Proceedings of the Edinburgh Mathematical Society (2010) **53**, 353–371 ©
DOI:10.1017/S0013091508000837 Printed in the United Kingdom

THE ULTRAMETRIC CORONA PROBLEM AND SPHERICALLY COMPLETE FIELDS

ALAIN ESCASSUT

Laboratoire de Mathématiques, UMR 6620, Université Blaise Pascal (Clermont-Ferrand), Les Cézeaux, 63177 Aubiere cedex, France
(alain.escassut@math.univ-bpclermont.fr)

(Received 9 August 2008)

Abstract Let K be a complete ultrametric algebraically closed field and let A be the Banach K-algebra of bounded analytic functions in the 'open' unit disc D of K provided with the Gauss norm. Let $\mathrm{Mult}(A, \|\cdot\|)$ be the set of continuous multiplicative semi-norms of A provided with the topology of simple convergence, let $\mathrm{Mult}_m(A, \|\cdot\|)$ be the subset of the $\phi \in \mathrm{Mult}(A, \|\cdot\|)$ whose kernel is a maximal ideal and let $\mathrm{Mult}_a(A, \|\cdot\|)$ be the subset of the $\phi \in \mathrm{Mult}(A, \|\cdot\|)$ whose kernel is a maximal ideal of the form $(x-a)A$ with $a \in D$. We complete the characterization of continuous multiplicative norms of A by proving that the Gauss norm defined on polynomials has a unique continuation to A as a norm: the Gauss norm again. But we find prime closed ideals that are neither maximal nor null. The Corona Problem on A lies in two questions: is $\mathrm{Mult}_a(A, \|\cdot\|)$ dense in $\mathrm{Mult}_m(A, \|\cdot\|)$? Is it dense in $\mathrm{Mult}(A, \|\cdot\|)$? In a previous paper, Mainetti and Escassut showed that if each maximal ideal of A is the kernel of a unique $\phi \in \mathrm{Mult}_m(A, \|\cdot\|)$, then the answer to the first question is affirmative. In particular, the authors showed that when K is strongly valued each maximal ideal of A is the kernel of a unique $\phi \in \mathrm{Mult}_m(A, \|\cdot\|)$. Here we prove that this uniqueness also holds when K is spherically complete, and therefore so does the density of $\mathrm{Mult}_a(A, \|\cdot\|)$ in $\mathrm{Mult}_m(A, \|\cdot\|)$.

Keywords: ultrametric Banach algebras; multiplicative spectrum; corona problem

2010 *Mathematics subject classification:* Primary 12J25
Secondary 46S10

1. Introduction and results

Let $B = H^\infty(D)$ be the unital Banach algebra of bounded analytic functions on the open unit disc D in the complex plane. Each $a \in D$ defines a multiplicative linear functional ϕ_a on B by 'point evaluation' i.e. $\phi_a(f) = f(a)$. If a function f lies in the kernel of all the ϕ_a, then clearly $f = 0$. This tells us that the set of all the ϕ_a is dense in the set $\Xi(B)$ of all non-zero multiplicative linear functionals on B in the hull-kernel topology that is lifted from the kernels of the functionals, which are the maximal ideals of B (each maximal ideal, being of codimension 1, is the kernel of a multiplicative linear functional).

The Corona Conjecture of Kakutani was that one also has density with respect to the weak-$*$ topology (or Gelfand topology) which is the topology of simple convergence on B, defined on the space $\Xi(B)$. This was famously proved by Carleson in [**4**]. The key fact is that if $f_1, \dots, f_n$ belong to B and if there exists $d > 0$ such that, for all $a \in D$, we have

$$|f_1(z)| + \cdots + |f_n(z)| > d,$$

then the ideal generated by the $f_1, \dots, f_n$ is the whole of B. People often transfer the name 'Corona Statement' to this key fact. Indeed, this Corona Statement implies that the Corona Conjecture is true, due to the fact that all maximal ideals of a $\mathbb{C}$-Banach algebra are of codimension 1.

Now consider the situation in the non-Archimedean context. Let K be an algebraically closed field complete with respect to an ultrametric absolute value $|\cdot|$. Given $a \in K$ and $r > 0$, we denote by $d(a, r)$ the disc $\{x \in K \mid |x - a| \leqslant r\}$, by $d(a, r^-)$ the disc $\{x \in K \mid |x - a| < r\}$, by $C(a, r)$ the circle $\{x \in K \mid |x - a| = r\}$ and set $D = d(0, 1^-)$. Let A be the K-algebra of bounded power series converging in D which is complete with respect to the Gauss norm defined as

$$\left\| \sum_{n=1}^{\infty} a_n x^n \right\| = \sup_{n \in \mathbb{N}} |a_n|;$$

we know that this norm is actually the norm of uniform convergence on D [**8**, **15**].

In [**19**] the Corona Problem was considered in a similar way to that for the field $\mathbb{C}$ [**4**, **14**]: the author asked whether the set of maximal ideals of A defined by the points of D (which are well known to be of the form $(x - a)A$) is dense in the whole set of maximal ideals with respect to a so-called Gelfand topology. In fact, as explained in [**11**], this makes no sense because the maximal ideals which are not of the form $(x - a)A$ are of infinite codimension [**11**]. Consequently, the *Corona Problem* should be defined in a different way, as explained in [**11**]. However, in [**19**] a 'Corona Statement' similar to that mentioned above was shown in our algebra A and it is useful in the present paper, as it was in [**11**].

Roughly, the 'Corona Statement' shows that each maximal ideal is just the ideal of elements of the algebra B, vanishing along an ultrafilter, on the domain D. Therefore, on $\mathbb{C}$, $f(z)$ has a limit along the ultrafilter and the limit defines a character which, by definition, lies in the closure of the set of characters defined by points of D. And there are no other characters. On the field K, although a similar 'Corona Statement' remains true [**19**], we cannot manage the problem in the same way because $f(x)$ has no limit along an ultrafilter (the field is not locally compact). But we may consider continuous multiplicative semi-norms and then $|f(x)|$ has a limit along an ultrafilter, which defines again a continuous multiplicative semi-norm. But do we get all continuous multiplicative semi-norms whose kernel is a maximal ideal in that way? That is the problem (solved when the field is strongly valued [**11**]). Here we will examine it when the field is spherically complete.

First, here we shall complete the characterization of continuous multiplicative norms on A (begun in [**11**]). Next, we shall look for prime closed ideals other than maximal ideals and the zero ideal: we shall show such ideals do exist. And finally, when the field is spherically complete, we shall show that each maximal ideal is the kernel of only one continuous multiplicative semi-norm, which implies the density of the set of continuous multiplicative semi-norms whose kernel is a maximal ideal of codimension 1 in the set of all of continuous multiplicative semi-norms whose kernel is a maximal ideal (it is well known that the field $\mathbb{C}_p$ is not spherically complete [**17**] but it admits a spherical completion). The main tool with which to solve this problem is the ultrametric holomorphic functional calculus [**7**, **9**], but we also examine ultrafilters.

Given a commutative K-algebra B with unity, provided with a K-algebra norm $\|\cdot\|$, the set of continuous multiplicative K-algebra semi-norms of B was studied in [**1**, **7**–**9**, **13**] and is usually denoted by $\mathrm{Mult}(B, \|\cdot\|)$ [**7**–**9**, **13**]. For each $\phi \in \mathrm{Mult}(B, \|\cdot\|)$, we denote by $\mathrm{Ker}(\phi)$ the closed prime ideal of the $f \in B$ such that $\phi(f) = 0$. The set of the $\phi \in \mathrm{Mult}(B, \|\cdot\|)$ such that $\mathrm{Ker}(\phi)$ is a maximal ideal is denoted by $\mathrm{Mult}_m(B, \|\cdot\|)$, the set of the $\phi \in \mathrm{Mult}(B, \|\cdot\|)$ such that $\mathrm{Ker}(\phi)$ is a maximal ideal of codimension 1 is denoted by $\mathrm{Mult}_a(B, \|\cdot\|)$ and, here, the set of continuous multiplicative norms of A will be denoted by $\mathrm{Mult}_o(B, \|\cdot\|)$.

We know [**9**, **13**] that

$$\sup\{\phi(f) | \phi \in \mathrm{Mult}(B, \|\cdot\|)\} = \lim_{n\to\infty} (\|f^n\|)^{1/n} \quad \forall f \in B.$$

On the other hand, $\mathrm{Mult}(B, \|\cdot\|)$ is provided with the topology of simple convergence and is compact for this topology.

We know that, for every $\mathcal{M} \in \max(B)$, there exists at least one $\phi \in \mathrm{Mult}_m(B, \|\cdot\|)$ such that $\mathrm{Ker}(\phi) = \mathcal{M}$, but in certain cases there exist infinitely many $\phi \in \mathrm{Mult}_m(B, \|\cdot\|)$ such that $\mathrm{Ker}(\phi) = \mathcal{M}$ [**6**, **9**]. A maximal ideal $\mathcal{M}$ of B is said to be *univalent* if there is only one $\phi \in \mathrm{Mult}_m(B, \|\cdot\|)$ such that $\mathrm{Ker}(\phi) = \mathcal{M}$, and the algebra B is said to be *multbijective* if every maximal ideal is univalent (so, non-multbijective commutative Banach K-algebras with unity do exist).

Thus, the ultrametric Corona Problem may be viewed at two levels.

(1) Is $\mathrm{Mult}_a(A, \|\cdot\|)$ dense in $\mathrm{Mult}_m(A, \|\cdot\|)$ (with respect to the topology of simple convergence)?

(2) Is $\mathrm{Mult}_a(A, \|\cdot\|)$ dense in $\mathrm{Mult}(A, \|\cdot\|)$ (with respect to the same topology)?

Actually, this way to set the Corona Problem on an ultrametric field is not really different from the original problem once considered on $\mathbb{C}$, because on a commutative $\mathbb{C}$-Banach algebra with unity all continuous multiplicative semi-norms are known to be of the form $|\chi|$, where χ is a character of A. Thus, the Corona Problem is equivalent to showing that the set of multiplicative semi-norms defined by the points of the open disc is dense inside the whole set of continuous multiplicative semi-norms, with respect to the topology of simple convergence.

Remark 1.1. Given a filter $\mathcal{G}$, if, for every $f \in A$, $|f(x)|$ admits a limit $\varphi_{\mathcal{G}}(f)$ along $\mathcal{G}$, the function $\varphi_{\mathcal{G}}$ obviously belongs to $\mathrm{Mult}(A, \|\cdot\|)$. Moreover, it clearly lies in the closure of $\mathrm{Mult}_a(A, \|\cdot\|)$. Consequently, if we can prove that every element of $\mathrm{Mult}_m(A, \|\cdot\|)$ is of the form $\varphi_{\mathcal{G}}$, with $\mathcal{G}$ a certain filter on D, problem (1) is solved. And similarly, if we could prove that every element of $\mathrm{Mult}(A, \|\cdot\|)$ is of the form $\varphi_{\mathcal{G}}$, problem (2) would be solved.

Studying such problems first requires knowledge of the nature of continuous multiplicative semi-norms on A. The first part of our study consists of studying continuous multiplicative norms on A by completing a study made in [**11**]. We recall the role of circular filters and of ultrafilters.

1.1. Definitions and notation

Let $a \in D$ and let $R \in]0, 1]$. Given $r, s \in \mathbb{R}$ such that $0 < r < s$, we set $\Gamma(a, r, s) = \{x \in K \mid r < |x - a| < s\}$.

We call a *circular filter of centre a and diameter R on D* the filter $\mathcal{F}$ that admits as a generating system the family of sets $\Gamma(\alpha, r', r'') \cap D$ with $\alpha \in d(a, R)$, $r' < R < r''$, i.e. $\mathcal{F}$ is the filter that admits for its basis the family of sets of the form

$$D \cap \left(\bigcap_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i) \right)$$

with $\alpha_i \in d(a, R)$, $r'_i < R < r''_i$, $1 \leqslant i \leqslant q$, $q \in \mathbb{N}$.

Recall that the field K is said to be *spherically complete* if every decreasing sequence of discs has a non-empty intersection (it is known that $\mathbb{C}_p$ is not spherically complete but it has a spherical completion).

In a field which is not spherically complete, one has to consider decreasing sequences of discs (D_n) with an empty intersection. We call the *circular filter with no centre, of canonical basis* (D_n), the filter admitting for its basis the sequence (D_n) and the number $\lim_{n\to\infty} \operatorname{diam}(D_n)$ is called the *diameter of the filter*.

Finally, the filter of neighbourhoods of a point $a \in D$ is called a *circular filter of the neighbourhoods of a on D* and its diameter is 0. Given a circular filter $\mathcal{F}$, its diameter is denoted by $\operatorname{diam}(\mathcal{F})$.

Here, we shall denote by $\mathcal{W}$ the circular filter on D of centre 0 and diameter 1 and by $\mathcal{Y}$ the filter admitting for its basis the family of sets of the form $\Gamma(0, r, 1) \setminus (\bigcup_{n=0}^{\infty} d(a_n, r_n^-))$ with $a_n \in D$, $r_n \leqslant |a_n|$ and $\lim_{n\to\infty} |a_n| = 1$.

An ultrafilter $\mathcal{U}$ on D will be called a *coroner ultrafilter* if it is thinner than $\mathcal{W}$. Similarly, a sequence (a_n) on D will be called a *coroner sequence* if its filter is a coroner filter, i.e. if $\lim_{n\to+\infty} |a_n| = 1$.

Let $\psi \in \operatorname{Mult}(A, \|\cdot\|)$. Then ψ will be said to be *coroner* if its restriction to $K[x]$ is equal to $\|\cdot\|$.

In [**11**] regular ultrafilters were defined: let $(a_n)_{n\in\mathbb{N}}$ be a coroner sequence in D. The sequence is called a *regular sequence* if

$$\inf_{j\in\mathbb{N}} \prod_{n\in\mathbb{N},\, n\neq j} |a_n - a_j| > 0.$$

An ultrafilter $\mathcal{U}$ is said to be *regular* if it is thinner than a regular sequence. Thus, by definition, a regular ultrafilter is a coroner ultrafilter.

Two coroner ultrafilters $\mathcal{F}, \mathcal{G}$ are said to be *contiguous* if, for every subset $F \in \mathcal{F}$, $G \in \mathcal{G}$ of D, the distance from F to G is null.

On $K[x]$, circular filters on K are known to characterize multiplicative semi-norms by associating to each circular filter $\mathcal{F}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$ defined as $\varphi_{\mathcal{F}}(f) = \lim_{\mathcal{F}} |f(x)|$ [**7**–**9**, **12**].

We know that every $f \in A$ is an analytic element in each disc $d(a, r)$ whenever $r \in]0, 1[$ [**8**]. Consequently, by classical results [**8**], several properties of polynomials have continuation to A: given a circular filter $\mathcal{F}$ on D of diameter less than 1, for every $f \in A$, $|f(x)|$ has a limit along $\mathcal{F}$ denoted by $\varphi_{\mathcal{F}}(f)$ and then $\varphi_{\mathcal{F}}$ is a continuous multiplicative semi-norm on A. In particular, given $a \in D$ and $r \in]0, 1[$, if we consider the circular filter $\mathcal{F}$ of centre a and diameter r, we denote by $\varphi_{a,r}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$ which actually is defined by $\varphi_{a,r}(f) = \lim_{|x-a| \to r} |f(x)|$ and is a norm whenever $\operatorname{diam}(\mathcal{F}) > 0$. So, if $\mathcal{F}$ is the circular filter of centre 0 and diameter r, we set $|f|(r) = \varphi_{\mathcal{F}}(f)$. Next, $\varphi_{\mathcal{W}}$ defines the Gauss norm on $K[x]$ and therefore admits a natural continuation to A as $\| \sum_{n=1}^{\infty} a_n x^n \| = \sup_{n \in \mathbb{N}} |a_n|$. However, by [**11**] we know that this continuation is far from unique.

So the problem is first to determine whether such multiplicative semi-norms defined on $K[x]$ by circular filters on D have a unique continuation to A.

Theorem 1.2 was proved in [**11**].

Theorem 1.2. *Let $\mathcal{F}$ be a circular filter on D of diameter $r < 1$. Then $\varphi_{\mathcal{F}}$ admits a unique continuation to A defined by $\varphi_{\mathcal{F}}(f) = \lim_{\mathcal{F}} |f(x)|$ and $\varphi_{\mathcal{F}}$ is a norm on A.*

Thus, the problem arising here is the continuation to A of the Gauss norm defined on $K[x]$, which is not that simple: we have to consider coroner ultrafilters.

Every ultrafilter $\mathcal{U}$ on D defines an element $\varphi_{\mathcal{U}}$ of $\operatorname{Mult}(A, \|\cdot\|)$ as $\varphi_{\mathcal{U}}(f) = \lim_{\mathcal{U}} |f(x)|$; such a limit does exist because each function $f \in A$ is bounded and therefore $|f(x)|$ takes values in the compact $[0, \|f\|]$.

Recall now the following theorem [**11**, Corollary 12.1].

Definitions. An element $f \in A$ is said to be *quasi-invertible* if it has finitely many zeros. Then we know that such an element is of the form Pg with $P \in K[x]$, P having all its zeros in D and g an invertible element of the algebra A.

Given a filter $\mathcal{F}$ on U, we denote by $\mathcal{J}(\mathcal{F})$ the ideal of the $f \in A$ such that $\lim_{\mathcal{F}}(f) = 0$.

A maximal ideal $\mathcal{M}$ of A will be said to be *coroner* (respectively, *regular*) if there exists a coroner (respectively, *regular*) ultrafilter $\mathcal{U}$ such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$.

Theorem 1.3. *Let $\mathcal{M}$ be a maximal ideal of A. Either $\mathcal{M}$ is of codimension 1 and is thus of the form $(x - a)A (a \in D)$, or it is of infinite codimension and thus is coroner. Moreover, if $\mathcal{M}$ is of infinite codimension, then*

(i) *$\varphi_{\mathcal{U}}$ belongs to the closure of* $\operatorname{Mult}_a(A, \|\cdot\|)$,

(ii) *given any $f \in \mathcal{M}$, f is not quasi-invertible.*

By [**11**, Theorem 23], we have the following result.

Theorem 1.4. *Let $\mathcal{U}$ be a regular ultrafilter. Then $\mathcal{J}(\mathcal{U})$ is a regular maximal ideal of A.*

Remark 1.5. Characterizing the coroner ultrafilters $\mathcal{U}$ such that $\mathcal{J}(\mathcal{U})$ is a maximal ideal appears to be very difficult. For instance, consider an ultrafilter $\mathcal{U}$ thinner than $\mathcal{Y}$. It is a coroner ultrafilter. But $\mathcal{J}(\mathcal{U}) = \{0\}$. Indeed, suppose a non-identically zero function f lies in $\mathcal{J}(\mathcal{U})$. Let (a_n) be its sequence of zeros, set $r_n = |a_n|$, $n \in \mathbb{N}$, and let

$$E = D \setminus \bigcup_{n=0}^{\infty} d(a_n, r_n^-).$$

Clearly, $|f(x)| = |f|(|x|)$ for all $x \in E$. However, E belongs to $\mathcal{Y}$, and therefore $\mathcal{U}$ is secant with E, a contradiction with the hypothesis $f \in \mathcal{J}(\mathcal{U})$.

On the other hand, the mapping $\mathcal{J}$ from the set of coroner ultrafilters to the set of ideals of A is not injective: as noted in [**11**], two contiguous coroner ultrafilters define the same ideal.

Now, two corollaries can be derived from Theorems 1.3 and 1.4.

Corollary 1.6. *If A is multbijective, then for every $\phi \in \mathrm{Mult}_m(A, \|\cdot\|)$ there exists a coroner ultrafilter $\mathcal{U}$ such that $\phi = \varphi_{\mathcal{U}}$.*

Corollary 1.7. *If A is multbijective, then $\mathrm{Mult}_a(A, \|\cdot\|)$ is dense in $\mathrm{Mult}_m(A, \|\cdot\|)$.*

Corollaries 1.6 and 1.7 have immediate applications to the case when K is strongly valued.

Definitions and notation. The field K is said to be *strongly valued* if at least its residue class field or its value group is not countable [**8**].

Theorem 1.8. *If K is strongly valued, every commutative K-Banach algebra with unity is multbijective [**7, 9**].*

Corollary 1.9. *If K is strongly valued, then for every $\phi \in \mathrm{Mult}_m(A, \|\cdot\|)$ there exists a coroner ultrafilter $\mathcal{U}$ such that $\phi = \varphi_{\mathcal{U}}$. Moreover, $\mathrm{Mult}_a(A, \|\cdot\|)$ is dense in $\mathrm{Mult}_m(A, \|\cdot\|)$.*

Thus, by Theorems 1.3 and 1.8, if an element $\psi \in \mathrm{Mult}(A, \|\cdot\|)$ is neither the Gauss norm nor of the form $\varphi_{\mathcal{F}}$ on the whole set A, with $\mathcal{F}$ a circular filter on D of diameter $r < 1$, then its restriction to $K[x]$ must be the Gauss norm on $K[x]$. So its kernel is a prime closed ideal included in a maximal ideal of the form $\mathcal{J}(\mathcal{U})$, with $\mathcal{U}$ a coroner ultrafilter.

Here we shall first examine the problem of the continuation of $\varphi_{\mathcal{W}}$ to A through multiplicative norms, which was not done in [**11**].

Notation. Let F be a field, let R be a commutative F-algebra with unity and let $\mathcal{D}$ be a derivation on R. Let J be an ideal of R. We will denote by $\tilde{J}$ the set $\{f \in R \mid \mathcal{D}^{(n)} \in J\ \forall n \in \mathbb{N}\}$.

On A we shall apply this notation to the usual derivation of functions. Let $\psi \in \mathrm{Mult}(A, \|\cdot\|)$. Here we set

$$\mathrm{Subker}(\psi) = \widetilde{\mathrm{Ker}(\psi)}.$$

https://doi.org/10.1017/S0013091508000837 Published online by Cambridge University Press

In [**11**], we asked if there exist prime closed ideals which are neither zero nor maximal ideals. We are now able to answer this question. To do this we first note quite an easy theorem, whose proof mainly holds in basic algebraic considerations.

Theorem 1.10. *Let F be a field, let R be a commutative F-algebra with unity and let $\mathcal{D}$ be a derivation on R. Let J be an ideal of R. Then $\tilde{J}$ is an ideal of R and $\widetilde{(\tilde{J})} = \tilde{J}$. Moreover, if F is of characteristic 0 and if J is prime, so is $\tilde{J}$.*

Since $\|f'\| \leqslant \|f\|$ for all $f \in A$, we can derive the following corollary.

Corollary 1.11. *Suppose that K is of characteristic 0. Let $\mathcal{P}$ be a prime ideal of A. Then $\tilde{\mathcal{P}}$ is a prime ideal of A such that $\widetilde{(\tilde{\mathcal{P}})} = \tilde{\mathcal{P}}$. Moreover, if $\mathcal{P}$ is closed, so is $\tilde{\mathcal{P}}$.*

Corollary 1.12. *Suppose that K is of characteristic 0. Let $\psi \in \mathrm{Mult}(A, \|\cdot\|)$. Then $\mathrm{Subker}(\psi)$ is a prime closed ideal.*

In order to prove Theorem 1.14 and give a counterexample to Theorem 1.10 when K is of characteristic $p \neq 0$, we shall state the following theorem.

Theorem 1.13. *There exist regular maximal ideals $\mathcal{M}$ of A and $f \in \mathcal{M}$, having a sequence of zeros of order 1 and no other zeros, such that $f' \notin \mathcal{M}$, and such that $\tilde{M} \neq \{0\}$.*

Theorem 1.14. *Suppose that K is spherically complete and let $\mathcal{M}$ be a regular maximal ideal of A. There exists $f \in \mathcal{M}$, having a sequence of zeros of order 1 and no other zeros, such that $f' \notin \mathcal{M}$.*

Remarks 1.15.

(i) Now, we may notice that when the field is of characteristic 2 it is easy to show that, for certain maximal ideals $\mathcal{M}$ of A, $\tilde{\mathcal{M}}$ is not prime. Indeed, by Theorem 1.13, there exist a regular maximal ideal $\mathcal{M}$ and $f \in \mathcal{M}$ such that $f' \notin \mathcal{M}$. Hence, f does not belong to $\tilde{\mathcal{M}}$. Now consider $g = f^2$. Then $g' = 2ff' = 0$; hence, $g^{(n)} \in \mathcal{M}$ for all $n \in \mathbb{N}$. If K is of characteristic 3, we can also construct a similar but less simple counter-example.

(ii) In the algebra of bounded complex holomorphic functions in the open unit disc of $\mathbb{C}$, the derivation is not an endomorphism. Consequently, ideals of the form $\tilde{P}$ do not exist.

Following Theorem 1.2, we can now complete the characterization of continuous multiplicative norms on A.

Theorem 1.16. *Let $\psi \in \mathrm{Mult}(A, \|\cdot\|)$ be coroner. Then $\mathrm{Subker}(\psi)$ is not null. Moreover, if K is spherically complete, then, for every $f \in A$ such that $\psi(f) < \|f\|$, there exists $g \in \mathrm{Subker}(\psi)$ admitting no zeros except zeros of f and admitting each zero of f as a zero of order superior or equal to its order as a zero of f.*

Corollary 1.17. *Let $\psi \in \mathrm{Mult}(A, \|\cdot\|)$ be coroner. Then ψ is not a norm.*

By Theorems 1.2 and 1.10, we now can state the following corollary.

Corollary 1.18. *Let $\psi \in \mathrm{Mult}(A, \|\cdot\|)$ be a norm. If ψ is not $\|\cdot\|$, there exists a circular filter $\mathcal{F}$ on D, of diameter $r < 1$, such that $\psi = \varphi_{\mathcal{F}}$.*

On the other hand, each coroner maximal ideal is the kernel of some coroner continuous multiplicative semi-norm of A. Consequently, we have the following.

Corollary 1.19. *Let $\mathcal{M}$ be a coroner maximal ideal of A. Then $\tilde{\mathcal{M}}$ is not null.*

Concerning the Corona Problem, we may note the following.

Corollary 1.20. $\mathrm{Mult}_o(A, \|\cdot\|)$ *is included in the closure of* $\mathrm{Mult}_a(A, \|\cdot\|)$.

By Theorems 1.14 and 1.16, we can derive the following theorem.

Theorem 1.21. *Let K be spherically complete and let $\mathcal{M}$ be a regular maximal ideal. Then $\tilde{\mathcal{M}}$ is neither null nor equal to $\mathcal{M}$.*

Remark 1.22. The prime closed ideal we shall construct, in the proof of Theorem 1.21, which is neither null nor maximal, does not seem to be the kernel of an element of $\mathrm{Mult}(A, \|\cdot\|)$. Recall that in [**3**] an example of a Banach K-algebra of analytic elements with no divisors of zero, admitting no continuous multiplicative norm, was constructed.

Corollary 1.23. *Suppose that K is of characteristic 0. Then A admits prime closed ideals that are neither null nor maximal ideals. Moreover, if K is spherically complete, then every regular maximal ideal $\mathcal{M}$ of A contains a prime closed ideal $\tilde{\mathcal{M}}$ that is neither null nor equal to $\mathcal{M}$.*

Next, we will consider continuous multiplicative semi-norms whose kernels are maximal ideals when the field is spherically complete and we can answer the first question asked at the beginning of the paper.

Theorem 1.24. *If K is spherically complete, then A is multbijective.*

Corollary 1.25. *If K is spherically complete, then for every $\phi \in \mathrm{Mult}_m(A, \|\cdot\|) \setminus \mathrm{Mult}_a(A, \|\cdot\|)$ there exists a coroner ultrafilter $\mathcal{U}$ such that $\phi = \varphi_{\mathcal{U}}$.*

Corollary 1.26. *If K is spherically complete, then* $\mathrm{Mult}_a(A, \|\cdot\|)$ *is dense in* $\mathrm{Mult}_m(A, \|\cdot\|)$.

In [**11**] we considered the following conjectures.

Conjecture 1.27. *A is multbijective no matter what the complete algebraically closed field K.*

This conjecture obviously implies that every $\phi \in \mathrm{Mult}_m(A, \|\cdot\|)$ is of the form $\varphi_{\mathcal{U}}$ with $\mathcal{U}$ an ultrafilter on D, and hence that $\mathrm{Mult}_a(A, \|\cdot\|)$ is dense in $\mathrm{Mult}_m(A, \|\cdot\|)$, independently of the complete algebraically closed field K. It seems very unlikely that A might be non-multbijective for non-spherically complete fields such as C_p. However, in Proposition 2.17, due to Lazard's problem [**16**], the hypothesis 'K is spherically complete' is crucial to factorize a function h in the form $\hat{h}\tilde{h}$.

Conjecture 1.28. $\mathrm{Mult}_a(A, \|\cdot\|)$ *is dense in* $\mathrm{Mult}(A, \|\cdot\|)$.

This seems much more difficult to prove. By Corollary 1.20 we prove that all continuous multiplicative norms are known and do belong to the closure of $\mathrm{Mult}_a(A, \|\cdot\|)$. And, by Corollary 1.25, we have proved that (provided K is spherically complete, or strongly valued) every element of $\mathrm{Mult}(A, \|\cdot\|)$ is defined by a limit of $|f(x)|$ on a filter, except maybe some elements whose kernel is a prime closed ideal that is neither zero nor maximal. Thus, if $\mathrm{Mult}_a(A, \|\cdot\|)$ were not dense in $\mathrm{Mult}(A, \|\cdot\|)$, this would only be due to such elements.

But on the other hand, by Theorem 1.13, we have seen that in characteristic 0 there exist prime closed ideals that are neither null nor maximal ideals. Could they be kernels of continuous multiplicative semi-norms? A natural candidate for such a semi-norm admitting for kernel $\mathrm{Subker}(\psi)$ would be defined as $\tilde{\psi}(f) = \sup_{n\in\mathbb{N}}(\psi(f^{(n)}))$. But this semi-norm is not multiplicative when the residue characteristic of K is $p \neq 0$. If $p = 0$, the answer to this question is not clear.

2. The proofs

Proof of Theorem 1.10. $\tilde{J}$ is obviously an ideal of R. For convenience, let us set $f^{(n)} = \mathcal{D}^{(n)}(f)$, $n \in \mathbb{N}$, with $f^{(0)} = f$. Now let $f \in \tilde{J}$ and consider $f^{(k)}$. Given any $n \in \mathbb{N}$, we have $(f^{(k)})^{(n)} = f^{(n+k)} \in J$; hence, $f^{(k)}$ belongs to $\tilde{J}$. Consequently, $f^{(k)}$ belongs to $\tilde{J}$ for every $k \in \mathbb{N}$. So $\tilde{J}$ is included in $(\tilde{J})\tilde{}$ and therefore is equal to this.

Now, suppose that F is of characteristic 0 and that J is prime. We will check that $\tilde{J}$ is a prime ideal. Let $f, g \in R \setminus \tilde{J}$. So there exist integers $k, l \in \mathbb{N}$ such that $f^{(j)} \in J$ for all $j < k$, $g^{(j)} \in J$ for all $j < l$ and $f^{(k)} \notin J$, $g^{(l)} \notin J$. Consider

$$(fg)^{(k+l)} = \sum_{j=0}^{k+l} \binom{k+l}{j} f^{(j)} g^{(k+l-j)}.$$

It is easily seen that $f^{(j)} g^{(k+l-j)} \in J$ for all $j < k$ and for all $j > k$ and that $f^{(k)} g^{(l)} \notin J$ because J is prime. Consequently, since F is of characteristic 0,

$$\sum_{j=0}^{k+l} \binom{k+l}{j} f^{(j)} g^{(k+l-j)} \notin J,$$

which shows that $\tilde{J}$ is prime. □

In the proof of the other theorems we shall need several basic results. Lemma 2.1 is immediate and Lemma 2.2 is well known [**8**].

Lemma 2.1. *Let $\sum_{n=0}^{\infty} u_n$ be a converging series with positive terms. There exists a sequence of strictly positive integers $t_n \in \mathbb{N}$ satisfying*

$$t_n \leqslant t_{n+1},\ n \in \mathbb{N}, \quad \lim_{n\to\infty} t_n = +\infty, \quad \sum_{m=0}^{\infty} t_m u_m < +\infty.$$

https://doi.org/10.1017/S0013091508000837 Published online by Cambridge University Press

Lemma 2.2. *Let $f \in A$ be non-quasi-invertible and let $(a_n)_{n\in\mathbb{N}}$ be the sequence of zeros with multiplicity q_n. Then the series*

$$\sum_{n=0}^{\infty} q_n \log(|a_n|)$$

converges to $\log(|f(0)|) - \log \|f\|$.

In Proposition 2.17, we shall use the following classical lemmas (Lemmas 2.3–2.6) [**8**].

Lemma 2.3. *Let $a \in K$ and $r > 0$ and $b \in d(a,r)$. Then $\varphi_{a,r} = \varphi_{b,r}$.*

Lemma 2.4. *Let $f \in A$. For every $r \in]0,1[$, f has finitely many zeros. Let $a \in C(0,r)$. If f has no zero in $d(a,r^-)$, then $|f(x)| = |f|(r)$ for all $x \in d(a,r^-)$.*
Moreover, the following three statements are equivalent:

(i) *f is invertible in A;*

(ii) *f has no zeros in D;*

(iii) *$|f(x)|$ is a constant in D.*

Lemma 2.5. *Let $f, g \in A$ be such that every zero a of f is a zero of g of order superior or equal to its order as a zero of f. Then there exists $h \in A$ such that $g = fh$.*

Lemma 2.6. *Let $f \in A$. Then*

$$|f'|(r) \leqslant \frac{|f|(r)}{r} \quad \forall r < 1.$$

By classical results on analytic functions we have the following lemma (see, for example, [**8**, Theorem 23.13]).

Lemma 2.7. *Let $f \in A$ admits zeros $a_1, \dots a_q$ of respective order k_j, $j = 1, \dots, q$, in $\Gamma(0, r', r'')$. Then*

$$|f|(r'') = |f|(r') \prod_{j=1}^{q} (r''|a_j|)^{k_j}.$$

In the proof of Theorem 1.14, we shall need the following lemma.

Lemma 2.8. *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \in H(C(0,r))$$

and assume that f has a unique zero α, of order 1*, in $C(0,r)$. Then $|f'(\alpha)| = |f'|(r)$.*

Proof. By hypothesis, $f(x)$ is of the form $(x-\alpha)h(x)$ with $h \in H(C(0,r))$, having no zero in $C(0,r)$. Then $|f|(r) = r|h|(r)$. Moreover, since h has no zero in $C(0,r)$, we have $|h(\alpha)| = |h|(r)$. And, by Lemma 2.6, $|f'|(r) \leqslant |f|(r)/r$. Therefore, we have

$$|f'(\alpha)| \leqslant |f'|(r) \leqslant \frac{|f|(r)}{r} = |h|(r) = |h(\alpha)| = |f'(\alpha)|$$

and hence $|f'(\alpha)| = |f'|(r)$. □

The following theorem is [**8**, Theorem 25.5].

Theorem 2.9. *Let $(a_j)_{j\in\mathbb{N}}$ be a sequence in $d(0,1^-)$ such that $0<|a_n|\leqslant|a_{n+1}|$ for every $n\in\mathbb{N}$ and $\lim_{n\to\infty}|a_n|=r$. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}^*$ and let $B\in\,]1,+\infty]$. There exists $f\in A(d(0,r^-))$ satisfying the following conditions:*

(i) $f(0)=1$;

(ii) $\|f\|\leqslant B\prod_{j=0}^{n}|a_n/a_j|^{q_j}$ *whenever* $n\in\mathbb{N}$;

(iii) *for each* $n\in\mathbb{N}$, a_n *is a zero of* f *of order* $z_n\geqslant q_n$.

As a corollary of this theorem, we can write the following.

Corollary 2.10. *Let $(a_j)_{j\in\mathbb{N}}$ be a sequence in $d(0,r^-)$ such that $0<|a_n|\leqslant|a_{n+1}|$ for every $n\in\mathbb{N}$, $\lim_{n\to\infty}|a_n|=r$ and let $(q_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}^*$ such that*

$$\prod_{j=0}^{n}\left(\frac{|a_n|}{r}\right)^{q_j}>0.$$

Let $B\in\,]1,+\infty]$. There exists $f\in A$ satisfying the following conditions:

(i) $f(0)=1$;

(ii) $\|f\|\leqslant B\prod_{j=0}^{\infty}(r/|a_n|)^{q_j}$ *whenever* $n\in\mathbb{N}$;

(iii) *for each* $n\in\mathbb{N}$, a_n *is a zero of* f *of order* $z_n\geqslant q_n$.

Lazard's Theorem is well known in the case when K is spherically complete [**10**, **16**]. Here we will just write it in the unit disc.

Theorem 2.11. *Let K be spherically complete. Let A' be the K algebra of all power series converging in D. Let $(a_j)_{j\in\mathbb{N}}$ be a coroner sequence and let (q_n) be a sequence of positive integers. There exists $f\in A'$ admitting each a_n as a zero of order q_n and having no other zeros.*

Consider a function $f\in A'$ such that $f(0)\neq 0$, admitting for zeros the a_n with order 1 and no other zeros. Then, by Lemma 2.7, we have

$$|f|(r)=|f(0)|\prod_{|a_j|<r}\left(\frac{1}{|a_j|}\right).$$

Consequently, such a function is bounded if and only if $\prod_{n=0}^{\infty}|a_n|>0$. Thus, we can derive the following corollary.

Corollary 2.12. *Let K be spherically complete. Let $(a_j)_{j\in\mathbb{N}}$ be a coroner sequence such that $\prod_{n=0}^{\infty}|a_n|>0$. There exist $f\in A$ admitting each a_n as a zero of order 1 and having no other zeros.*

Proof of Theorem 1.13. By [**8**, Theorem 23.15] we know that there exist bounded sequences $(a_n)_{n\in\mathbb{N}}$ in D such that the sequence $|a_n/a_{n+1}|$ is strictly increasing and then the function $f(x)=\sum_{n=0}^{\infty}a_nx^n$ admits a sequence of zeros $(\alpha_n)_{n\in\mathbb{N}^*}$ satisfying $|\alpha_n|=|a_n/a_{n+1}|$. Thus, in particular, if we set $r_n=|a_n/a_{n+1}|$, then f admits exactly one unique zero in each circle $C(0,r_n)$, each of order 1, and has no other zero in D. Consequently, by Lemma 2.8, we can see that $|f'(\alpha_n)|=|f'|(r_n)$ for all $n\in\mathbb{N}^*$. Now, let $\mathcal{U}$ be an ultrafilter thinner than the sequence $(\alpha_n)_{n\in\mathbb{N}^*}$. On the other hand, we can check that the sequence (α_n) is regular; hence, $\mathcal{U}$ is a regular ultrafilter. Consequently, by Theorem 1.4, $\mathcal{J}(\mathcal{U})$ is a maximal ideal of A. Now, f belongs to $\mathcal{J}(\mathcal{U})$ but f' does not.

Now, for each $n\in\mathbb{N}$, let $u_n=\log(r_n)$. Then, by Lemma 2.2, we have $\log(\|f\|)=-\sum_{n=0}^{\infty}u_n$. By Lemma 2.1, there exists an increasing sequence $(s_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{n\to\infty}s_n=+\infty$ and such that the series $\sum_{n=0}^{\infty}s_nu_n$ converges.

By Corollary 2.10, there exist $g\in A$ (not identically zero) such that, for each $n\in\mathbb{N}$, α_n is a zero of g of order $z_n\geqslant s_n$. And since $\lim_{n\to\infty}s_n=+\infty$, for every fixed $k\in\mathbb{N}$, we can see that $f^{(k)}(\alpha_n)=0$ when n is sufficiently large; therefore, $g^{(k)}$ belongs to $\mathcal{M}$. Consequently, $\tilde{\mathcal{M}}$ is not null, which ends the proof. □

Lemma 2.13. *Let $(a_n)_{n\in\mathbb{N}}$ be a regular sequence, let*

$$\delta=\inf_{k\in\mathbb{N}}\prod_{n\neq k,\,n\in\mathbb{N}}|a_n-a_k|$$

and let $\rho=\inf_{k\neq n,k,\,n\in\mathbb{N}}|a_n-a_k|$. Let $f\in A$ admit each a_n as a zero of order 1 and have no other zero. Then

$$|f'(x)|\geqslant\|f\|\delta^2\quad\text{for all } x\in\bigcup_{n=0}^{\infty}d(a_n,(\delta\rho)^-).$$

Proof. Let us fix $t\in\mathbb{N}$, let $r=|a_t|$ and let $E=d(a_t,r-)$. Set $u=x-a_t$, $g(u)=f(x)$ and consider $|g|(\delta)$. The zeros of g in $D\setminus d(0,\delta^-)$ are the a_n-a_t with $n\neq t$. Consequently, by Lemma 2.7, we can check that $|g|(\rho^-)=\|g\|\prod_{n\neq t}|a_n-a_t|$ and therefore

$$|g|(\rho)\geqslant\|g\|\delta=\|f\|\delta. \tag{2.1}$$

Consider now $|g'(u)|$ inside $d(0,\rho^-)$. Since g has a unique zero at 0 in this disc, by Lemma 2.7 we have $|g|(r)=(r/\rho)|g|(\rho)$. Now, $g(u)$ is of the form $uh(u)$, with $h\in A$ having no zero in $d(0,\rho^-)$. Thus, h is of the form $c(1+l(u))$ with $l\in A$ and $|l(u)|<1$ for all $u\in d(0,\rho^-)$, and hence $|l|(r)<1$ for all $r<\rho$. Now, $g'(u)=c(1+l(u)+ul'(u))$. Now, in $d(0,\rho^-)$, since $|l(u)|(r)<1$, by Lemma 2.6 we have $|l'(u)|(r)<1/r$ and hence $|ul'(u)|(r)<1$. Consequently, $|1+l(u)+ul'(u)|=1$ for all $u\in d(0,\rho^-)$ and therefore

$$|g'(u)|=|c|\quad\forall u\in d(0,\rho^-). \tag{2.2}$$

Now, $|g|(\rho)=|uh(u)|(\rho)=\rho|h|(\rho)$. Since $|h(u)|$ is the constant $|c|$ inside $d(0,\rho^-)$, by (2.1) we can see that $|c|\geqslant\|f\|\delta/\rho$. Hence, by (2.2) we have $|g'(u)|\geqslant\|f\|\delta/\rho$ for all

$u \in d(0, \rho^-)$, i.e. $|f'(x)| \geqslant \|f\|\delta/\rho$ for all $x \in d(a_t, \rho^-)$. This is true for every zero a_t of f and, therefore, by setting

$$E = \bigcup_{n=0}^{\infty} d(a_n, \delta^-),$$

we have $|f'(x)| \geqslant \|f\|\delta/\rho$ for all $x \in E$. But clearly E lies in $\mathcal{U}$ and therefore f' does not belong to $\mathcal{J}(\mathcal{U})$. □

Proof of Theorem 1.14. Since $\mathcal{M}$ is a regular maximal ideal, there exists a regular sequence (a_n) and a regular ultrafilter $\mathcal{U}$ thinner than the sequence (a_n) such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$. Since the sequence is regular, we have

$$\delta = \inf_{k\in\mathbb{N}} \prod_{n\neq k,\, n\in\mathbb{N}} |a_n - a_k| > 0 \quad \text{and} \quad \rho = \inf_{k\neq n, k,n\in\mathbb{N}} |a_n - a_k| > 0.$$

Since K is spherically complete, since $\delta > 0$ we may apply Corollary 2.12 showing there exists $f \in A$ admitting each a_n as a zero of order 1. Now, by Lemma 2.13 we have

$$|f'(x)| \geqslant \|f\|\delta^2 \quad \text{for all } x \in \bigcup_{n=0}^{\infty} d(a_n, (\delta\rho)^-),$$

which shows that $\varphi_{\mathcal{U}}(f') > 0$. Consequently, f' does not belong to $\mathcal{M}$. □

Lemma 2.14. *Let $\psi \in \mathrm{Mult}(A, \|\cdot\|)$ satisfy $\psi(P) = \|P\|$ for all $P \in K[x]$. Every quasi-invertible element $f \in A$ also satisfies $\psi(f) = \|f\|$.*

Proof. First, suppose that $f \in$ invertible in A. Then $1 = \psi(f)\psi(f^{-1})$. But $\psi(f) \leqslant \|f\|$, $\psi(f^{-1}) \leqslant \|f^{-1}\|$; hence, both inequalities must be equalities. Now, let $f = Pg \in A$ be quasi-invertible, with $P \in K[x]$ and $g \in A$, invertible in A. Then $\psi(f) = \psi(P)\psi(g) = \|P\|\|g\| = \|Pg\| = \|f\|$. □

Proof of Theorem 1.16. The proof takes advantage of the proof of a theorem in [**2**]. Suppose the claim is wrong. Let $\psi \in \mathrm{Mult}(A, \|\cdot\|)$ be an absolute value on A which is different from the Gauss norm $\|\cdot\|$ on A. Thus, there exists a circular filter $\mathcal{F}$ on D, of diameter $r \leqslant 1$, such that $\psi(P) = \varphi_{\mathcal{F}}(P)$ for all $P \in K[x]$. But, by Theorem 1.2, we know that $r = 1$, and hence the restriction of ψ to $K[x]$ is the Gauss norm. Now, since ψ is not the Gauss norm on A, there exists $f \in A$ such that $\psi(f) < \|f\|$. Actually, without loss of generality, we can choose $f \in A$ such that $\psi(f) < 1 \leqslant \|f\|$. Let $\rho = \psi(f)$. And, up a change of origin, we can also assume that $f(0) \neq 0$. By Lemma 2.14, f is not quasi-invertible; hence, f has a sequence of zeros $(a_n)_{n\in\mathbb{N}}$ in D, with $|a_n| \leqslant |a_{n+1}|$. For each $n \in \mathbb{N}$, let q_n be the multiplicity order of a_n. By Lemma 2.2 we know that $\sum_{n=0}^{\infty} -q_n \log|a_n| < +\infty$. Consequently, by Lemma 2.1 there exists a sequence t_n of strictly positive integers satisfying

$$t_n \leqslant t_{n+1},\ n \in \mathbb{N}, \quad \lim_{n\to\infty} t_n = +\infty, \quad \sum_{n=0}^{\infty} t_n q_n \log(|a_n|) < +\infty.$$

By Corollary 2.10 there exists a function $g \in A$ admitting each a_n as a zero of order $s_n \geqslant t_n q_n$, such that

$$|g|(|a_n|) \leqslant 2\left|\prod_{k=0}^{n}\left(\frac{a_n}{a_k}\right)^{t_n q_n}\right| \quad \forall n \in \mathbb{N}$$

and, consequently, g belongs to A.

Now, for each $n \in \mathbb{N}$ and for each $k = 0, \dots, n$, let $u_{n,k} = \max(0, t_n q_k - s_k)$ and let

$$P_n(x) = \prod_{k=0}^{n}(x - a_k)^{u_{n,k}}.$$

Clearly, all coefficients of P_n lie in D except for the leading coefficient, which is 1. Consequently, $\|P_n\| = 1$ for all $n \in \mathbb{N}$ and therefore

$$\|P_n g\| = \|g\|. \tag{2.3}$$

On the other hand, since the sequence t_n is increasing, we can check that, for each fixed $n \in \mathbb{N}$, each zero a_k of f^{t_n} is a zero of $P_n g$ of order greater than or equal to $t_k q_k$. Consequently, by Lemma 2.6, in the ring A we can write $P_n g$ in the form $f^{t_n}\sigma_n$, with $\sigma_n \in A$.

By (2.3), we have $\|\sigma_n\|\|f^{t_n}\| = \|g\|$; hence, since $\|f\| \geqslant 1$, we can see that $\|\sigma_n\| \leqslant \|g\|$. But now, since the restriction of ψ to $K[x]$ is $\|\cdot\|$, we have $\psi(P_n) = 1$; hence, $\psi(P_n g) = \psi(P_n)\psi(g) = \psi(g)$ and therefore

$$\psi(g) = \psi(f^{t_n}\sigma_n) = \psi(f)^{t_n}\psi(\sigma_n) \leqslant \rho^{t_n}\|g\|. \tag{2.4}$$

Relation (2.4) holds for every $n \in \mathbb{N}$; hence, $\lim_{n\to+\infty} \rho^{t_n}\|g\| = 0$. Consequently, $\psi(g) = 0$, which is a contradiction. This ends the proof of Theorem 1.16. □

In Propositions 2.18 and 2.19 we will denote by $|\cdot|_\infty$ the Archimedean absolute value on $\mathbb{R}$.

Let B be a commutative K-algebra with unity and let $f \in B$. We denote by $sp(f)$ the set of the $\alpha \in K$ such that $f - \alpha$ is not invertible in B.

Proving Proposition 2.16 requires the knowledge of monotonous filters on an infraconnected set, particularly T-filters. Let E be a closed bounded infraconnected subset of K and let $\tilde{E}$ be the smallest disc of the form $d(0, r)$ containing E. Let $a \in \tilde{E}$ and $R \in \mathbb{R}_+^*$ be such that $\Gamma(a, r, R) \cap E \neq \emptyset$ whenever $r \in \left]0, R\right]$ (respectively, $\Gamma(a, R, r) \cap D \neq \emptyset$ whenever $r > R$). We call an *increasing (respectively, a decreasing) filter of centre a and diameter R, on E* the filter $\mathcal{F}$ on E that admits for its basis the family of sets $\Gamma(a, r, R) \cap E$ (respectively, $\Gamma(a, R, r) \cap E$). For every sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_n < r_{n+1}$ (respectively, $r_n > r_{n+1}$) and $\lim_{n\to\infty} r_n = R$, it is seen that the sequence $\Gamma(a, r_n, R) \cap E$ (respectively, $\Gamma(a, R, r_n) \cap E$) is a basis of $\mathcal{F}$ and such a basis will be called *a canonical basis*.

For convenience, in a field that is not spherically complete, we also call a *decreasing filter with no centre, of canonical basis* $(D_n)_{n\in\mathbb{N}}$ *and diameter* $R > 0$, *on E* the circular filter on E that admits for its basis a sequence $(D_n)_n \in \mathbb{N}$ in the form $D_n = d(a_n, r_n) \cap D$ with $D_{n+1} \subset D_n$, $r_{n+1} < r_n$, $\lim_{n\to\infty} r_n = R$ and $\bigcap_{n\in\mathbb{N}} d(a_n, r_n) = \emptyset$.

Now, given a monotonous filter $\mathcal{F}$ on E, it is called a T-*filter* if the holes of E satisfy a certain arithmetical condition [**5**, **8**] also defined by T-sequences [**7**, **18**].

From the classical Krasner–Mittag–Leffler Theorem, here we can state the following proposition [**8**, **16**].

Proposition 2.15. *Let E be a set of the form $d(0,R) \setminus \bigcup_{i\in J} d(a_i, r_i^-)$ (where J is a set of indices). Then any element $h \in H(E)$ has a unique decomposition of the form $\sum_{n=0}^{\infty} h_n$, whereas $h_0 \in H(d(0,R))$ and, for each $n \geqslant 1$, $h_n \in H(K \setminus d(a_{i_n}, r_{i_n}^-))$ and $\lim_{|x|\to+\infty} h_n(x) = 0$, where the sequence of holes $(d(a_{i_n}, r_{i_n}^-))_{n\in\mathbb{N}}$ of E is linked to h and defines its decomposition. Then*

$$\|h\|_E = \max\Big(\|h_0\|_{d(0,R)}, \sup_{n\geqslant 1}(\|h_n\|_{K\setminus(d(a_{i_n}, r_{i_n}^-))})\Big).$$

Furthermore, h_0 is of the form $\sum_{j=0}^{\infty} a_{0,j}x^j$ with $\|h_0\|_{d(0,R)} = \sup_{j\geqslant 0} |a_{0,j}|R^j$ and, for $n \geqslant 1$, h_n is of the form

$$\sum_{j=1}^{\infty} a_{n,j}(x - a(r_{i_n}))^{-j}$$

with $\|h_n\|_{K\setminus(d(a_{i_n}, r_{i_n}^-))} = \sup_{j\geqslant 1} |a_{n,j}|(r_{i_n})^{-j}$.

Proposition 2.16. *Let $(B, \|\cdot\|)$ be a commutative ultrametric K-Banach algebra with unity. Suppose that there exist $f \in B$, $\phi, \psi \in \mathrm{Mult}(B, \|\cdot\|)$ such that $\psi(f) < \phi(f)$, $sp(f) \cap \Gamma(0, \psi(f), \phi(f)) = \emptyset$ and there exists $\varepsilon \in\,]0, \phi(f) - \psi(f)[$ further satisfying $\|(f-a)^{-1}\| \leqslant M$ for all $a \in \Gamma(0, \psi(f), \phi(f) - \varepsilon)$. Then there exists $\gamma \in B$ such that $\psi(\gamma) = 1$, $\phi(\gamma) = 0$.*

Proof. Let $\|\cdot\|_{si}$ be the spectral norm defined on B defined as

$$\|f\|_{si} = \lim_{n\to\infty} \sqrt[n]{\|f^n\|}.$$

Then we know that $\|f\|_{si} = \sup\{\theta(f) \mid \theta \in \mathrm{Mult}(B, \|\cdot\|)\}$.

Let $s = \psi(f)$, $t = \phi(f) - \varepsilon$ and $R = \|f\|$. We first check that

$$M \geqslant \frac{1}{s}. \tag{2.5}$$

Let $a \in \Gamma(0, s, t)$. Since $\psi(f) < |a|$, we have $\psi(f - a) = |a|$; hence,

$$\psi((f-a)^{-1}) = \frac{1}{|a|} > \frac{1}{s}$$

and therefore

$$\|(f-a)^{-1}\|_{si} \geqslant \frac{1}{s}.$$

But of course $\|g\|_{si} \leqslant \|g\|$ for all $g \in B$, so (2.5) holds.

https://doi.org/10.1017/S0013091508000837 Published online by Cambridge University Press

Now, let $L = 1/M$. For each $r \in |K| \cap]s,t[$, we choose $a(r) \in C(0,r)$ and denote by E the set $d(0,R) \setminus (\bigcup_{r \in]s,t[} d(a(r), L^-))$. We notice that $d(a(r), L^-) \subset C(0,r)$. Now, there exists a natural homomorphism σ from $H(E)$ into B such that $\sigma(x) = f$. And since $R = \|f\|$ and $\|(f-b)^{-1}\| \leqslant M$ for all $b \in \Gamma(0,s,t)$, the topological properties of the Krasner-Mittag–Leffler Theorem recalled above show that σ is clearly continuous with respect to the norms $\|\cdot\|_E$ of $H(E)$ and $\|\cdot\|$ of B. Now, let $\psi' = \psi \circ \sigma$, $\phi' = \phi \circ \sigma$. Then both ϕ' and ψ' belong to $\mathrm{Mult}(H(E), \|\cdot\|)$ and satisfy $\psi'(x) = s$, $\phi'(x) = t$. So ψ' is of the form $\varphi_{\mathcal{F}}$, with $\mathcal{F}$ a circular filter secant with $d(0,s)$, and ϕ' is of the form $\varphi_{\mathcal{G}}$, with $\mathcal{G}$ a circular filter secant with $d(0,R) \setminus d(0,t^-)$.

By the properties of T-filters [**8, 9, 18**], we know that, for every $r \in]s,t[$, E admits an increasing idempotent T-sequence $d(a_n, L^-)$ of centre 0 and diameter r, such that $|a_n| < |a_{n+1}|$. Consequently, E admits an increasing T-filter of centre 0 and diameter r (and, similarly, it admits a decreasing T-filter of centre 0 and diameter r) [**18**]. Then by [**8**, Theorem 37.2] there exists $h \in H(E)$ meromorphic on each hole $d(a_n, L^-)$ such that $h(u) = 0$ for all $u \in d(0,R) \setminus d(0,r^-)$, further admitting each a_n as a simple pole or a holomorphic point and no other pole in $d(0,R)$ [**8**]. Moreover, since h is a quasi-invertible element in $H(d(0,s))$, we may choose h having no zero in $d(0,s)$. Consequently, $|h(x)|$ is then constant in $d(0,s)$ and may be taken equal to 1.

Let $\gamma = \sigma(h)$. Then $\psi(\gamma) = \psi'(h) = 1$ and $\phi(\gamma) = \phi'(h) = 0$, which ends the proof. □

By [**19**, Theorem 3.2], we have the following proposition.

Proposition 2.17. *Let $f_1, \dots, f_q \in A$ satisfy $\|f_j\| < 1$ for all $j = 1, \dots, q$ and*

$$\inf\Big\{ \max_{j=1,\dots,q}(|f_j(x)|) \;\Big|\; x \in D \Big\} = \omega > 0.$$

There exist $g_1, \dots, g_q \in A$ such that $\sum_{j=1}^{q} g_j f_j = 1$ and $\max_{j=1,\dots,q} \|g_j\| < \omega^{-2}$.

Proposition 2.18. *Let $\mathcal{M}$ be a non-principal maximal ideal of A and let $\mathcal{U}$ be an ultrafilter on D such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$. Let $f \in A \setminus \mathcal{M}$ be non-invertible in A and let $g \in A$, $h \in \mathcal{M}$ such that $fg = 1 + h$. Let $\lambda = \varphi_{\mathcal{U}}(f)$, let $\varepsilon \in]0, \min(\lambda, 1)[$ and let*

$$\Lambda = \{x \in D \mid ||f(x)g(x)| - 1|_\infty < \varepsilon,\ ||f(x)| - \lambda|_\infty < \varepsilon\}.$$

Suppose that there exist a function $\tilde{h} \in A$ admitting for zeros in D the zeros of h in $D \setminus \Lambda$ and a function $\hat{h} \in A$ admitting for zeros the zeros of h in Λ, each counting multiplicities, so that $h = \tilde{h}\hat{h}$. Then $|\tilde{h}(x)|$ has a strictly positive lower bound in Λ and $\hat{h}$ belongs to $\mathcal{M}$.

Moreover, there exists $\omega \in]0, \lambda[$ such that $\omega \leqslant \inf\{\max(|f(x)|, |\hat{h}(x)|) \mid x \in D\}$. Furthermore, for every $a \in d(0, (\lambda - \varepsilon))$, we have $\omega \leqslant \inf\{\max(|f(x) - a|, |\hat{h}(x)|) \mid x \in D\}$.

Proof. Let $u \in \Lambda$ and let s be the distance of u from $K \setminus \Lambda$. So the disc $d(u, s^-)$ is included in Λ, and hence fg has no zero inside this disc. Consequently, $|f(x)g(x)|$ is a constant b in $d(u, s^-)$. Consider the family F_u of radii of circles $C(u, r)$, containing at least one zero of fg. By Lemma 2.4, F_u has no cluster point different from 1. Consequently, there exists $\rho \geqslant s$ such that fg admits at least one zero in $C(u, \rho)$ and admits no zero in

$d(u,\rho^-)$; then $|f(x)g(x)|$ is a constant c in $d(u,\rho^-)$. However, at u we then see that $b=c$, and therefore $d(u,\rho^-)$ is included in Λ. Hence, $\rho=s$ and therefore fg admits at least one zero α in $C(u,s)$. Thus, at α we have $h(\alpha)=-1$. Therefore, in the disc $d(\alpha,s^-)$ we can check that $\varphi_{\alpha,s}(h)\geqslant 1$. But, by Lemma 2.3, $\varphi_{\alpha,s}(h)=\varphi_{u,s}(h)$; hence, $\varphi_{u,s}(h)\geqslant 1$.

Now,

$$\frac{\|h\|}{\varphi_{u,s}(h)}=\frac{\|\tilde h\|}{\varphi_{u,s}(\tilde h)}\frac{\|\hat h\|}{\varphi_{u,s}(\hat h)}\geqslant\frac{\|\tilde h\|}{\varphi_{u,s}(\tilde h)}.$$

Therefore, since $\varphi_{u,s}(h)\geqslant 1$, we obtain

$$\frac{\|\tilde h\|}{\varphi_{u,s}(\tilde h)}\leqslant\|h\|. \tag{2.6}$$

But, since by definition $d(u,s^-)$ is included in Λ, $\tilde h$ has no zero in this disc; hence, $|\tilde h(x)|$ is constant and equal to $\varphi_{u,s}(\tilde h)$. Consequently, by (2.6) we obtain $\|\tilde h\|/|\tilde h(u)|\leqslant\|h\|$ and therefore we have

$$|\tilde h(u)|\geqslant\frac{\|\tilde h\|}{\|h\|}\quad\forall u\in\Lambda.$$

This shows that $\tilde h$ does not belong to $\mathcal{M}$. Consequently, $\hat h$ does belong to $\mathcal{M}$.

Now, by hypothesis, we have $fg-\hat h\tilde h=1$. Since both g and $\tilde h$ belong to A and therefore are bounded in D, it is obvious that $\inf\{\max(|f(x)|,|\hat h(x)|)\mid x\in D\}>0$. So we may obviously choose $\omega\in\,]0,\lambda-\varepsilon[$ such that $\omega\leqslant\inf\{\max(|f(x)|,|\hat h(x)|)\mid x\in D\}$.

Let $\Lambda'=\{x\in D\mid |f(x)|\geqslant\lambda-\varepsilon\}$ and let $a\in d(0,(\lambda-\varepsilon)^-)$. When β lies in Λ', we have $|f(\beta)|>|a|$; hence, $\max(|f(\beta)-a|,|\hat h(\beta)|)\geqslant\omega$. Now, let β lie in $D\setminus\Lambda'$ and let τ be the distance from β to Λ'. Since $D\setminus\Lambda'$ is open, τ is greater than 0. Thus, we have $\varphi_{\beta,\tau}(f)\geqslant\lambda-\varepsilon$. If f has no zero in $d(\beta,\tau^-)$, we would have $\varphi_{\beta,\tau}(f)=|f(\beta)|<\lambda-\varepsilon$, which a contradiction. Hence, f must have a zero γ in $d(\beta,\tau^-)$. Then $|\hat h(\gamma)|\geqslant\omega$. But since, by definition, $\Lambda\subset\Lambda'$, the zeros of $\hat h$ belong to Λ'. And since $d(\beta,\tau^-)\cap\Lambda'=\emptyset$, actually $\hat h$ has no zero in $d(\beta,\tau^-)$. Consequently, $|\hat h(x)|$ is constant in $d(\beta,\tau^-)$ and hence $|\hat h(\beta)|\geqslant\omega$, which completes the proof. □

By Propositions 2.15–2.17, we can easily state the following proposition.

Proposition 2.19. *Suppose that K is spherically complete. Let $\mathcal{M}$ be a non-principal maximal ideal of A and let $\mathcal{U}$ be an ultrafilter on D such that $\mathcal{M}=\mathcal{J}(\mathcal{U})$. Let $f\in A\setminus\mathcal{M}$ satisfy $\|f\|<1$, let $\lambda=\varphi_{\mathcal{U}}(f)$ and let $\varepsilon\in\,]0,\lambda[$. There exists $\omega>0$ such that, for every $a\in d(0,\lambda-\varepsilon)$, there exists $g_a\in A$ satisfying $(f-a)g_a-1\in\mathcal{M}$ and $\|g_a\|\leqslant\omega^{-2}$.*

Proof. Suppose first that f is invertible in A. Then, by Lemma 2.4, $|f(x)|$ is a constant and hence is equal to λ. Therefore, $|f(x)-a|=\lambda$ for all $a\in d(0,\lambda-\varepsilon)$. Consequently, $f-a$ is invertible and its inverse g_a satisfies $\|g_a\|=\lambda^{-1}$. Thus, we only have to show the claim when f is not invertible.

Since f does not belong to $\mathcal{M}$, we can find $g\in A$ and $h\in\mathcal{M}$ such that $fg=1+h$. Let

$$\Lambda=\{x\in D\mid \big||f(x)g(x)|-1\big|_\infty<\varepsilon,\ \big||f(x)|-\lambda\big|_\infty<\varepsilon\}.$$

https://doi.org/10.1017/S0013091508000837 Published online by Cambridge University Press

Since K is spherically complete, we can factorize h in the form $\tilde{h}\hat{h}$, where $\tilde{h} \in A$ is a function admitting for zeros in D the zeros of h in $D\setminus\Lambda$ and $\hat{h} \in A$ is a function admitting for zeros the zeros of h in Λ, each counting multiplicities, so that $h = \tilde{h}\hat{h}$. Moreover, we can choose $\tilde{h}\hat{h}$ so that $\|\hat{h}\| < 1$. Thus, we have $fg - (\hat{h})(\tilde{h}) = 1$, with $\|f\| < 1$, $\|\hat{h}\| < 1$. By Proposition 2.18, there exists $\omega > 0$ such that $\omega \leqslant \inf\{\max(|f(x)|, |\hat{h}(x)|) \mid x \in D\}$ and that, for every $a \in d(0, (\lambda - \varepsilon))$, we have $\omega \leqslant \inf\{\max(|f(x) - a|, |\hat{h}(x)|) \mid x \in D\}$. Now, we notice that $\|f - a\| < 1$ for every $a \in d(0, \lambda - \varepsilon)$, so we may apply Proposition 2.17 and obtain $g_a, h_a \in A$ such that $(f - a)g_a + \hat{h}h_a = 1$, with $\|g_a\| < \omega^{-2}$, $\|h_a\| < \omega^{-2}$. Since $\hat{h}h_a$ belongs to $\mathcal{M}$, $(f - a)g_a - 1$ belongs to $\mathcal{M}$, which completes the proof. □

Proof of Theorem 1.21. Suppose that A is not multbijective and let $\mathcal{M}$ be a maximal ideal that is not univalent. Let F be the quotient field $A/\mathcal{M}$, let θ be the canonical surjection from A onto F and let $\|\cdot\|_q$ be the K-Banach algebra quotient norm of F. By [**11**, Corollary 12.1] there exists an ultrafilter $\mathcal{U}$ on D such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$. Thus, there exists $\psi \in \mathrm{Mult}(A, \|\cdot\|)$ such that $\mathrm{Ker}(\psi) = \mathcal{M}$ and $\psi \neq \varphi_{\mathcal{U}}$. Consequently, there exists $f \in A$ such that $\psi(f) \neq \varphi_{\mathcal{U}}(f)$, with $\psi(f) \neq 0$, $\varphi_{\mathcal{U}}(f) \neq 0$. We shall check that we may also assume that $\psi(f) < \varphi_{\mathcal{U}}(f)$. Indeed, suppose that $\psi(f) > \varphi_{\mathcal{U}}(f)$. Let $g \in A$ be such that $\theta(g) = \theta(f)^{-1}$. Then we can see that $\psi(g) = \psi(f)^{-1}$, $\varphi_{\mathcal{U}}(g) = (\varphi_{\mathcal{U}}(f))^{-1}$. Therefore, $\psi(g) < \varphi_{\mathcal{U}}(g)$. Thus, we may assume that $\psi(f) < \varphi_{\mathcal{U}}(f)$ without loss of generality. Similarly, we may obviously assume that $\|f\| < 1$. By Lemma 2.14, we know that f is not invertible.

Let $\lambda = \varphi_{\mathcal{U}}(f)$ and let $\varepsilon \in]0, \lambda[$. By Proposition 2.19, there exists $\omega > 0$ such that, for every $a \in d(0, \lambda - \varepsilon)$, there exists $g_a \in A$ satisfying $(f - a)g_a - 1 \in \mathcal{M}$ and $\|g_a\| \leqslant \omega^{-2}$. Now, $\theta(g_a) = (\theta(f - a))^{-1}$. Thus, $\|(\theta(f - a))^{-1}\|_q \leqslant \omega^{-2}$ for all $a \in d(0, \lambda - \varepsilon)$. Now, by applying Proposition 2.16 to the K-Banach algebra F, we can see that there exists $y \in F$ such that $\varphi_{\mathcal{U}}(y) = 0$, $\psi(y) = 1$, which is a contradiction to the hypothesis that $\mathrm{Ker}(\varphi_{\mathcal{U}}) = \mathrm{Ker}(\psi)$. This finishes the proof, by showing that A is multbijective. □

Acknowledgements. I am grateful to Marie-Claude Sarmant and Kamal Boussaf for many remarks and to the editors and the referee for suggestions on § 1.

References

1. V. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, AMS Surveys and Monographs, Volume 33 (American Mathematical Society, Providence, RI, 1990).
2. K. Boussaf, Identity theorem for bounded meromorphic p-adic functions, *Bull. Sci. Math.*, in press.
3. K. Boussaf and A. Escassut, Absolute values on algebras of analytic elements, *Annales Math. Blaise Pascal* **2** (1995), 15–23.
4. L. Carleson, Interpolation by bounded analytic functions and the Corona Problem, *Annals Math.* **76** (1962), 547–559.
5. A. Escassut, T-filtres, ensembles analytiques et transformations de Fourier p-adique, *Annales Inst. Fourier* **25** (1975), 35–80.
6. A. Escassut, Spectre maximal d'une algèbre de Krasner, *Colloq. Math.* **38** (1978), 339–357.
7. A. Escassut, The ultrametric spectral theory, *Period. Math. Hungar.* **11**(1) (1980), 7–60.

8. A. Escassut, *Analytic elements in p-adic analysis* (World Scientific, Singapore 1995).
9. A. Escassut, *Ultrametric Banach algebras* (World Scientific, Singapore, 2003).
10. A. Escassut, p-adic value distribution, in *Some topics on value distribution and differentiability in complex and p-adic analysis* (ed. A. Escassut, W. Tutschke and C. C. Yang), Mathematics Monograph Series, Volume 11, Chapter 2 (China Scientific, Beijing, 2008).
11. A. Escassut and N. Mainetti, About the ultrametric Corona Problem, *Bull. Sci. Math.* **132** (2008), 382–394.
12. G. Garandel, Les semi-normes multiplicatives sur les algèbres d'éléments analytiques au sens de Krasner, *Indagationes Math.* **37** (1975), 327–341.
13. B. Guennebaud, Sur une notion de spectre pour les algèbres normées ultramétriques, thèse (Université de Poitiers, 1973).
14. K. Hoffman, *Banach spaces of analytic functions* (Prentice-Hall, Englewood Cliffs, NJ, 1962).
15. M. Krasner, Prolongement analytique uniforme et multiforme dans les corps valués complets, in *Les tendances géométriques en algèbre et théorie des nombres*, Colloques Internationaux du CNRS Paris, Volume 143, pp. 94–141 (Éditions du Centre National de la Recherche Scientifique, Paris, 1966).
16. M. Lazard, Les zéros des fonctions analytiques sur un corps valué complet, *Publ. Math. IHES* **14** (1962), 47–75.
17. A. Robert, *A course in p-adic analysis*, Graduate Texts in Mathematics, Volume 198 (Springer, 2000).
18. M.-C. Sarmant and A. Escassut, T-suites idempotentes, *Bull. Sci. Math.* **106** (1982), 289–303.
19. M. Van Der Put, The non-archimedean corona problem, in *Table Ronde d'Analyse non Archimedienne*, Mémoires de la Société Mathématique de France, Volume 39, pp. 287–317 (Société Mathématique de France, Paris, 1974).