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Characterizations of invertible, unitary, and normal composition operators

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Let C_ϕ be a composition operator on $L^2(\lambda)$, where λ is a σ-finite measure on a set X . Conditions under which C_ϕ is invertible, unitary, and normal are investigated in this paper.

1. Preliminaries

Let (X, S, λ) be a σ-finite positive measure space, and let ϕ be a measurable non-singular $\{\lambda\phi^{-1}(E) = 0$ whenever $\lambda(E) = 0\}$ transformation from X into itself. Then we define a linear transformation C_ϕ on the Hilbert space $L^2(\lambda)$ into the space of all complex-valued functions on X by $C_\phi f = f \circ \phi$ for f in $L^2(\lambda)$. In case C_ϕ is a bounded operator with range in $L^2(\lambda)$, we call it a composition operator induced by ϕ . The Radon-Nikodym derivative of the measure $\lambda\phi^{-1}$ with respect to the measure λ will be denoted by f_0 .

If E and F are sets in S , then $E \Delta F = (E-F) \cup (F-E)$. The notation '$E \subset F$' will mean $\lambda(E-F) = 0$. The sets E and F are said to be equivalent, in symbols '$E = F$' , if $\lambda(E \Delta F) = 0$. Two sigma-subalgebras S_1 and S_2 contained in S will be called equivalent, if to every set E in either one of them there corresponds a set F in the

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other so that $E = F$.

Every essentially bounded complex-valued measurable function θ on X induces the multiplication operator M_θ on $L^2(\lambda)$, which is defined by $M_\theta f = \theta \cdot f$ for all f in $L^2(\lambda)$.

Let $p = \{p_1, p_2, p_3, \ldots\}$ be a sequence of strictly positive numbers. Then $l^2(p)$ denotes the Hilbert space of all sequences $\{x_n\}$ of complex numbers such that $\sum_{i=1}^{\infty} p_i |x_i|^2 < \infty$. The Banach algebra of all bounded linear operators on $L^2(\lambda)$ will be denoted by $B(L^2(\lambda))$.

THEOREM 1.1. *Let* $C_\phi \in B(L^2(\lambda))$. *Then* C_ϕ *is bounded away from zero, if and only if there exists an* $M > 0$ *such that* $\lambda\phi^{-1}(E) \geq M\lambda(E)$ *for every* $E \in S$. *Also*

$$\inf_{\|f\| \neq 0} \|C_\phi f\|^2/\|f\|^2 = m(C_\phi) = \sup\{M : \lambda\phi^{-1}(E) \geq M\lambda(E) \text{ for all } E \in S\} .$$

The proof is dual to the proof of [7, Theorem 1] or [5, Theorem 2.1.1]. //

COROLLARY 1.1. *If* $C_\phi \in B(L^2(\lambda))$, *then* $m(C_\phi) = {}_\infty\|f_0\|$, *where* ${}_\infty\|f_0\|$ *denotes the essential infimum of* f_0 . //

LEMMA 1.2. *Let* $M_\theta \in B(L^2(\lambda))$. *Then* M_θ *is one-to-one if and only if* $\theta \neq 0$ *almost everywhere.* //

2. Invertible composition operators

DEFINITION. Let (X, S, λ) be a measure space. Then a measurable transformation ϕ on X into itself is said to be one-to-one (or left invertible) if there exists a measurable transformation ψ on X into itself such that $(\psi \circ \phi)(x) = x$ almost everywhere. It is said to be onto (or right invertible) if there exists a measurable transformation ω such that $(\phi \circ \omega)(x) = x$ almost everywhere. It is said to be invertible if there is a measurable transformation ψ such that $(\phi \circ \psi)(x) = (\psi \circ \phi)(x) = x$ almost everywhere. Such a ψ is called the inverse of ϕ, and is denoted by ϕ^{-1}.

DEFINITION. Let f be a complex-valued measurable function on X. Then

$$\text{ess. range } f = \{c : c \in C,\ \lambda(f^{-1}(F)) \neq 0 \text{ for every neighborhood } F \text{ of } c\} .$$

The following theorem characterizes one-to-one composition operators.

THEOREM 2.1. *Let* $C_\phi \in B(L^2(\lambda))$. *Then the following statements are equivalent:*

(i) C_ϕ *is one-to-one;*

(ii) ess. range f = ess. range $C_\phi f$ *for every* $f \in L^2(\lambda)$ *;*

(iii) $\lambda(E) = 0$ *, whenever* $\lambda\phi^{-1}(E) = 0$ *for* $E \in S$ *;*

(iv) $f_0 \neq 0$ *almost everywhere, where* $f_0 = d\lambda\phi^{-1}/d\lambda$.

Proof. *(i)* ⇒ *(ii)*. In view of [7, Theorem 1], it is always true that ess. range $C_\phi f \subset$ ess. range f. To show the reverse inclusion let $c \in$ ess. range f and F be a neighborhood of c. Then $f^{-1}(F)$ is a non-null set and since (by hypothesis) C_ϕ is one-to-one, $\phi^{-1}(f^{-1}(F))$ is a non-null set. Hence $c \in$ ess. range $C_\phi f$.

(ii) ⇒ *(iii)*. Let $\lambda\phi^{-1}(E) = 0$. Then ess. range $C_\phi X_E = \{0\}$, where X_E denotes the characteristic function of E, and hence ess. range $X_E = \{0\}$. This implies that $\lambda(E) = 0$.

(iii) ⇒ *(iv)*. Since $\lambda\phi^{-1}(E) = \int_E f_0 d\lambda$ for every $E \in S$, it follows that $f_0 \neq 0$ almost everywhere.

(iv) ⇒ *(i)*. If $f_0 \neq 0$, then by Lemma 1.2, M_{f_0} is one-to-one.

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Since $C_\phi^* C_\phi = M_{f_0}$ (see [6]), it follows that $C_\phi^* C_\phi$ is one-to-one, and hence C_ϕ is one-to-one. //

COROLLARY 2.1. *Let* C_ϕ *be a one-to-one composition operator on* $L^2(\lambda)$. *Then* $C_\phi f$ *is a characteristic function, if and only if* f *is a characteristic function.*

Proof. If f is a characteristic function, then clearly $C_\phi f$ is a characteristic function. Conversely, suppose $C_\phi f$ is a characteristic function. Then, since ess. range f = ess. range $C_\phi f$ by Theorem 2.1, it follows that f is a characteristic function. //

COROLLARY 2.2. *If* X *is a non-atomic measure space, then the nullity of* C_ϕ *is either zero or infinite.* //

COROLLARY 2.3. *Let* $C_\phi \in B(l^2(p))$. *Then* C_ϕ *is one-to-one if and only if* ϕ *is onto.* //

THEOREM 2.2. *Let* $C_\phi \in B(L^2(\lambda))$. *Then* C_ϕ *is one-to-one if* ϕ *is onto and a right inverse of* ϕ *is a non-singular transformation.*

Proof. Since ϕ is onto, there exists a measurable transformation ω such that $(\phi \circ \omega)(x) = x$ almost everywhere. Now let $E \in S$. Then $\omega^{-1}(\phi^{-1}(E)) = (\phi \circ \omega)^{-1}(E) = E$. Since ω is non-singular, $\lambda(E) = 0$ whenever $\lambda(\phi^{-1}(E)) = 0$. Hence, by Theorem 2.1, C_ϕ is one-to-one. //

The converse of the above theorem is not true in general, as is shown by the following example.

EXAMPLE 2.1. Let S be the set of all subsets of N , the set of natural numbers, and let λ be the counting measure. Then, if ϕ is the mapping defined by $\phi(n) = n$ if n is odd and $\phi(n) = n - 1$ if n is even, the operator C_ϕ is one-to-one on $L^2(N, S_1, \lambda)$, where $S_1 = \phi^{-1}(S) = \{\phi^{-1}(E) : E \in S\}$. But ϕ is clearly not onto. //

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The following theorem characterizes surjective composition operators.

THEOREM 2.3. *Let* $C_\phi \in B\left(L^2(\lambda)\right)$. *Then* C_ϕ *is onto, if and only if there exists an* $\alpha \in R$ *such that* $f_0 \geq \alpha > 0$ *on* X^{f_0} , *and* $\phi^{-1}(S) = S$, *where* $X^{f_0} = \{x : x \in X \text{ and } f_0(x) \neq 0\}$ *and* $\phi^{-1}(S) = \{\phi^{-1}(E) : E \in S\}$.

We first prove the following lemma.

LEMMA 2.4. *Let* $C_\phi \in B\left(L^2(\lambda)\right)$. *Then the range of* C_ϕ *is dense in* $L^2\left(X, \phi^{-1}(S), \lambda\right)$.

Proof. Let f belong to the range of C_ϕ . Then $f = C_\phi g$ for some g in $L^2(\lambda)$. Since the set of all simple functions is dense in $L^2(\lambda)$, there exists a sequence $\{g_n\}$ of simple functions such that $g_n \to g$. The boundedness of C_ϕ implies that $C_\phi g_n \to C_\phi g = f$. Clearly $C_\phi g_n$ belongs to $L^2\left(X, \phi^{-1}(S), \lambda\right)$ for all n ; it follows that f belongs to $L^2\left(X, \phi^{-1}(S), \lambda\right)$. Since X is σ-finite, we can write $X = \bigcup_{i=1}^{\infty} X_i$, where the X_i's are disjoint and $\lambda\left(X_i\right) < \infty$ for every i . Let χ_E be in $L^2\left(X, \phi^{-1}(S), \lambda\right)$. Then

$$\chi_E = \chi_{\phi^{-1}(F)} = \sum_{i=1}^{\infty} \chi_{\phi^{-1}\left(F_i\right)} = \sum_{i=1}^{\infty} C_\phi \chi_{F_i} ,$$

where $F_i = F \cap X_i$. The sum on the right-hand side converges to χ_E almost everywhere. By the Lebesgue Dominated Convergence Theorem, it converges to χ_E in L^2-norm, and hence χ_E is in the closure of the range of C_ϕ . This is enough to show that the range of C_ϕ is dense in $L^2\left(X, \phi^{-1}(S), \lambda\right)$. //

Proof of the theorem. Suppose $\phi^{-1}(S) = S$. Then by the above lemma the range of C_ϕ is dense in $L^2(\lambda)$, and if $f_0 \geq \alpha > 0$ on X^{f_0} , then

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by [8, Theorem 2.2] the range of C_ϕ is closed. Hence C_ϕ is onto.

Conversely, suppose C_ϕ is onto. Then C_ϕ has closed range, and hence f_0 is bounded away from zero on X^{f_0} [8, Theorem 2.2]. The set $\phi^{-1}(S)$ is always a sub-set of S. To show the reverse inclusion, let $E \in S$, and assume $\lambda(E) < \infty$. Since C_ϕ is onto, there exists g in $L^2(\lambda)$ such that $C_\phi g = \chi_E$. Let $F = \{x : x \in X \text{ and } g(x) = 1\}$. Then clearly $C_\phi \chi_F = \chi_E$, and hence $\phi^{-1}(F) = E$. Thus $E \in \phi^{-1}(S)$. This shows that $\phi^{-1}(S) = S$. //

THEOREM 2.5. *Let* $C_\phi \in B(L^2(\lambda))$. *Then* C_ϕ *has dense range, if and only if* $\phi^{-1}(S) = S$.

Proof. The theorem follows from the fact that the range of C_ϕ is dense in $L^2(X, \phi^{-1}(S), \lambda)$. //

COROLLARY 2.4. *Let* $C_\phi \in B(L^2(\lambda))$. *Then* C_ϕ *has dense range, if* ϕ *is one-to-one.*

Proof. Suppose ϕ is one-to-one. Then there exists a measurable transformation ω such that $(\omega \circ \phi)(x) = x$ almost everywhere. If $E \in S$, then $(\omega \circ \phi)^{-1}(E) = \phi^{-1}(\omega^{-1}(E)) = E$. Let $F = \omega^{-1}(E)$. Then $\phi^{-1}(F) = E$. Hence $E \in \phi^{-1}(S)$. //

The converse of the above corollary is not true in general, as is evident from Example 2.1, where C_ϕ is onto, but ϕ is not one-to-one.

COROLLARY 2.5. *Let* $C_\phi \in B(l^2(p))$. *Then* C_ϕ *has dense range, if and only if* ϕ *is one-to-one.*

Proof. Sufficiency follows from Corollary 2.4.

To prove the necessary part, suppose ϕ is not one-to-one. Then $\phi(n_1) = \phi(n_2)$ for some $n_1 \neq n_2$. It is easy to show that neither $\{n_1\}$

nor $\{n_2\}$ belongs to $\phi^{-1}(S)$. Hence, by Theorem 2.5, C_ϕ does not have dense range. //

COROLLARY 2.6. *Let* $\inf p_i = \alpha > 0$ *and* $\sup p_i = \beta < \infty$. *Then* C_ϕ *is onto, if and only if* ϕ *is one-to-one.*

Proof. This follows from [8, Theorem 2.5 and corollary]. //

Now we proceed to give a characterization of the invertible composition operators.

THEOREM 2.6. *Let* $C_\phi \in B(L^2(\lambda))$. *Then* C_ϕ *is invertible if and only if there exists an* $\alpha \in R$ *such that* $f_0 \geq \alpha > 0$ *almost everywhere on* X, *and* $\phi^{-1}(S) = S$.

Proof. This follows from Theorem 2.1 and Theorem 2.3. //

REMARK. If the underlying measure algebra has only two elements, that is $S = \{\emptyset, X\}$, then every composition operator on $L^2(\lambda)$ is invertible.

COROLLARY 2.7. *Let* $C_\phi \in B(l^2(p))$. *Then* C_ϕ *is one-to-one with dense range, if and only if* ϕ *is invertible.* //

COROLLARY 2.8. *Let* $\inf p_i = \alpha > 0$ *and* $\sup p_i < \beta < \infty$. *Then* $C_\phi \in B(l^2(p))$ *is invertible, if and only if* ϕ *is invertible.* //

Example 2.1 shows that the invertibility of C_ϕ does not necessarily imply the invertibility of ϕ ; in general the invertibility of ϕ and the non-singularity of ϕ^{-1} does not imply invertibility of C_ϕ , as is shown in the following examples.

EXAMPLE 2.2. Let X be the set of natural numbers and $p = \{1, 2, 3, \ldots\}$. Let ϕ be the mapping defined by $\phi(n) = n^2$ when $n = a_n$, where $a_n = (a_{n-1})^2 + 1$ with $a_0 = 0$, and $\phi(n) = n - 1$ otherwise. Consider the sequence of characteristic functions $X_{\{a_n^2\}}$.

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Then $\|C_\phi \chi_{\{a_n^2\}}\|^2 / \|\chi_{\{a_n^2\}}\|^2 = 1/a_n$. This shows that C_ϕ is not bounded below, and hence C_ϕ is not invertible.

EXAMPLE 2.3. Let $X = [0, 1]$ and S be the σ-algebra of Borel sets of the unit interval with Lebesgue measure. If $\phi(x) = \sqrt{x}$, then C_ϕ is a bounded operator [7, Theorem 3]. It is clear that ϕ is invertible and ϕ^{-1} is non-singular. But, since $\|C_\phi \chi_{[0,1/n]}\|^2 / \|\chi_{[0,1/n]}\|^2 = 1/n$, C_ϕ is not bounded away from zero, and hence it is not invertible.

Now it is clear from the above examples that characterization of the invertibility of C_ϕ in terms of the invertibility of ϕ (and *vice-versa*) is not possible in general. From what is done so far, it is evident that the underlying σ-algebra of measurable sets plays an important role in the invertibility of C_ϕ . For some suitable σ-algebra the invertibility of C_ϕ can be characterized in terms of the invertibility of ϕ .

DEFINITION. A topological space X is called an absolute Borel space if it is homeomorphic to a Borel subset of the Hilbert cube. An absolute Borel space with a σ-finite measure on its Borel subsets is called an absolute measure space.

THEOREM 2.7. *Let X be an absolute measure space, and let C_ϕ be a bounded operator on $L^2(\lambda)$. Then C_ϕ is invertible, if and only if ϕ is invertible with non-singular inverse and ϕ^{-1} induces a composition operator on $L^2(\lambda)$.*

First we shall prove the following lemma.

LEMMA 2.8. *If the composition operator C_ϕ on $L^2(\lambda)$ is invertible, then C_ϕ^{-1} takes characteristic functions into characteristic functions.*

Proof. Let χ_E be in $L^2(\lambda)$. Then, since C_ϕ is onto, there

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exists a function g in $L^2(\lambda)$ such that $C_\phi g = \chi_E$. Since C_ϕ is one-to-one, by Corollary 2.1, g is a characteristic function. //

Proof of the theorem. Suppose ψ is the non-singular inverse of ϕ and $C_\psi \in B(L^2(\lambda))$. Then $C_\phi C_\psi = C_{\psi\circ\phi} = I = C_{\phi\circ\psi} = C_\psi C_\phi$, where I denotes the identity operator. This shows that C_ϕ is invertible.

Conversely, suppose C_ϕ is invertible. Then, by Lemma 2.8, C_ϕ^{-1} takes characteristic functions into characteristic functions. On the quotient σ-algebra $[S]$ of S modulo sets of measure zero we define a mapping h as $h([E]) = [F]$ when $C_\phi^{-1}\chi_E = \chi_F$ (or, equivalently, $F = \text{support } C_\phi^{-1}f$, for $E = \text{support } f = \{x : f(x) \neq 0\}$).

If $E_1 \cap E_2 = \emptyset$, then

$$C_\phi^{-1}\chi_{E_1\cup E_2} = C_\phi^{-1}\chi_{E_1} + C_\phi^{-1}\chi_{E_2} .$$

From this it follows that $h([E_1])$ and $h([E_2])$ are disjoint and

$$h([E_1] \cup [E_2]) = h([E_1]) \cup h([E_2]) .$$

For all $[E_1]$ and $[E_2]$, we have

$$(1) \qquad h([E_1]) = h([E_1] \cap [E_2]) \cup h([E_1]-[E_2]) ,$$

$$(2) \qquad h([E_2]) = h([E_2] \cap [E_1]) \cup h([E_2]-[E_1]) .$$

From (1) and (2), it can be shown that

$$h([E_1] \cap [E_2]) = h([E_1]) \cap h([E_2])$$

and

$$h([E_1]-[E_2]) = h([E_1]) - h([E_2]) .$$

If $\{X_i\}$ is an increasing sequence of measurable sets of finite measure such that $X = \bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty} \phi^{-1}(X_i)$, then

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$$\bigcup_{i=1}^{\infty} h\left(\left[\phi^{-1}(X_i)\right]\right) = \bigcup_{i=1}^{\infty} [X_i] = [X] \subseteq h([X]) \subseteq [X] .$$

Hence

$$h([X]) = [X] .$$

This shows that h is a homomorphism. Let $\{E_n\}$ be a sequence of disjoint measurable sets of finite measure. Then, since C_ϕ^{-1} is bounded, we have $C_\phi^{-1}f = \sum_{n=1}^{\infty} C_\phi^{-1}\left(\alpha_n \cdot X_{E_n}\right)$, where $f = \sum_{n=1}^{\infty} \alpha_n X_{E_n}$ and $\alpha_n = 1/n \cdot \lambda\left(E_n\right)$. This, together with the fact that $\bigcup_{n=1}^{\infty} E_n$ = support f and $\left\{\text{support } C_\phi^{-1}\left(\alpha_n \cdot X_{E_n}\right)\right\}$ is a sequence of disjoint measurable sets, implies that

$$h\left(\bigcup_{n=1}^{\infty} [E_n]\right) = \bigcup_{n=1}^{\infty} h([E_n]) ,$$

which proves that h is a σ-homomorphism. Since $h([E_1]) = h([E_2])$ implies that $[E_1] = [E_2]$ and $h\left(\left[\phi^{-1}(E)\right]\right) = [E]$ for every $[E] \in [S]$, we conclude that h is an automorphism. By [4, p. 139] there exists a point mapping ψ from X into itself such that $h([E]) = \left[\psi^{-1}(E)\right]$. This shows that ψ is a non-singular transformation, and $C_\phi^{-1}X_E = X_{\psi^{-1}(E)} = C_\psi X_E$. Now C_ψ is bounded on the characteristic functions; it follows from [7, Theorem 1] that C_ψ is a bounded operator. Since $C_{\phi\circ\psi} = I = C_{\psi\circ\phi}$, we get $(\phi \circ \psi)(x) = (\psi \circ \phi)(x) = x$ almost everywhere. This shows that ϕ is invertible, and ϕ^{-1} induces a composition operator. //

The above theorem is true for all such spaces where every automorphism or σ-homomorphism is induced by a unique point mapping. The following are some examples of measure spaces where the theorem is valid.

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EXAMPLE 2.4 [*1*, Lemma 5, p. 112]. X is a compact metric space and λ is a finite Borel measure on X.

EXAMPLE 2.5 [*4*, §32.1]. (X, S, λ) is any measure space such that S is σ-perfect and reduced. (A measure space (X, S, λ) is said to be reduced, if for any two different points x, y in X there exists a non-null set $E \in S$ such that $x \in E$ and $y \notin E$.)

3. Unitary composition operators

LEMMA 3.1. *Let* $C_\phi \in B(L^2(\lambda))$. *Then* $C_\phi = M_\theta$ *for some* θ *implies that* $\theta(x) = 1$ *almost everywhere.*

Proof. Let $X = \bigcup_{n=1}^{\infty} E_n$, where $\lambda(E_n) < \infty$ for each n, and $E_n \subset E_m$ if $m > n$. If $f_n = X_{E_n}$, then we have $C_\phi f_n = M_\theta f_n$ for all n; equivalently $X_{\phi^{-1}(E_n)} = \theta \cdot X_{E_n}$ for all n. Since $\bigcup_{n=1}^{\infty} \phi^{-1}(E_n) = X$, we conclude that $\theta(x) = 1$ almost everywhere. //

THEOREM 3.1. *Let* X *be an absolute measure space, and let* C_ϕ *be a bounded operator on* $L^2(\lambda)$. *Then the following statements are equivalent:*

(i) C_ϕ *is unitary;*

(ii) $f_0 = 1$ *almost everywhere and* ϕ *is one-to-one;*

(iii) $f_0 = 1$ *almost everywhere and* C_ϕ *is invertible;*

(iv) C_ϕ^* *is a composition operator.*

Proof. *(i)* ⇒ *(ii)*. Suppose C_ϕ is unitary. Then C_ϕ is invertible, and it follows immediately from Theorem 2.7 that ϕ is one-to-one. We know that $M_{f_0} = C_\phi^* C_\phi = I$; hence $f_0 = 1$ almost everywhere.

(ii) ⇒ *(iii)*. Whenever ϕ is one-to-one, C_ϕ has dense range, and hence C_ϕ is invertible (because C_ϕ is an isometry).

(iii) ⇒ (iv). Since $C_\phi^* = M_{f_0} C_\phi^{-1} = C_\phi^{-1} = C_{\phi^{-1}}$, C_ϕ^* is a composition operator.

(iv) ⇒ (i). Suppose C_ϕ^* is a composition operator. Then there exists a measurable transformation ψ such that $C_\phi^* = C_\psi$. Since $C_\phi^* C_\phi = M_{f_0}$, we get $C_{\phi\circ\psi} = M_{f_0}$. By Lemma 3.1, $f_0 = 1$ almost everywhere. In view of Theorem 2.5 it is enough to show that $\phi^{-1}(S) = S$. For this let $E \in S$ with $\lambda(E) < \infty$. Then, if X_E is in the range of C_ϕ, $X_E = C_\phi h$ for some h in $L^2(\lambda)$. Since C_ϕ is one-to-one, by Corollary 2.1, $h = X_F$ for some $F \in S$. Hence $X_E = C_\phi X_F = X_{\phi^{-1}(F)}$, which yields $E = \phi^{-1}(F)$. This shows that $E \in \phi^{-1}(S)$. In case X_E is not in the range of C_ϕ, then we can write $X_E = f + g$, where $f \in (\text{range } C_\phi)^\perp$, the orthogonal complement of the range of C_ϕ, and $g \in$ range C_ϕ. If we take $g = C_\phi g_1$, then, since C_ϕ^* is a composition operator and $C_\phi^* X_E = C_\phi^* f + C_\phi^* g = C_\phi^* g = C_\phi^* C_\phi g_1 = g_1$, it follows that $g = X_G$ for some $G \in S$. This gives $f = X_E - X_G = X_{E-G} - X_{G-E}$. The fact that $-\lambda((G-E) \cap G) = \langle f, g\rangle = 0$ implies that $G \subset E$. Now let $F_1 = \phi^{-1}(F_2)$ for some $F_2 \in S$ (that is $X_{F_1} \in$ range C_ϕ) such that $F_1 \supset E - G$. Then $\lambda((E-G) \cap F_1) = \langle f, X_{F_1}\rangle = 0$, which implies that $E \subset G$. Thus we get $X_E = X_G = g$, and hence $E \in \phi^{-1}(S)$. //

EXAMPLE 3.1. Let $X = R$ and $\phi(x) = x + c$. Then C_ϕ is a unitary composition operator.

4. Normal composition operators

THEOREM 4.1. *Let C_ϕ be a bounded operator on $L^2(\lambda)$. Then C_ϕ is normal, if and only if C_ϕ has dense range and $f_0 \circ \phi = f_0$ almost*

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everywhere.

Proof. Suppose $C_\phi^* C_\phi = C_\phi C_\phi^*$. Then

$$\ker C_\phi = \ker C_\phi^* C_\phi = \ker M_{f_0} = \ker C_\phi C_\phi^* = \ker C_\phi^* = \left(\text{range } C_\phi\right)^\perp .$$

If $\lambda(E) \neq 0$, where $E = \{x : f_0(x) = 0\}$, then for every $E' \subseteq E$ with $\lambda(E') < \infty$ we can find an element F of finite measure in S such that $\langle \chi_{E'}, C_\phi \chi_F\rangle = \langle \chi_{E'}, \chi_{\phi^{-1}(F)}\rangle \neq 0$, which is a contradiction. Hence $\lambda(E) = 0$. This shows that C_ϕ is one-to-one, and consequently it has dense range. Furthermore,

$$C_\phi^* C_\phi \chi_{\phi^{-1}(E_i)} = f_0 \cdot \chi_{\phi^{-1}(E_i)} ,$$

and

$$C_\phi C_\phi^* \chi_{\phi^{-1}(E_i)} = C_\phi C_\phi^* C_\phi \chi_{E_i} = C_\phi\left(f_0 \cdot \chi_{E_i}\right) = f_0 \circ \phi \cdot \chi_{\phi^{-1}(E_i)} ,$$

where $\bigcup_{i=1}^{\infty} E_i = X$ and $E_i \subset E_j$ for $i < j$ and $\lambda\left(E_i\right) < \infty$ for all i . Therefore $f_0 = f_0 \circ \phi$ almost everywhere.

Conversely, suppose C_ϕ has dense range and $f_0 = f_0 \circ \phi$. Then $C_\phi C_\phi^* f = f_0 \circ \phi \cdot f = f_0 \cdot f = C_\phi^* C_\phi f$ for all f in the range of C_ϕ . Since $C_\phi^* C_\phi$ and $C_\phi C_\phi^*$ are equal on a dense set, we have $C_\phi^* C_\phi = C_\phi C_\phi^*$. Hence C_ϕ is normal. //

COROLLARY 4.1. *Let* $C_\phi \in B\left(l^2(N)\right)$. *Then* C_ϕ *is normal, if and only if* ϕ *is invertible, where* $l^2(N) = \left\{\{x_n\} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\}$. //

COROLLARY 4.2. *Let* $p = \{p_1, p_2, p_3, \ldots\}$ *be a strictly increasing (or strictly decreasing) sequence. Then* $C_\phi \in B\left(l^2(p)\right)$ *is normal, if and only if* ϕ *is the identity.* //

EXAMPLE 4.1. Let $X = R$, and $\phi(x) = ax + b$. Then C_ϕ is a normal composition operator.

Now we give a typical example of a normal composition operator C_ϕ which is not onto.

EXAMPLE 4.2. Let $N_i = \{1_i, 2_i, 3_i, \ldots\}$ and let $X = \bigcup_{i=1}^{\infty} N_i$. Let the measure λ be defined as

$$\lambda(n_i) = \begin{cases} i^{(n-1)/2} & \text{, when } n \text{ is odd,} \\ 1/i^{n/2} & \text{, when } n \text{ is even.} \end{cases}$$

If ϕ is the mapping defined by

$$\phi(n_i) = \begin{cases} (n+2)_i & \text{, when } n \text{ is odd,} \\ 1_i & \text{, when } n = 2 \text{ ,} \\ (n-2)_i & \text{, when } n \text{ is even and greater than } 2 \text{ ,} \end{cases}$$

then $f_0(n_i) = 1/i$ for all $n_i \in N_i$. Since ϕ is one-to-one, C_ϕ has dense range. Also it is clear that $f_0 = f_0 \circ \phi$. Hence C_ϕ is normal. If m is fixed, then $\|C_\phi X_{\{m_i\}}\|^2/\|X_{\{m_i\}}\|^2 = 1/i$, $i \in N$. This shows that C_ϕ is not onto.

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