 2.5]. The co-multiplications $\Delta : A \to M(A \otimes_{\min} A)$, resp. $\widehat{\Delta} : \widehat{A} \to M(\widehat{A} \otimes_{\min} \widehat{A})$ of $E(2)$, resp. $\widehat{E}(2)$, is given by $\Delta(x) = W(x \otimes \mathbb{1})W^*$ for $x \in A$, resp. $\widehat{\Delta}(y) := W^*(\mathbb{1} \otimes y)W$ for $y \in \widehat{A}$. The duality relation between $E(2)$ and $\widehat{E}(2)$ is opposite: $\widehat{E}(2) = \widehat{E(2)}^{\mathrm{op}}$ [25, Proposition 2.8.21], but since $\mathbb{T}$ is commutative, that is meaningless for our purposes.

The unbounded operators a, a^{-1}, b are affiliated with $\widehat{A}$ in the sense of C^*-algebras, and a is "group like", that is, $\widehat{\Delta}(a) = a \otimes a$, where the left-hand side is interpreted as a nondegenerate $*$-homomorphism acting on an affiliated element. This makes $\mathbb{T}$ a closed quantum subgroup of $E(2)$: identifying $\ell_\infty(\mathbb{Z}) \cong \{f(a) : f \in C_b(\mathbb{R}(\mu^{1/2}))\}$ (recall that a is injective), the embedding $\gamma : \ell_\infty(\mathbb{Z}) \hookrightarrow M(\widehat{A})$ is given by mapping $g \in \ell_\infty(\mathbb{Z})$ to $f(a)$, where $f(\mu^{k/2}) := g(k)$, $k \in \mathbb{Z}$ [25, Subsection 2.8.5]. Denote by p the projection $k \mapsto \delta_{k,0}$ in $\ell_\infty(\mathbb{Z})$. Then $\gamma(p)$ is the projection onto $\{e_{2l,l} : l \in \mathbb{Z}\}$.

To establish the separation property, consider all $y \in M(\widehat{A})$ satisfying $\widehat{\Delta}(y)(\gamma(p) \otimes \mathbb{1}) = (\gamma(p) \otimes y)$. This means that $\mathbb{1} \otimes y$ commutes with $W(\gamma(p) \otimes \mathbb{1})$, or equivalently, that y commutes with $(\omega_{\zeta, \eta} \otimes \mathrm{id})(W)$ for every $\zeta \in \mathrm{Im}\,\gamma(p)$ and $\eta \in \mathcal{H}$. Substituting $q - l$ for t in (6.9), this amounts to y commuting with each of the operators $x_{l,t} \in B(\mathcal{H})$, $l, t \in \mathbb{Z}$, given by $x_{l,t}e_{m,n} := B(t, l - n + 1)e_{m+2t, n+t}$ for $m, n \in \mathbb{Z}$.

Let $l, t \in \mathbb{Z}$. Clearly, $x_{l,t}$ commutes with a. For $m, n \in \mathbb{Z}$ and $s \in \mathbb{R}$,

$$|b|^{is} x_{l,t} e_{m,n} = B(t, l - n + 1)|b|^{is} e_{m+2t,n+t} = \mu^{is(m+2t)/2} B(t, l - n + 1) e_{m+2t,n+t},$$

$$x_{l,t}|b|^{is} e_{m,n} = \mu^{ism/2} x_{l,t} e_{m,n} = \mu^{ism/2} B(t, l - n + 1) e_{m+2t,n+t},$$

so that $|b|^{is} x_{l,t} = \mu^{ist} x_{l,t} |b|^{is}$ for every $s \in \mathbb{R}$, or formally $|b|x_{l,t} = \mu^t x_{l,t}|b|$. This implies that for every $g \in \mathbb{B}(F)$ we have

$$(6.10) \qquad\qquad g(a, |b|) x_{l,t} = x_{l,t} g_t(a, |b|),$$

where $g_t \in \mathbb{B}(F)$ is defined by $g_t(\alpha, \beta) := g(\alpha, \mu^t \beta)$. Moreover, for $k, m, n \in \mathbb{Z}$,

$$(6.11) \quad u^k x_{l,t} e_{m,n} = B(t, l - n + 1) u^k e_{m+2t,n+t} = B(t, l - n + 1) e_{m+2t,n+t+k},$$

$$x_{l,t} u^k e_{m,n} = x_{l,t} e_{m,n+k} = B(t, l - n - k + 1) e_{m+2t,n+t+k}.$$

Lemma 6.12 *Let $g \in \mathbb{B}(F)$ and $k \in \mathbb{Z}$. Assume that $u^k g(a, |b|)$ commutes with the operators $(x_{l,t})_{l,t \in \mathbb{Z}}$. If $k \neq 0$, then $g(a, |b|) = 0$; if $k = 0$, then g is the restriction of $h \otimes \mathbb{1}$ for some $h \in \mathbb{B}(\mathbb{R}(\mu^{1/2}))$.*

Proof Both cases will use the following computation. Let $t, m, n \in \mathbb{Z}$. Since $\mathbb{C}e_{m,n}$ is invariant under both a and $|b|$, it is invariant under $g(a, |b|)$ and $g_t(a, |b|)$. Let $\gamma, \gamma_t \in \mathbb{C}$ be such that $g(a, |b|) e_{m,n} = \gamma e_{m,n}$ and $g_t(a, |b|) e_{m,n} = \gamma_t e_{m,n}$. By assumption, for all $l \in \mathbb{Z}$ we have $x_{l,t} u^k g(a, |b|) = u^k g(a, |b|) x_{l,t} = u^k x_{l,t} g_t(a, |b|)$ from (6.10), so using (6.11), $x_{l,t} u^k g(a, |b|) e_{m,n} = \gamma x_{l,t} u^k e_{m,n} = \gamma B(t, l - n - k + 1) e_{m+2t,n+t+k}$ is equal to $u^k x_{l,t} g_t(a, |b|) e_{m,n} = \gamma_t u^k x_{l,t} e_{m,n} = \gamma_t B(t, l - n + 1) e_{m+2t,n+t+k}$, that is,

$$(6.12) \qquad\qquad \gamma B(t, l - n - k + 1) = \gamma_t B(t, l - n + 1).$$

Suppose that $k = 0$. Let $t, m, n \in \mathbb{Z}$ and let $\gamma, \gamma_t \in \mathbb{C}$ be as above. Then for every $l \in \mathbb{Z}$, we have $\gamma B(t, l - n + 1) = \gamma_t B(t, l - n + 1)$ from (6.12). Choosing l such that $B(t, l - n + 1) \neq 0$, which is possible by [25, Corollary A.11], we get $\gamma = \gamma_t$. As m, n were arbitrary, we deduce that $g(a, |b|) = g_t(a, |b|)$, hence $g = g_t$. By the definition of F, as t was arbitrary, g is of the form $h \otimes \mathbb{1}$.

Suppose that $k \neq 0$. Since $(B(t, 0))_{t \in \mathbb{Z}}$ are the Fourier coefficients of a non-constant function [25, Definition A.4], we can fix $0 \neq t \in \mathbb{Z}$ with $B(t, 0) \neq 0$. Assuming that $g(a, |b|) \neq 0$, fix $m, n \in \mathbb{Z}$ such that $g(a, |b|) e_{m,n} \neq 0$. Let $\gamma, \gamma_t \in \mathbb{C}$ be as above; then $\gamma \neq 0$. Replacing $l - n + 1$ by l in (6.12) for convenience, we get $\gamma B(t, l - k) = \gamma_t B(t, l)$ for all $l \in \mathbb{Z}$, and in particular, $\gamma_t \neq 0$ (take $l = k$). Hence $B(t, sk) = (\gamma_t/\gamma)^{-s} B(t, 0)$ for all $s \in \mathbb{Z}$. From [25, Proposition A.9], since $t \neq 0$, we have $B(t, l) \xrightarrow[|l| \to \infty]{} 0$, a contradiction. ∎

Let $\widehat{M}$ be the strong closure of $\widehat{A}$ in $B(\mathcal{H})$. We need a certain expansion of elements of $\widehat{M}$.

Lemma 6.13 *Every $y \in \widehat{M}$ possesses a (unique) sequence of functions $(g_k)_{k \in \mathbb{Z}}$ in $\mathbb{B}(F)$ such that*

$$y = \text{strong-lim}_{N \to \infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) u^k g_k(a, |b|)$$

Proof For each $\lambda \in \mathbb{T}$, define a unitary $w_\lambda \in B(\ell_2(\mathbb{Z}))$ by $w_\lambda(e_l) := \lambda^l e_l$ ($l \in \mathbb{Z}$), and a unitary $W_\lambda \in B(\mathcal{H})$ by $W_\lambda := 1 \otimes w_\lambda$. Then W_λ commutes with $a, |b|$ and $W_\lambda u W_\lambda^* = \lambda u$. For every $k \in \mathbb{N}$ and $g_k \in \mathbb{B}(F)$ we thus get

(6.13) $$\mathrm{Ad}(W_\lambda)(u^k g_k(a, |b|)) = \lambda^k u^k g_k(a, |b|).$$

Given $n \in \mathbb{Z}$, define the "Fourier coefficient" contraction $\Upsilon_n \in B(B(\mathcal{H}))$ by

$$\Upsilon_n(y) := \frac{1}{2\pi} \int_{\mathbb{T}} \lambda^{-n} \mathrm{Ad}(W_\lambda)(y)|\mathrm{d}\lambda| \quad (y \in B(\mathcal{H})),$$

where the integral converges strongly. The operator Υ_n is continuous in the bounded strong operator topology. Letting $\{K_N\}_{N=1}^\infty$ denote Fejér's kernel, we have for $y \in B(\mathcal{H})$ and $N \in \mathbb{N}$,

$$\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right)\Upsilon_n(y) = \int_{\mathbb{T}} \frac{1}{2\pi} K_N(\lambda)\mathrm{Ad}(W_\lambda)(y)|\mathrm{d}\lambda|.$$

Thus, the sequence $\left\{\sum_{n=-N}^{N}(1 - \frac{|n|}{N+1})\Upsilon_n(y)\right\}_{N=1}^\infty$ is bounded by $\|y\|$, and it converges strongly to y.

On account of (6.13), if y has the form $\sum_{k=-N}^{N} u^k g_k(a, |b|)$, then

$$\Upsilon_n(y) = u^n g_n(a, |b|)$$

for $-N \le n \le N$ and 0 otherwise. Every element y of $\widehat{M}$ is the strong limit of a bounded net (y_i) of elements of the form $y_i = \sum_{k \in \mathbb{Z}} u^k g_{ki}(a, |b|)$, where $g_{ki} \ne 0$ for only finitely-many values of k for every i [25, Theorem 2.5.21]. Consequently, $\Upsilon_n(y_i) = u^n g_{ni}(a, |b|) \to \Upsilon_n(y)$ strongly for all n. As u is unitary, we infer that the net $(g_{ni}(a, |b|))_i$ converges strongly for all n, necessarily to $g_n(a, |b|)$ for some $g_n \in \mathbb{B}(F)$. By the foregoing, $y = \lim_N \sum_{n=-N}^{N}(1 - \frac{|n|}{N+1})u^n g_n(a, |b|)$ strongly. For uniqueness, have Υ_n act on both sides of the equation. ∎

Lemma 6.14 *Let $y \in \widehat{M}$, and let (g_k) be the functions corresponding to y as in Lemma 6.13. If y commutes with all the operators $x_{l,t} \in B(\mathcal{H})$, $l, t \in \mathbb{Z}$, then so does $u^k g_k(a, |b|)$ for every $k \in \mathbb{Z}$.*

Proof For $m, n \in \mathbb{Z}$, denote by $p_{m,n}$ the projection of $\mathcal{H}$ onto $\mathbb{C}e_{m,n}$. Fix $l, t \in \mathbb{Z}$. Clearly, $g(a, |b|)$ commutes with $p_{m,n}$ for every $g \in \mathbb{B}(F)$, $up_{m,n} = p_{m,n+1}u$, and $x_{l,t}p_{m,n} = p_{m+2t,n+t}x_{l,t}$. By assumption, we have

$$\lim_{N \to \infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) x_{l,t} u^k g_k(a, |b|) = \lim_{N \to \infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) u^k g_k(a, |b|)x_{l,t},$$

both limits being in the strong operator topology. Fix $k_0 \in \mathbb{Z}$. For every $m, n \in \mathbb{Z}$, we get

$$\lim_{N \to \infty} \sum_{k=-N}^{N} (1 - \frac{|k|}{N+1}) p_{m+2t,n+t+k_0} x_{l,t} u^k g_k(a, |b|)p_{m,n}$$

$$= \lim_{N \to \infty} \sum_{k=-N}^{N} (1 - \frac{|k|}{N+1}) p_{m+2t,n+t+k_0} u^k g_k(a, |b|)x_{l,t}p_{m,n}.$$

As a result, with $z := x_{l,t} u^{k_0} g_{k_0}(a, |b|) - u^{k_0} g_{k_0}(a, |b|) x_{l,t}$, we have

$$z p_{m,n} = p_{m+2t,n+t+k_0} z p_{m,n} = 0.$$

Summing over all $m, n \in \mathbb{Z}$, we get the desired commutation relation. ∎

We are now ready to prove that $\mathbb{T}$ has the separation property in $E(2)$. If $y \in \widehat{M}$ with corresponding functions (g_k) as in Lemma 6.13 commutes with all the operators $(x_{l,t})_{l,t \in \mathbb{Z}}$, then by Lemma 6.14, $u^k g_k(a, |b|)$ commutes with $(x_{l,t})_{l,t \in \mathbb{Z}}$ for every $k \in \mathbb{Z}$. Lemma 6.12 implies that $g_k(a, |b|) = 0$ for $k \neq 0$ and that $g_0 = h \otimes \mathbb{1}$ for a suitable h. This precisely means that y is a function of a, namely $y \in \mathrm{Im}\, \gamma$. So we established (6.2) in our setting, and the proof is complete.

Acknowledgements We thank Matthew Daws for his interest and helpful comments. We are grateful to the referee for carefully reading the paper and making very useful suggestions.

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