THE STRUCTURE OF CYCLIC PAIRED-COMPARISON DESIGNS

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1. Introduction and summary

When n objects are to be compared in pairs, a complete experiment requires $N = \binom{n}{2}$ comparisons. There are frequent occasions when it is desirable to make only a fraction F of the possible comparisons, either because N is large or because even an individual comparison is laborious. The problem of what constitutes a satisfactory subset of the comparisons has been considered by Kendall [5] who lays down the following two minimum requirements:

(a) every object should appear equally often;

(b) the design should be "connected" so that it is impossible to split the objects into two sets with no comparison made between objects in one set and objects in the other.

For a limited number of combinations of n and F it is possible to find partially balanced incomplete block (PBIB) designs with two associate classes and of block size two (Clatworthy [1]), designs which possess a high degree of symmetry and for which conditions (a) and (b) are automatically satisfied. However, a more flexible class of experimental arrangements is called for. Such a class is provided by the "cyclic" designs described in § 2 which include the PBIB designs (with blocks of size two) of the group divisible and cyclic types. Cyclic designs were introduced, in a special case, by Kempthorne [4] and have also been studied by McKeon [6]. Both authors were primarily interested in the analysis and were able to develop methods of analysis corresponding to two important applications.

In the present paper, the emphasis is on design rather than analysis. We shall say that two designs are equivalent if one can be obtained from the other by a re-labelling of the n objects. It is shown how the cyclic designs for given n may be enumerated and effectively reduced in number in view of equivalences which exist among designs of the same size. These results are quite independent of the background to the experiment and hold equally whether actual measurements are made on the objects or whether all that is available is some expression of preference for one object in each pair.

However, in order to allow some comparison with the results of Kempthorne, Clatworthy, and McKeon, a table of "efficiencies" of the designs is given. On this criterion, it turns out that some of our designs are superior to any so far proposed, while in other cases our methods show that arrangements previously advocated are optimal in the class of cyclic designs. Group divisible PBIB designs are optimal when they exist; other types of PBIB designs may or may not have higher efficiencies than the best cyclic design of the same size.

A review of work on the design of paired-comparison experiments is given in [2]. It will be convenient to represent the n objects in the experiment by the numbers $0, 1, 2, \ldots, n-1$.

2. Cyclic paired-comparison designs

Consider first the case $n = 5$. The 20 pairwise comparisons which can be made when order of presentation is taken into account may be set out in four cyclic sets:

$\{1\}$:	01	12	23	34	40
$\{2\}$:	02	13	24	30	41
$\{3\}$:	03	14	20	31	42
$\{4\}$:	04	10	21	32	43

Here set $\{s\}$ $(s = 1, 2, \cdots, n-1)$ is of the form

$$0s \quad 1, s+1 \quad \cdots \quad t, s+t \quad \cdots \quad n-1, s+n-1$$

where $t + s$ is reduced modulo n when necessary. Note that in all sets the order of presentation has been balanced out. Set $\{3\}$ is equivalent to $\{2\}$ (briefly $\{3\} \sim \{2\}$) if we ignore the reversal in the order of presentation, as we shall; likewise $\{1\} \sim \{4\}$. Clearly the smallest experiment satisfying Kendall's conditions (a) and (b) consists of 5 comparisons which may be taken to be set $\{1\}$ by suitable labelling of the objects. For $n = 5$ set $\{2\}$ would do equally well since the re-numbering (permutation) of the objects

$$(1) \qquad R(5, 2) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \end{pmatrix}$$

changes $\{1\}$ to $\{2\}$ and $\{2\}$ to $\{4\} \sim \{1\}$. Thus $\{1\}$ and $\{2\}$ differ only in the arbitrary numbers assigned to the objects.

For $n = 6$ the situation is rather different. The 15 distinct comparisons can be broken down into two cyclic sets:

$\{1\}$:	01	12	23	34	45	50
$\{2\}$:	02	13	24	35	40	51

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and the "half-set":

$$\{3\}: \quad 03 \quad 14 \quad 25$$

$\{1\}$—$\{3\}$ all satisfy condition (a), but only $\{1\}$ meets (b). $\{2\}$ is not connected and separates into two subsets 02 24 40 and 13 35 51, the reason being, of course, that 2 is a factor of 6. However, $\{1\}$ may be combined with $\{2\}$, or even with $\{3\}$ if order within a pair is ignored.

Some generalizations are apparent. When n is odd, the $\binom{n}{2}$ distinct pairs may be divided into $\frac{1}{2}(n-1)$ sets of n; for n even into $\frac{1}{2}n-1$ sets of n and one set of $\frac{1}{2}n$. $\{s\} \sim \{n-s\}$ and we shall henceforth let s run from 1 to m, where $m = \frac{1}{2}(n-1)$ or $\frac{1}{2}n$ according as n is odd or even. The leading set $\{1\}$ will always satisfy (a) and (b); so will $\{s\}$ provided s and n are relative primes. If s and n have greatest common divisor (g.c.d.) f, then $\{s\}$ separates into f subsets. A combination of sets $\{s_1\}$, $\{s_2\}$, $\cdots$ gives a connected design if the positive integers n, s_1, s_2, $\cdots$ have g.c.d. unity. Among connected designs of a given size we may intuitively prefer the "most connected" design which can be constructed with a minimum of repetition. This point will be further discussed in § 5.

3. The permutations R(n, u)

The permutation $R(5, 2)$ of equation (1) is a special case of the permutation $R(n, u)(u = 1, 2, \cdots, m)$ which acting on object t $(t = 0, 1, \cdots, n-1)$ changes it to $tu(\text{mod}\, n)$. Correspondingly, under $R(n, u)$

$$\text{set } \{s\} \to \text{set } \{su\} \qquad (s = 1, 2, \cdots, m),$$

where su has to be reduced modulo n and also for the equivalence $\{s\} \sim \{n-s\}$. In particular, $R(n, 1)$ is the identity permutation.

EXAMPLE 1. Under repeated application of $R(17, 8)$

$$\{1\} \to \{8\} \to \{4\} \to \{2\} \to \{1\},$$

where $\{8\} \to \{4\}$ since $8.8 = 64 \equiv 13 \equiv 4$. We write

$$64 \equiv 4(\text{mod } 17; E).$$

Regarded as operations on the sets $\{s\}$ the permutations

$$R(n, 1), R(n, 2), \cdots, R(n, m),$$

are seen to be isomorphic with the integers

$$1, 2, \cdots, m,$$

under multiplication with reduction $(\text{mod}\, n; E)$. We will now study this system of integers more closely.

THEOREM. *If p is a prime, then the integers*

$$G\colon 1, 2, \cdots, m$$

form a multiplicative group under reduction (mod p; E).

PROOF. Let g be a primitive root of p; that is, g is a positive integer $< p$ such that $g^x \not\equiv 1 \pmod{p}$ for $x = 1, 2, \cdots, p-2$, but $g^{p-1} \equiv 1 \pmod{p}$. Then the cyclic group H of order $p-1$

$$H\colon \quad 1, g, g^2, \cdots, g^{p-2}, \qquad \pmod{p}$$

is the complete system of residues mod p, viz.,

$$H\colon \quad 1, 2, \cdots, 2m (= p-1).$$

Clearly the elements

$$G^*\colon \quad 1, g^2, g^4, \cdots, g^{p-3}, \qquad \pmod{p}$$

form a subgroup G^* (the alternating group) of H. Now under the double reduction (mod p; E) we have

$$g^{\frac{1}{2}(p-1)} \equiv 1,$$

and G^* becomes the group

$$1, g, g^2, \cdots, g^{\frac{1}{2}(p-3)}, \qquad (\mathrm{mod}\ p;\ E)$$

which is the G of the theorem.

When n is not prime, the integers less than n and relatively prime to n are said to constitute a reduced system of residues mod n, and form a multiplicative group. The number of such integers, whether n is prime or not, is given by Euler's function $\varphi(n)$ and is even except for the trivial cases $n = 1, 2$. By a simple extension of the above theorem, it is clear that the first $\frac{1}{2}\varphi(n)$ integers relatively prime to n form a multiplicative group, G', under reduction (mod p; E).

The definition of a primitive root may be generalized from p to n: g is a primitive root of n if $g^{\varphi(n)}$ is the smallest power of g equal to one. We have tacitly assumed that every prime has a primitive root. Actually a more general result is well known (e.g. Dickson [3], Theorem 26): There exist primitive roots of n if and only if n equals 2, 4, p^h, or $2p^h$, where p is any prime > 2 and h any integer. In these cases it follows as before that the group G' is given by

$$G'\colon \quad 1, g, g^2, \cdots, g^{\frac{1}{2}\varphi(n)-1} \qquad (\mathrm{mod}\ n;\ E).$$

The generation of G' by powers of some element $g' \in G'$ is in fact possible in a wider set of circumstances. As long as these powers are distinct and not equal to one (mod n; E), but $g'^{\frac{1}{2}\varphi(n)} \equiv 1$ (mod n; E), g' will serve its purpose. We shall call the smallest such integer the least primitive root (mod n; E) of n and denote it by d'. Values of d' are easily obtained for small n and a short list is given in Table 1. Tables of (ordinary) least primitive roots d of primes are of assistance; for we have

$$d' \equiv \min (d, d^2) \qquad (\text{mod } n;\, E).$$

EXAMPLE 2. (a) $n = 14$. G' consists of 1, 3, 5, so that $\frac{1}{2}\varphi(14) = 3$. We have

$$3^2 = 9 \equiv 5, \qquad 3^3 \equiv 5.3 \equiv 1 \qquad (\text{mod } 14;\, E).$$

Thus $d' = 3$.

(b) $n = 23$. From Uspensky and Heaslet [7] we find $d = 5$. Hence

$$d' \equiv \min (5, 25) \equiv 2 \qquad (\text{mod } 23;\, E).$$

4. Enumeration of cyclic paired-comparison designs

We shall now demonstrate the usefulness of the results of § 3 in the enumeration of distinct incomplete paired-comparison designs.

TABLE 1

Elements of G' and least primitive root d' (mod n; E)

n	Elements of G'	d'
5	1, 2	2
6	1	1
7	1, 2, 3	2
8	1, 3	3
9	1, 2, 4	2
10	1, 3	3
11	1, 2, 3, 4, 5	2
12	1, 5	5
13	1, 2, 3, 4, 5, 6	2
14	1, 3, 5	3
15	1, 2, 4, 7	2
16	1, 3, 5, 7	3
17	1, 2, 3, 4, 5, 6, 7, 8	3
18	1, 5, 7	5
19	1, 2, 3, 4, 5, 6, 7, 8, 9	2
20	1, 3, 7, 9	3
21	1, 2, 4, 5, 8, 10	2
22	1, 3, 5, 7, 9	3
23	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	2
24	1, 5, 7, 11	–
25	1, 2, 3, 4, 6, 7, 8, 9, 11, 12	2
26	1, 3, 5, 7, 9, 11	7
27	1, 2, 4, 5, 7, 8, 10, 11, 13	2
28	1, 3, 5, 9, 11, 13	5
29	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	2
30	1, 7, 11, 13	7

EXAMPLE 3. $n = 13$. From Table 1 the least primitive root (mod 13; E) is 2. Successive powers of $R(13, 2)$ therefore generate all the permutations

$R(13, 1)$, $R(13, 2)$, $\cdots$, $R(13, 6)$ so that consideration may be restricted to $R(13, 2)$. Under repeated application of $R(13, 2)$ we obtain

$$\{1\} \to \{2\} \to \{4\} \to \{5\} \to \{3\} \to \{6\} \to \{1\},$$

showing that all individual sets are equivalent. Turning now to pairs of sets we can easily establish the following equivalences

$$\begin{aligned}&\{1, 2\} \sim \{2, 4\} \sim \{4, 5\} \sim \{5, 3\} \sim \{3, 6\} \sim \{6, 1\},\\ &\{1, 3\} \sim \{2, 6\} \sim \{4, 1\} \sim \{5, 2\} \sim \{3, 4\} \sim \{6, 5\},\\ &\{1, 5\} \sim \{2, 3\} \sim \{4, 6\}.\end{aligned}$$

Thus the 15 pairs of sets have been reduced to 3 distinct pairs. All comparisons of pairs of sets, that is, of designs in which each object occurs $r = 4$ times, can consequently be confined to comparisons of the 3 prototype designs $\{1, 2\}$, $\{1, 3\}$, $\{1, 5\}$. In the same way, triples of sets ($r = 6$) may be reduced to the prototypes $\{123\}$, $\{124\}$, $\{125\}$, $\{134\}$, the first three representing 6 members, the last one two. For $r = 8$ there are again 3 prototypes corresponding to the omission of two pairs from the complete design ($r = 12$). There is only one distinct design for $r = 10$ resulting from the absence of any one set in the complete experiment.

Similar arguments apply whenever n is prime and continue to hold for all connected sets even when n is composite. Unconnected sets $\{s_1\}$, $\{s_2\}$, $\cdots$, $\{s_u\}$ are equivalent if and only if each has the same greatest common divisor f (say) with n. It is clear that the structure of these u sets is the same as that of the sets $\{s_1/f\}$, $\{s_2/f\}$, $\cdots$, $\{s_u/f\}$, with n replaced by n/f. The distinct connected designs listed in Table 2 for $n \leqq 15$ have been enumerated with the help of these considerations, and the list could easily be extended. Before reduction for equivalences and removal of unconnected designs, the total number of arrangements of size (n, r) is

$$\binom{\frac{1}{2}(n-1)}{\frac{1}{2}r} \quad \text{for } n \text{ odd},$$

$$\binom{\frac{1}{2}n-1}{\frac{1}{2}r} \quad \text{for } n \text{ even, } r \text{ even},$$

$$\binom{\frac{1}{2}n-1}{\frac{1}{2}(r-1)} \quad \text{for } n \text{ even, } r \text{ odd}.$$

5. Efficiencies of incomplete paired-comparison designs

Kempthorne [4] and McKeon [6] have put cyclic designs to two distinct uses, requiring different analyses, but nevertheless leading to expressions for efficiency which differ only by a proportionality factor. It is, therefore, of interest to compute these efficiencies for the designs of Table 2 in order to see how previously proposed designs compare with our larger collection. We use McKeon's general definition of efficiency, E_f, of any paired comparison design as the ratio of the average between-object variance for the complete design to the average between-object variance for the incomplete design. Denote a particular cyclic incomplete paired-comparison design by $[g_1, g_2, \cdots, g_m]$ with $g_s = 1$ or 0 according as set $\{s\}$ is or is not included in the design. Then the efficiency of this design is (with a slight change of McKeon's notation)

$$E_f = \frac{(n-1)^2}{rnb_0},$$

where

$$b_0 = \sum_{l=1}^{(\frac{1}{2}n-1)} \left(\frac{1}{\lambda_l}\right) + \frac{1}{2\lambda_{\frac{1}{2}n}} \qquad n \text{ even,}$$

$$= \sum_{l=1}^{\frac{1}{2}(n-1)} \frac{1}{\lambda_l} \qquad n \text{ odd,}$$

and

$$\lambda_l = \tfrac{1}{2}r - \sum_{k=1}^{\frac{1}{2}(n-1)} g_k \cos 2\pi(kl/n) - (-1)^l \tfrac{1}{2}g_{\frac{1}{2}n} \qquad n \text{ even,}$$

$$= \tfrac{1}{2}r - \sum_{k=1}^{\frac{1}{2}(n-1)} g_k \cos 2\pi(kl/n) \qquad n \text{ odd.}$$

For a complete design $(r = n-1) E_f = 1$.

The following are some comments on Table 2:

(a) Considerable variation in E_f may exist among designs of the same size (n, r), especially when r/n is relatively small. The choice of design is therefore important. For given n it may happen that a smaller design is more efficient than a poorly chosen larger design. Maximal efficiencies for each size are in bold type.

(b) Unconnected designs have zero efficiency and are not shown. Apart from this, however, there appears to be no simple relationship between efficiency and the "degree of connectedness." For example, when $n = 9$, $r = 4$ the design $\{1, 2\}$ consisting of two connected sets is less efficient than $\{1, 3\}$ which contains the unconnected set $\{3\}$.

(c) The maxima of E_f for $(n, r) = (6, 3)$, $(6, 4)$, $(8, 4)$, $(8, 6)$, $(9, 6)$, $(10, 5)$, $(10, 8)$, $(12, 6)$, $(12, 8)$, $(12, 9)$, $(12, 10)$, $(14, 7)$, and $(15, 10)$ correspond to all group divisible PBIB designs with two associate classes which exist in the range of Table 2 (Clatworthy* [1]).

* Clatworthy's and also Kempthorne's figures for efficiency need to be multiplied up by $2(n-1)/n$ to convert them to values of E_f.

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TABLE 2

Efficiencies of connected cyclic paired-comparison designs

			$n = 5$				
$r =$	2	{1}	.800*				
			$n = 6$				
$r =$	2	{1}	.714				
	3	{1, 3}	.926	{2, 3}	.887*		
	4	{1, 2}	.962				
			$n = 7$				
$r =$	2	{1}	.643*				
	4	{1, 2}	.929*				
			$n = 8$				
$r =$	2	{1}	.583				
	3	{1, 4}	.853				
	4	{1, 2}	.887*	{1, 3}	.942		
	5	{1, 2, 4}	.950*	{1, 3, 4}	.948		
	6	{1, 2, 3}	.980				
			$n = 9$				
$r =$	2	{1}	.533*				
	4	{1, 2}	.847	{1, 3}	.905*		
	6	{1 ,2, 3}	.963*	{1, 2, 4}	.970		
			$n = 10$				
$r =$	2	{1}	.491				
	3	{1, 5}	.785	{2, 5}	.783*		
	4	{1, 2}	.809	{1, 3}	.897	{1, 4}	.900
	5	{1, 2, 5}	.931*	{1, 3, 5}	.953	{1, 4, 5}	.884
		{2, 4, 5}	.866				
	6	{1, 2, 3}	.945	{1, 2, 4}	.944		
	7	{1, 2, 3, 5}	.972	{1, 2, 4, 5}	.972*		
	8	{1, 2, 3, 4}	.988				
			$n = 11$				
$r =$	2	{1}	.455*				
	4	{1, 2}	.772	{1, 3}	.885*		
	6	{1, 2, 3}	.922	{1, 2, 4}	.947*		
	8	{1, 2, 3, 4}	.977*				

* Asterisks refer to designs proposed by Kempthorne [4].

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TABLE 2 (continued)

$n = 12$

$r =$ 2	{1}	.423				
3	{1, 6}	.723*				
4	{1, 2}	.738	{1, 3}	.863	{1, 4}	.842
	{1, 5}	.852	{2, 3}	.879	{3, 4}	.840
5	{1, 2, 6}	.899	{1, 3, 6}	.895	{1, 4, 6}	.920*
	{1, 5, 6}	.820	{2, 3, 6}	.895	{3, 4, 6}	.887
6	{1, 2, 3}	.900	{1, 2, 4}	.924	{1, 2, 5}	.935
	{1, 3, 4}	.922	{1, 3, 5}	.960	{1, 4, 5}	.924
	{2, 3, 4}	.942				
7	{1, 2, 3, 6}	.950	{1, 2, 4, 6}	.940	{1, 2, 5, 6}	.949
	{1, 3, 4, 6}	.958*	{1, 3, 5, 6}	.960	{1, 4, 5, 6}	.945
	{2, 3, 4, 6}	.939				
8	{1, 2, 3, 4}	.967	{1, 2, 3, 5}	.970	{1, 2, 4, 5}	.976
	{1, 3, 4, 5}	.965				
9	{1, 2, 3, 4, 6}	.982*	{1, 2, 3, 5, 6}	.984	{1, 2, 4, 5, 6}	.981
	{1, 3, 4, 5, 6}	.982				
10	{1, 2, 3, 4, 5}	.992				

$n = 13$

$r =$ 2	{1}	.396*				
4	{1, 2}	.706	{1, 3}	.845*	{1, 5}	.865
6	{1, 2, 3}	.878	{1, 2, 4}	.918	{1, 2, 5}	.938*
	{1, 3, 4}	.923				
8	{1, 2, 3, 4}	.955	{1, 2, 3, 5}	.957	{1, 2, 3, 6}	966*.
10	{1, 2, 3, 4, 5}	.983*				

$n = 14$

$r =$ 2	{1}	.371				
3	{1, 7}	.668	{2, 7}	.668*		
4	{1, 2}	.676	{1, 3}	.827	{1, 4}	.852
	{1, 6}	.805				
5	{1, 2, 7}	.874	{1, 3, 7}	.901	{1, 4, 7}	.905*
	{1, 6, 7}	.762	{2, 4, 7}	.857		
6	{1, 2, 3}	.855	{1, 2, 4}	.901	{1, 2, 5}	.928
	{1, 2, 6}	.909	{1, 3, 4}	.909	{1, 3, 5}	.940
	{1, 4, 6}	.928				
7	{1, 2, 3, 7}	.936	{1, 2, 4, 7}	.950*	{1, 2, 5, 7}	.940
	{1, 2, 6, 7}	.918	{1, 3, 4, 7}	.918	{1, 3, 5, 7}	.966
	{1, 4, 6, 7}	.940				
8	{1, 2, 3, 4}	.941	{1, 2, 3, 5}	.954	{1, 2, 3, 6}	.956
	{1, 2, 4, 6}	.936	{1, 2, 5, 6}	.963		
9	{1, 2, 3, 4, 7}	.965	{1, 2, 3, 5, 7}	.971	{1, 2, 3, 6, 7}	.972*
	{1, 2, 4, 6, 7}	.964	{1, 2, 5, 6, 7}	.963		
10	{1, 2, 3, 4, 5}	.978	{1, 2, 3, 4, 6}	.978		
11	{1, 2, 3, 4, 5, 7}	.987	{1, 2, 3, 4, 6, 7}	.987*		
12	{1, 2, 3, 4, 5, 6}	.994				

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Table 2 (continued)

$n = 15$						
$r =$ 2	{1}	.350*				
4	{1, 2}	.649	{1, 3}	.808*	{1, 4}	.842
	{1, 5}	.781	{1, 6}	.840	{3, 5}	.781
6	{1, 2, 3}	.834	{1, 2, 4}	.889	{1, 2, 5}	.913
	{1, 2, 6}	.921	{1, 3, 4}	.897	{1, 3, 5}	.931*
	{1, 3, 6}	.888	{1, 4, 5}	.931	{1, 4, 6}	.933
	{1, 5, 6}	.874	{3, 5, 6}	.883		
8	{1, 2, 3, 4}	.927	{1, 2, 3, 5}	.940	{1, 2, 3, 6}	.945
	{1, 2, 3, 7}	.954	{1, 2, 4, 7}	.955	{1, 2, 5, 6}	.951*
	{1, 2, 5, 7}	.951	{1, 3, 4, 5}	.952	{1, 3, 4, 6}	.954
	{1, 3, 5, 6}	.951	{1, 4, 5, 6}	.919		
10	{1, 2, 3, 4, 5}	.970	{1, 2, 3, 4, 6}	.973	{1, 2, 3, 4, 7}	.973
	{1, 2, 3, 5, 6}	.976*	{1, 2, 3, 5, 7}	.971	{1, 2, 4, 5, 7}	.980
	{1, 3, 4, 5, 6}	.971				
12	{1, 2, 3, 4, 5, 6}	.989*	{1, 2, 3, 4, 5, 7}	.989	{1, 2, 3, 4, 6, 7}	.990

There are only two other cases which are PBIB designs, viz. (5, 2) and the arrangement {1, 3, 4} for $n = 13$, $r = 6$. These designs are not group divisible and are formally classified as cyclic PBIB designs. The second is not optimal.

(d) Triangular PBIB designs exist for (6, 4) (10, 3), (10, 6), (15, 6), (15, 8) with respective efficiencies .962, .818, .947, .925, .942. There is only one Square PBIB design in the range of Table 2, viz. (9, 4) with $E_f = .889$. In general, Triangular and Square designs are not cyclic. The efficiency of such a design may or may not exceed that of the best cyclic design of the same size.

(e) Kempthorne's designs exist only when $n+r$ is odd. They are of the form $\{\frac{1}{2}(n-r+1), \frac{1}{2}(n-r+2), \cdots, m\}$, where $m = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd. Each of these designs is represented in Table 2 by an equivalent one. They sometimes have maximal efficiency.

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