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UNCERTAINTY PRINCIPLES INVARIANT UNDER THE FRACTIONAL FOURIER TRANSFORM

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Abstract

Uncertainty principles like Heisenberg's assert an inequality obeyed by some measure of joint uncertainty associated with a function and its Fourier transform. The more groups under which that measure is invariant, the more that measure represents an intrinsic property of the underlying object represented by the given function. The Fourier transform is imbedded in a continuous group of operators, the fractional Fourier transforms, but the Heisenberg measure of overall spread turns out not to be invariant under that group. A new family is developed of measures that are invariant under the group of fractional Fourier transforms and that obey associated uncertainty principles. The first member corresponds to Heisenberg's measure but is generally smaller than his although equal to it in special cases.

1. Introduction

Uncertainty principles assert a reciprocal relation between the spread of a function f and the spread of its Fourier transform $\hat{f}$. The Heisenberg uncertainty principle or "the Heisenberg-Pauli-Weyl inequality" [3] uses the standard deviation of $|f|^2$ as a measure $\Delta(f)$ of the spread of f and the same measure $\Delta(\hat{f})$ for $\hat{f}$; that is, defining

$$\langle f, g\rangle = (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} \overline{f}(x)g(x)\,dx, \qquad \|f\|_2^2 = \langle f, f\rangle \tag{1}$$

and

$$\hat{f}(t) = (\mathcal{F} f)(t) = \langle \exp(ist), f(s)\rangle \tag{2}$$

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and taking zero as the centroid of $|f|^2$ and of $|\hat{f}|^2$,

$$\Delta(f) = \|xf(x)\|_2/\|f\|_2. \tag{3}$$

In units in which Planck's constant equals 2π, Heisenberg's uncertainty principle is:

$$1/2 \leq \Delta(f)\Delta(\hat{f}). \tag{4}$$

Unless both f and $\hat{f}$ decrease to zero at infinity faster than $|x|^{-3/2}$, one of $\Delta(f)$ or $\Delta(\hat{f})$ is infinite and then Heisenberg's principle, (4), is uninformative about the minimum size of the other. Considerable research has been done to circumvent this limitation on the usefulness of Heisenberg's principle by generalising to other measures of spread and by relaxing the requirement that the same measure be used for both f and $\hat{f}$.

The theorem of Cowling and Price in [3, 4] and [14] generalises the results of Heisenberg-Pauli-Weyl, of Hirschman [7], and others. (For a recent bibliography see [9, 15]). A corollary to their theorem is that for $1 \leq p$, $q \leq \infty$, if $\theta, \phi \geq 0$ satisfy $\theta > 1/p^\sharp$ and $\phi > 1/q^\sharp$ where $t^\sharp = 2t/(t-2)$ and α satisfies $\alpha(\theta - 1/p^\sharp) = (1-\alpha)(\phi - 1/q^\sharp)$ then there is $K = K(p, q, \theta, \phi)$ such that for all locally-integrable tempered distributions f,

$$\|f\|_2 \leq K\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\||x|^\theta f(x)\|_p^\alpha \||y|^\phi \hat{f}(y)\|_q^{1-\alpha}. \tag{5}$$

Clearly (4) corresponds to the particular case $p = q = 2$, $\theta = \phi = 1$ (and $K_{\min} = 1/\sqrt{2}$).

In the context of signal analysis, Pollak and Slepian [13], Landau and Pollak [8], and others use a measure of energy concentration on an interval rather than of spread. Letting χ_T be the characteristic function of the interval $[-T, T]$ they define $\alpha^2 = \theta(T, f) = \|\chi_T f\|_2^2/\|f\|_2^2$ and similarly $\beta^2 = \theta(\Omega, \hat{f})$, the fractions of the signal energy within the time interval $[-T, T]$ and the frequency interval $[-\Omega, \Omega]$. Clearly for all f $0 \leq \alpha^2, \beta^2 \leq 1$. They determine a function $\gamma^2 : \mathbb{R}^+ \to [0, 1]$ such that the possible pairs (α, β) lie in that subset of $[0, 1]^2$ (excluding $(0, 1)$ and $(1, 0)$) described by

$$\cos^{-1}(\gamma(\Omega T)) \leq \cos^{-1}\alpha + \cos^{-1}\beta. \tag{6}$$

The results (4–6) can all be put in the general form

$$c(p) \leq \sigma(p, f), \tag{7}$$

where p is a vector of parameters and $\sigma(p, f)$ is some p-measure of "overall" spread of f, composed from measures of spread both of f and of $\hat{f}$. In each of (4–6) the corresponding σ has certain invariance properties: for example for (4) σ is symmetric in f and $\hat{f}$ but for (5) and (6) it is not; for

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(4) and (5) σ is invariant under the normalised dilatation $f \to S_\alpha f$ where $(S_\alpha f)(t) = |a|^{\frac{1}{2}} f(at)$ $(a \in \mathbb{R}^*)$ but for (6) it is not.

In this paper I examine the invariance of the measure σ of overall spread in the Heisenberg principle, (4), (and so of the generalisations (5)) under the fractional Fourier transform $\mathscr{F}_\theta$ that I have recently discussed [11], first introduced by Condon in 1937 [2] and extended by Bargmann in 1961 [1]. The Lie group of fractional Fourier transforms $\{\mathscr{F}_\theta\}_{\theta\in\mathbb{T}}$ (where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, the circle group) is the natural continuous imbedding into the circle $\mathbb{T}$ (in d dimensions into the d-torus $\mathbb{T}^d$) of the cyclic group $\{\mathscr{F}^k\}_{k\in\mathbb{Z}_4}$ of order 4 formed by the powers of Fourier-Plancherel operator $\mathscr{F}$ defined in (2). In particular $\mathscr{F}_0 = \mathscr{I}$, the identity operator, $\mathscr{F}_{\pi/2} = \mathscr{F}$, $\mathscr{F}_{-\pi/2} = \mathscr{F}^{-1}$ and $\mathscr{F}_\pi = \mathscr{F}^2 = \mathscr{T}$ the parity operator.

In an uncertainty principle of the form (7), the more groups under which a non-trivial measure σ is invariant the more it can be said to measure an intrinsic property of the object represented by f. For all $\theta \in \mathbb{T}$ the object represented by f (that is, by $\mathscr{F}_0 f$) is equally well represented by $\mathscr{F}_\theta f$, so for σ to represent an intrinsic property of the whole class $\{\mathscr{F}_\theta f\}_{\theta\in\mathbb{T}}$, it should be invariant under the group $\{\mathscr{F}_\theta\}_{\theta\in\mathbb{T}}$; that is, σ should satisfy

$$\forall\theta \in \mathbb{T} \quad \sigma(p, \mathscr{F}_\theta f) = \sigma(p, f). \tag{8}$$

I show that the Heisenberg measure, implicit in (4), does not satisfy this requirement, and introduce a family of measures that do. The first member of this family reduces to the Heisenberg measure in special cases but is generally *smaller*, so the corresponding uncertainty principle not only has the desirable quality of invariance under $\mathscr{F}_\theta$ but is quantitatively stronger as well.

2. The fractional Fourier transform $\mathscr{F}_\theta$ and the operators $\mathscr{J}$ and $\mathscr{J}^\pm$

One construction [1, 11] of the fractional Fourier transform $\mathscr{F}_\theta (\theta \in \mathbb{T})$ is based on the observation that the set of normalised Hermite functions $\{h_n\}_{n\in\mathbb{N}} (\mathbb{N} = \{0, 1, 2, 3, \ldots\})$ defined by

$$h_n(t) = (2^{n-\frac{1}{2}} n!)^{-\frac{1}{2}} \exp(-t^2/2) H_n(t) \tag{1}$$

(where H_n is the nth Hermite polynomial, defined by $H_n(t) = \exp \times (t^2)(-d/dt)^n \exp(-t^2))$ is a complete orthonormal set of eigenfunctions under (1.1) for the Fourier operator $\mathscr{F}$, satisfying

$$\mathscr{F} h_n = e^{-in\pi/2} h_n, \tag{2}$$

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[5] so that if f has the Fourier-Hermite series

$$f = \sum_{n \in \mathbf{N}} \langle h_n , f \rangle h_n \tag{3}$$

its Fourier transform is

$$\hat{f} = \mathscr{F} f = \sum_{n \in \mathbf{N}} \langle h_n , f \rangle e^{-\mathrm{i} n\pi/2} h_n$$

and so a "fractional" Fourier transform $\mathscr{F}^{\alpha} (\alpha \in \mathbb{R})$ is naturally defined

$$\mathscr{F}^{\alpha} f = \sum_{n \in \mathbf{N}} \langle h_n , f \rangle e^{-\mathrm{i} n\alpha\pi/2} h_n$$

that is, writing $\mathscr{F}^{\alpha} = \mathscr{F}_{\theta}$ where $\theta = \alpha\pi/2 \quad (\theta \in \mathbb{T})$

$$\mathscr{F}_{\theta} f = \sum_{n \in \mathbf{N}} \langle h_n , f \rangle e^{-\mathrm{i} n\theta} h_n. \tag{4}$$

Interchanging the order of integration and summation in (4) (provided $\theta/\pi \notin \mathbb{Z}$) one gets for $0 < |\theta| < \pi$

$$(\mathscr{F}_{\theta} f)(t) = \langle K_{\theta}(s , t) , f(s) \rangle \quad \text{where } K_{\theta}(s , t) = \sum_{n \in \mathbf{N}} e^{\mathrm{i} n\theta} h_n(s) \overline{h}_n(t). \tag{5}$$

The series for the kernel $K_{\theta}(s , t)$ of the integral operator in (5) can be evaluated in closed form by using the generating function for h_n and a unitary and isomorphic mapping between the Hilbert space of entire analytic functions of a complex variable and $L^2(\mathbb{R})$ [1, 11], yielding, eventually, the definition of $\mathscr{F}_{\theta}$ for $0 < |\theta| < \pi$ as

$$(\mathscr{F}_{\theta} f)(t) = \frac{e^{-\frac{\mathrm{i}}{2}(\frac{\pi}{2} \operatorname{sgn} \theta - \theta)}}{\sqrt{2\pi |\sin \theta|}} \int_{\mathbf{R}} \exp \left[-\mathrm{i} \frac{-(s^2 + t^2) \cos \theta + 2st}{2 \sin \theta} \right] f(s) \, ds \tag{6}$$

while for $\theta = 0$ and π one has $\mathscr{F}_0 = \mathscr{I}$ and $\mathscr{F}_{\pi} = \mathscr{T}$.

Defining operators, $\mathscr{D}$, $\mathscr{X}$, $\mathscr{J}^{\pm}$ and $\mathscr{J}$ by

$$\left. \begin{array}{l} (\mathscr{D} f)(t) = (d/dt) f(t), \qquad (\mathscr{X} f)(t) = t f(t) \\ \mathscr{J}^{\pm} = 2^{-\frac{1}{2}} (\pm \mathscr{D} - \mathscr{X}) \quad \text{and} \quad \mathscr{J} = \mathscr{J}^{+} \mathscr{J}^{-} = 2^{-1} (-\mathscr{D}^2 + \mathscr{X}^2 - \mathscr{I}) \end{array} \right\} \tag{7}$$

then it is well known [6, 10, 16] that the h_n are the eigenfunctions of $\mathscr{J}$ and that

$$\mathscr{J}^{+} h_n = -\sqrt{n+1} h_{n+1} ; \quad \mathscr{J}^{-} h_n = -\sqrt{n} h_{n-1} \quad \text{and} \quad \mathscr{J} h_n = n h_n. \tag{8}$$

It is easy to show that under the inner product (1.1)

$$\mathscr{J}^{+*} = \mathscr{J}^{-} ; \quad \mathscr{J}^{-*} = \mathscr{J}^{+} \quad \text{and} \quad \mathscr{J}^{*} = \mathscr{J} \tag{9}$$

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where $*$ denotes the adjoint.The operators obey the commutator relations

$$\begin{aligned}&[\mathscr{J}^+, \mathscr{J}^-] = -\mathscr{I}; \quad [\mathscr{I}, \mathscr{J}^+] = \mathscr{J}^+; \quad [\mathscr{I}, \mathscr{J}^-] = -\mathscr{J}^- \\ &[\mathscr{I}, \mathscr{J}^+] = [\mathscr{I}, \mathscr{J}^-] = [\mathscr{I}, \mathscr{I}] = \mathscr{O} \text{(the additive identity)}\end{aligned} \tag{10}$$

and so constitute a basis for an irreducible representation of a complex 4-dimensional Lie algebra.

I have shown [11] that $-i\mathscr{J}$ is the infinitesimal generator of the Lie group $\{\mathscr{F}_\theta\}_{\theta\in\mathbf{T}}$ and so $\mathscr{J}$ certainly commutes with $\mathscr{F}_\theta$ (as can also be seen from (4) and (8)) although $\mathscr{J}^+$ and $\mathscr{J}^-$ do not. The following propositions state some invariance relations involving 2-norms and inner products of $(\mathscr{J}^+)^k f$ and $(\mathscr{J}^-)^k f$ $(k \in \mathbb{N})$ that are used to construct $\mathscr{F}_\theta$-invariant measures of overall spread. (Since throughout this paper I deal only with the 2-norm I shall drop the subscript "2" from now on.)

Proposition 2.1. *For all $k \in \mathbb{N}$ $\|(\mathscr{J}^+)^k f\|$ and $\|(\mathscr{J}^-)^k f\|$ are $\mathscr{F}_\theta$-invariant; that is to say*

$$\forall\theta \in \mathbb{T} \quad \|(\mathscr{J}^+)^k \mathscr{F}_\theta f\| = \|(\mathscr{J}^+)^k f\| \quad \textit{and} \quad \|(\mathscr{J}^-)^k \mathscr{F}_\theta f\| = \|(\mathscr{J}^-)^k f\|. \tag{11}$$

Proof. By (9)

$$\|(\mathscr{J}^-)^k \mathscr{F}_\theta f\|^2 = \langle \mathscr{F}_\theta f, (\mathscr{J}^+)^k (\mathscr{J}^-)^k \mathscr{F}_\theta f\rangle$$

so if $(\mathscr{J}^+)^k(\mathscr{J}^-)^k$ commutes with $\mathscr{F}_\theta$ then (since $\mathscr{F}_\theta$ is unitary) $\|(\mathscr{J}^-)^k f\|$ is $\mathscr{F}_\theta$-invariant. From the commutation relations (10) one can show (by induction) that

$$(\mathscr{J}^+)^k(\mathscr{J}^-)^k = \mathscr{J}(\mathscr{J} - \mathscr{I})(\mathscr{J} - 2\mathscr{I}) \cdots (\mathscr{J} - (k-1)\mathscr{I}),$$

which is a polynomial in $\mathscr{J}$. As $\mathscr{J}$ commutes with $\mathscr{F}_\theta$ so does every polynomial in $\mathscr{J}$ and therefore so does $(\mathscr{J}^+)^k(\mathscr{J}^-)^k$. This completes the proof that $\|(\mathscr{J}^-)^k f\|$ is $\mathscr{F}_\theta$-invariant. The $\mathscr{F}_\theta$-invariance of $\|(\mathscr{J}^+)^k f\|$ is proved similarly.

Proposition 2.2. *For all $k \in \mathbb{N}$*

$$\forall\theta \in \mathbb{T} \quad \langle(\mathscr{J}^+)^k \mathscr{F}_\theta f, (\mathscr{J}^-)^k \mathscr{F}_\theta f\rangle = e^{-i2k\theta}\langle(\mathscr{J}^+)^k f, (\mathscr{J}^-)^k f\rangle. \tag{12}$$

Proof. Let $f = \sum_{n\in\mathbf{N}} f_n h_n$ where $f_n = \langle h_n, f\rangle$ so $\mathscr{F}_\theta f = \sum_{n\in\mathbf{N}} e^{-in\theta} f_n h_n$ (as in (4)) and so, by (8),

$$(\mathscr{J}^+)^k \mathscr{F}_\theta f = \sum_{n\in\mathbf{N}} e^{-in\theta} f_n \left(\prod_{l=1}^{k} -\sqrt{n+l}\right) h_{n+k}$$

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and

$$(\mathscr{J}^{-})\mathscr{F}_\theta f = \sum_{m\in\mathbf{N}} e^{\mathrm{i}m\theta} f_m \left(\prod_{s=1}^{k} -\sqrt{m+1-s}\right) h_{m-k}.$$

Using the last two results and the orthonormality of $\{h_n\}$ one gets

$$\begin{aligned}\langle(\mathscr{J}^{+})^k\mathscr{F}_\theta f\,,(\mathscr{J}^{-})^k\mathscr{F}_\theta f\rangle\\ = \sum_{n,m}\overline{f}_n f_m e^{-\mathrm{i}\theta(m-n)}\left(\prod_{l=1}^{k}\sqrt{(n+l)(m+1-l)}\right)\delta_{n+k\,,m-k}\end{aligned}$$

that is

$$\langle(\mathscr{J}^{+})^k\mathscr{F}_\theta f\,,(\mathscr{J}^{-})^k\mathscr{F}_\theta f\rangle = e^{-\mathrm{i}2\theta k}\sum_{n\in\mathbf{N}}\overline{f}_n f_{n+2k}\prod_{l=1}^{k}\sqrt{(n_l)(n+2k+1-l)}. \tag{13}$$

Putting $\theta=0$ and using $\mathscr{F}_0=\mathscr{I}$ in (13):

$$\langle(\mathscr{J}^{+})^k f\,,(\mathscr{J}^{-})^k f\rangle = \sum_{n\in\mathbf{N}}\overline{f}_n f_{n+2k}\prod_{\ell=1}^{k}\sqrt{(n+l)(n+2k+1-l)}.$$

Using the last result on the right-hand side of (13) gives (12) as required.

COROLLARY. *$|\langle(\mathscr{J}^{+})^k f\,,(\mathscr{J}^{-})^k f\rangle|$ is $\mathscr{F}_\theta$-invariant.*

3. The Heisenberg measure of spread

Denote by $\sigma_H(f)$ the *square* of the measure of overall spread of f appearing on the right-hand side of (1.4); that is

$$\sigma_H(f) = \Delta^2(f)\Delta^2(\hat{f}). \tag{1}$$

Using the operators $\mathscr{X}$ and $\mathscr{D}$ defined in Section 2 then

$$\Delta^2(f) = \|f\|^{-2}\|\mathscr{X}f\|^2 - \tilde{t}^2 \quad \text{where } \tilde{t} = \|f\|^{-2}\langle\mathscr{X}f\,,f\rangle,$$

the (real) centroid of $|f|^2$, and $\Delta^2(\hat{f}) = \|\hat{f}\|^{-2}\|\mathscr{X}\hat{f}\|^2 - \tilde{\omega}^2$ where $\tilde{\omega} = \|\hat{f}\|^{-2}\langle\mathscr{X}\hat{f}\,,\hat{f}\rangle$, the (real) centroid of $|\hat{f}|^2$. By a familiar device one can always transform f and $\hat{f}$ to a transform pair having an equal pair of values of Δ^2 but with their centroids both at zero; so I suppose for convenience and with no loss of generality that that has already been done, so

$$\Delta^2(f) = \|f\|^{-2}\|\mathscr{X}f\|^2 \tag{2}$$

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and by the unitarity of $\mathscr{F}$ and the fundamental result $\mathscr{F}\mathscr{D} = \mathrm{i}\mathscr{X}\mathscr{F}$:

$$\Delta^2(\hat{f}) = \|f\|^{-2}\|\mathscr{D}f\|^2. \tag{3}$$

From (2.7) $\mathscr{X}$ and $\mathscr{D}$ can be written in terms of $\mathscr{J}^{\pm}$ as

$$\mathscr{X} = 2^{-\frac{1}{2}}(-\mathscr{J}^+ - \mathscr{J}^-) \quad \text{and} \quad \mathscr{D} = 2^{-\frac{1}{2}}(\mathscr{J}^+ - \mathscr{J}^-)$$

from which one can show

$$\|\mathscr{X}f\|^2 = 2^{-1}\{\|\mathscr{J}^+ f\|^2 + \|\mathscr{J}^- f\|^2 + 2\Re\langle\mathscr{J}^+ f, \mathscr{J}^- f\rangle\} \tag{4}$$

and

$$\|\mathscr{D}f\|^2 = 2^{-1}\{\|\mathscr{J}^+ f\|^2 + \|\mathscr{J}^- f\|^2 - 2\Re\langle\mathscr{J}^+ f, \mathscr{J}^- f\rangle\}. \tag{5}$$

PROPOSITION 3.1. *The sum of the variances,* $\Delta^2(f) + \Delta^2(\hat{f})$, *of the energy densities of a function and its Fourier transform is* $\mathscr{F}_\theta$*-invariant.*

PROOF. Adding (4) and (5) and using (2) and (3) one gets

$$\Delta^2(f) + \Delta^2(\hat{f}) = \|f\|^{-2}(\|\mathscr{J}^+ f\|^2 + \|\mathscr{J}^- f\|^2). \tag{6}$$

Proposition 2.1 shows that the right-hand side of (6) is $\mathscr{F}_\theta$-invariant and the result then follows.

PROPOSITION 3.2. *The Heisenberg measure of spread of* f, $\sigma_H(f)$, *is not invariant under the fractional Fourier transform* $\mathscr{F}_\theta$ *but depends on* θ *according to the formula*:

$$\sigma_H(\mathscr{F}_\theta f) = \frac{1}{4}\left\{[\Delta^2(f) + \Delta^2(\hat{f})]^2 - [2\|f\|^{-2}\Re e^{-\mathrm{i}2\theta}\langle\mathscr{J}^+ f, \mathscr{J}^- f\rangle]^2\right\}. \tag{7}$$

PROOF. Multiply (4) and (5) together and use (1)–(3) and (6) to get

$$\sigma_H(f) = \frac{1}{4}\left\{[\Delta^2(f) + \Delta^2(\hat{f})]^2 - [2\|f\|^{-2}\Re\langle\mathscr{J}^+ f, \mathscr{J}^- f\rangle]^2\right\}, \tag{8}$$

then use Propositions 2.1, 2.2 and 3.1 to get the result (7).

4. $\mathscr{F}_\theta$-invariant measures of spread and their uncertainty principles

Looking at the Heisenberg measure σ_H in the form (3.8) in the light of the results of Propositions 3.2, 2.1, 2.2 and its corollary leads one to construct modified and generalised " k th-order measures of overall spread", σ_k.

DEFINITION 4.1. The k th-order measures of overall spread of f and $\hat{f}$ are the functions $\sigma_k(f)$ $(k = 1, 2, \ldots)$ where

$$\sigma_k(f) = \frac{1}{4}\|f\|^{-4}\left\{[\|(\mathscr{J}^+)^k f\|^2 + \|(\mathscr{J}^-)^k f\|^2]^2 - 4|\langle(\mathscr{J}^+)^k f, (\mathscr{J}^-)^k f\rangle|^2\right\}. \tag{1}$$

From Proposition 2.1 and the Corollary to Proposition 2.2 one can see immediately that the σ_k are $\mathscr{F}_\theta$-invariant.

THEOREM 4.1. *For all f the k-measures of its overall spread satisfy the uncertainty principles*

$$\sigma_k(f) \geq 4^{-1}\|f\|^{-4}\{\|(\mathscr{J}^+)^k f\|^2 - \|(\mathscr{J}^-)^k f\|^2\}^2. \tag{2}$$

PROOF. Use the Cauchy-Schwarz-Bunyakovski inequality on the inner-product term in (1).

To justify the inequalities (2) being called "uncertainty principles" one needs to see that they include a recognisable example as a special case.

THE CASE $k = 1$. Equations (3.2)–(3.5) give

$$\|f\|^{-2} 2\Re\langle\mathscr{J}^+ f, \mathscr{J}^- f\rangle = \Delta^2(f) - \Delta^2(\hat{f}) \tag{3}$$

and an elementary calculation gives

$$\|f\|^{-2} 2\Im\langle\mathscr{J}^+ f, \mathscr{J}^- f\rangle = 2\|f\|^{-2}\langle\mathscr{X} f, f\mathscr{D}\arg f\rangle. \tag{4}$$

Using these, (3.1) and (3.6) in the definition (1) with $k = 1$ gives

$$\sigma_1(f) = \sigma_H(f) - \nu^2(f) \tag{5}$$

where $\nu(f)$ is the real number defined by

$$\nu(f) = \|f\|^{-2}\Im\langle\mathscr{X} f, \mathscr{D} f\rangle = \|f\|^{-2}\langle\mathscr{X} f, f\mathscr{D}\arg f\rangle \tag{6}$$

so the first-order $\mathscr{F}_\theta$-invariant measure of overall spread is the Heisenberg measure σ_H *less* a nonnegative quantity ν^2 that is zero only in special cases (for example, if f has constant argument).

A simple calculation and an integration by parts shows that $\|\mathscr{J}^+ f\|^2 - \|\mathscr{J}^- f\|^2 = \|f\|^2$ so for $k = 1$ the inequality (2) becomes

$$\sigma_1(f) = \sigma_H(f) - \nu^2(f) \geq 1/4. \tag{7}$$

One can show by the usual argument that equality holds in (7) if and only if $f(x) = a\exp(-bx^2 + cx)$ where $a, b, c \in \mathbb{C}$ and $\Re b > 0$.

The inequality (7) is superior to the corresponding Heisenberg one ($\sigma_H(f) \geq 1/4$) for the quantitative reason that it is stronger, in the sense that for all

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f, $\sigma_H(f) \geq \sigma_1(f) \geq 1/4$ and for some f $\sigma_H(f) > \sigma_1(f)$, and, but more significantly, for the qualitative reason that the measure $\sigma_1(f)$ is $\mathscr{F}_\theta$-invariant and so measures a more intrinsic property of the "signal" represented both by f and $\hat{f}$ than does σ_H.

Remark 4.1. The familiar Weyl derivation of the Heisenberg inequality (as in [5], p. 119, for example) has as its key step the application of the Cauchy-Schwarz-Bunyakovski inequality to the inner product $\langle \mathscr{X} f, \mathscr{D} f\rangle$:

$$\|\mathscr{X} f\|^2 \|\mathscr{D} f\|^2 \geq |\langle \mathscr{X} f, \mathscr{D} f\rangle|^2; \tag{8}$$

that is, using $\|\mathscr{D} f\|^2 = \|\mathscr{X} \hat{f}\|^2$ and multiplying throughout by $\|f\|^{-4}$,

$$\sigma_H(f) \geq (\|f\|^{-2}\Re\langle \mathscr{X} f, \mathscr{D} f\rangle)^2 + (\|f\|^{-2}\Im\langle \mathscr{X} f, \mathscr{D} f\rangle)^2. \tag{9}$$

The first of the two terms on the right integrates to yield the constant $1/4$ and the second is the $\nu^2(f)$ of equation (5) and which is tacitly jettisoned in the usual next step of using $|\phi|^2 \geq (\Re\phi)^2$. The inequality (7), referred to as "stronger" than Heisenberg's, could therefore have been derived directly from (9), however there are an infinite number of valid inequalities that can be derived from a valid inequality. One could correctly conclude, for example, that for all f, $\sigma_H(f) - \frac{1}{2}\nu^2(f) \geq \frac{1}{4}$, but the left-hand side of this inequality is *not* $\mathscr{F}_\theta$-invariant and so does not represent an intrinsic property as well as σ_1.

The purpose of this paper is not merely to generate valid inequalities but to construct valid inequalities between measures that are $\mathscr{F}_\theta$-invariant and so state something intrinsic about the underlying signal, rather than something contingent on its particular representation. The construction of explicitly $\mathscr{F}_\theta$-invariant measures σ_k has led for $k = 1$ to an inequality that requires, in effect, the restoration and rearrangement of a dropped term in the familiarly derived inequality. That familiar derivation understandably gives no clue as to how the inequality (9) should be rearranged or manipulated so as to state an inequality obeyed by an $\mathscr{F}_\theta$-invariant quantity. Moreover attempting something similar in a higher-order case fails: there is *no* rearrangement of the terms that result from applying the Cauchy-Schwarz-Bunyakovski inequality to $\langle \mathscr{X}^2 f, \mathscr{D}^2 f\rangle$ that will yield an $\mathscr{F}_\theta$-invariant measure.

Remark 4.2. The function $\nu(f)$ has been defined by (6) under the convenient assumptions that

$$\tilde{t} = \|f\|^2\langle \mathscr{X} f, f\rangle = 0 \quad \text{and} \quad \tilde{\omega} = \|f\|^{-2}\langle \mathscr{X} \hat{f}, \hat{f}\rangle = 0.$$

It is easy to show that in the general case the appropriate definition is

$$\nu(f) = \Im\langle \mathscr{X} f, \mathscr{D} f\rangle - \tilde{t}\tilde{\omega}. \tag{10}$$

From this it is easy to show that $\nu(f) = -\nu(\hat{f})$.

REMARK 4.3. Defining translation and modulation operators T_a and M_a by $T_a f(t) = f(t-a)$ and $M_a f(t) = \exp(iat)f(t)$ $(a \in \mathbb{R})$ it is easy to see or to show that both σ_H and σ_1 are invariant under the groups $\{S_a\}_{a\in\mathbf{R}^*}$, $\{T_a\}_{a\in\mathbf{R}}$ and $\{M_a\}_{a\in\mathbf{R}}$.

REMARK 4.4. Defining "chirp" operators C_b by

$$C_b f(t) = \exp(ibt^2/2)f(t) \qquad (b \in \mathbb{R}) \tag{11}$$

it is easy to show that σ_1 is invariant under the group $\{C_b\}_{b\in\mathbf{R}}$ but that σ_H is not and in fact

$$\sigma_H(C_b f) = \sigma_H(f) + \Delta^2(f)[2b\nu(f) + b^2\Delta^2(f)] \tag{12}$$

and so

$$\min_{b\in\mathbf{R}} \sigma_H(C_b f) = \sigma_H(C_{b^*} f) = \sigma_1(f) \quad \text{where } b^* = -\nu(f)/\Delta^2(f). \tag{13}$$

REMARK 4.5. I have shown elsewhere [12] that the energy density $|\mathscr{F}_\theta f(r)|^2$ of the fractional Fourier transform of f (regarded as a function in polar coordinates (r, θ) on the plane) is the Radon transform of $W_f(t, \omega)$, the Wigner distribution of f. Let M_f be the central moment of inertia tensor of the normalised W_f. Its two invariants under rotation of coordinates are $\operatorname{tr} M_f$ and $\det M_f$ and it is not hard to show that $\operatorname{tr} M_f = \Delta^2(f) + \Delta^2(\hat{f})$ (see Proposition 3.1) and that $\det M_f = \sigma_1(f)$. Multiplication of f by C_b alters the inertia tensor of its Wigner distribution, rotating its principal axes and changing its trace but leaving its determinant, σ_1, unchanged. The trace is minimised by the same b that minimises $\Delta^2(\widehat{C_b f})$ and $\sigma_H(C_b f)$, that is the b^* of (13).

EXAMPLE. Take $f(t) = \exp(ict^4 - t^2/2)$ $(c \in \mathbb{R})$. One finds $\Delta^2(f) = 1/2$ (as for h_0), $\Delta^2(\hat{f}) = 1/2 + 30c^2$ and $\nu(f) = 3c$ so $\sigma_H(f) = 1/4 + 15c^2$ and $\sigma_1(f) = 1/4 + 6c^2 < \sigma_H(f)$. Following Remark 4.4 one notices that multiplying f by the operator C_b^* where $b^* = -\nu(f)/\Delta^2(f) = -6c$ will result in a function whose Heisenberg measure of spread is now reduced to $\sigma_1(f)$, the $\mathscr{F}_\theta$-invariant and C_b-invariant measure of the original f:

$$\sigma_H[\exp(-3it^2)f(t)] = \sigma_1[f(t)] = 1/4 + 6c^2,$$

a reduction of $9c^2$.

5. Conclusion

The construction I have given of a family of measures σ_k of overall spread that (unlike the Heisenberg measure σ_H) are invariant under the group $\{\mathscr{F}_\theta\}$ of fractional Fourier transforms has turned out in the case $k = 1$ to lead to a measure also invariant under the group $\{C_b\}$ of chirp operators and to an inequality that is stronger than Heisenberg's, σ_1 being generally smaller than σ_H but having the same sharp lower bound. Unfortunately for $k \geq 2$ the σ_k (like some other measures mentioned in Section 1) are not invariant under the group $\{S_a\}$ of normalised dilatations. The σ_k share a common weakness with σ_H: they are infinite unless f and $\hat{f}$ decay fairly rapidly at infinity. Perhaps an extension of the concept of $\mathscr{F}_\theta$-invariance to the ideas in the results of Hirschman, of Cowling and Price and of Landau, Pollak and Slepian will be successful in eliminating or reducing this weakness.

One of the Landau-Pollak-Slepian theorems relates to the asymptotic dimension of a class of signals that is "almost" band-limited and duration-limited, that is for which

$$\|f - \chi_T f\|^2 < \epsilon \quad \text{and} \quad \|\hat{f} - \chi_\Omega \hat{f}\|^2 < \epsilon, \qquad \text{as } \Omega T \to \infty.$$

I conjecture that using an $\mathscr{F}_\theta$-invariant definition of the class of signals to be considered (for example $S(A, \epsilon) = \{f \mid \min_{\{E\}} \|W_f - \chi_{E(A)} W_f\|^2_{\mathbb{R}^2} < \epsilon\}$ where W_f is the Wigner distribution of f and $\chi_{E(A)}$ is the characteristic function on $\mathbb{R}^2$ of the ellipse of area A) would be more appropriate and will lead to useful and interesting results.

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