Canad. Math. Bull. Vol. 18 (5), 1975

DEGREES OF VERTICES IN A FRIENDSHIP GRAPH

BY
ANTON KOTZIG

ABSTRACT. A friendship graph is a graph in which every two distinct vertices have exactly one common adjacent vertex (called a neighbour). Finite friendship graphs have been characterized by Erdős, Rényi and Sós [2]: Each finite friendship graph F_n which consists of n edge disjoint triangles such that all $n>1$ triangles have one vertex in common (F_1 is a triangle i.e. the complete graph with three vertices). Thus F_n has $2n+1$ vertices, $2n$ of them being of degree two and the remaining one (the common vertex of n triangles if $n>1$) being of degree $2n$.

Infinite friendship graphs have been constructed by Chvátal, Kotzig, Rosenberg and Roy O. Davies [1]. The purpose of this paper is to prove the following theorem on degrees of vertices in an infinite friendship graph G:

THEOREM. *Let G be a friendship graph. Then either G contains a vertex which is adjacent to each other vertex of G and then each other vertex of G is of degree two or G does not contain any such vertex and then each vertex of G is of infinite degree.*

The graphs considered in this paper are undirected, without loops or multiple edges and we use throughout this paper the following notation: If G is a graph and u and v are vertices in G, then we denote by $V(G)$ (or $E(G)$, respectively) the vertex-set (or edge-set, resp.) of G, by $d_G(u)$ the degree of u in G and by $\delta_G(u, v)$ the distance between u and v in G. If u is a vertex of $V(G)$ then N_u denotes the neighbourhood of u, i.e. the subgraph of G such that $V(N_u)=\{x|\delta_G(u, x)=1\}$ and $E(N_u)$ contains all the edges and only edges $[x, y]$ of $E(G)$ with the property that $\{x, y\}\subset V(N_u)$. (If an edge $e\in E(G)$ is incident to the vertices x and y then we put $e=[x, y]$). In a friendship graph any two vertices $u\neq v$ have exactly one common neighbour which will be denoted by $c_{u,v}$. One can easily show the following trivial consequences of the definition of a friendship graph:

LEMMA 1. *The smallest friendship graph is isomorphic to a triangle. Let G be a friendship graph with $|V(G)|>3$. Then (i) G is of diameter two; (ii) G does not contain any circuit of length four; (iii) each edge of G belongs to exactly one triangle. (=circuit of length three).*

Received by the editors November 13, 1974 and, in revised form February 17, 1975.
Research supported by Grant DGES-FCAC-74.

https://doi.org/10.4153/CMB-1975-120-x Published online by Cambridge University Press

COROLLARY 1. *A friendship graph G is uniquely decomposible into triangles. If G contains a vertex v of finite degree, then* $d_G(v)\equiv 0(mod\, 2)$.

COROLLARY 2. *The neighbourhood* $N_v=F$ *of each vertex v of a friendship graph G is a* 1*-regular graph* (*in sense of Harary* [3], *because we have for each* $w\in V(F)$ *that the vertices v and w have exactly* $d_F(w)$ *common neighbours in G; thus* $d_F(w)=1$ *for each w of* $V(F)$).

LEMMA 2. *If G is an infinite friendship graph and x a vertex of G with* $2<d_G(x)=2n<\infty$, *then each neighbour of x is of infinite degree.*

Proof. Clearly (see corollary 1) x belongs to exactly n edge disjoint triangles $T_1, T_2, \ldots, T_n$. Denote by $2i$-1 and $2i$ ($i=1,2,\ldots,n$) the vertices of T_i different from x. If we put $X_k=\{u|\delta_G(u,x)=k\}$ then we easily obtain: $X_0=\{x\}$; $X_1=\{1,2,\ldots,2n\}$; $X_0\cup X_1\cup X_2=V(G)$ (remember that G is of diameter two-see Lemma 1, (i)); $\Rightarrow|X_2|=\infty$. Each vertex w of X_2 has exactly one neighbour $c_{x,w}$ in common with x and clearly $c_{x,w}$ belongs to X_1. Denote by $W_i(i=1,2,\ldots 2n)$ the set of all vertices in X_2 adjacent to the vertex i of X_1. Then obviously $W_1\cup W_2\cup\ldots\cup W_{2n}=X_2$ and $W_i\cap W_j=\varnothing$ if $i\neq j$ (because G does not contain any circuit of length four-see Lemma 1, (ii)). From $|X_2|=\infty$ we obtain: At least one set W_i is infinite. Without loss of generality we can suppose that W_1 is an infinite set ($\Rightarrow d_G(1)=\infty$). Let a, b and y be vertices of G such that $\{a\neq b\}\subset W_1$ and $y\in\{3,4,\ldots,2n\}$. Then $c_{a,y}\neq c_{b,y}$ (because otherwise a and b have, in addition to 1, a common neighbour $c_{a,v}=c_{b,y}$, which is not possible in G). This implies: W_y is infinite for each $y\in\{3,4,\ldots,2n\}$ and by the same argument (considering another infinite set W_y, say W_3, instead of W_1) we obtain as well $|W_2|=\infty$. Thus each one of the sets $W_1, W_2,\ldots, W_{2n}$ is infinite and $d_n(i)=\infty$ for each $i\in\{1,2,\ldots,2n\}$, Q.E.D.

LEMMA 3. *Let x and v be two adjacent vertices both of infinite degree in a friendship graph G. Then each neighbour of x* (*or of v, respectively*) *is of infinite degree.*

Proof. If we put $X_k=\{u|\delta_G(u,x)=k\}$ (as in the proof of Lemma 1) then again $X_0\cup X_1\cup X_2=V(G)$; $X_0=\{x\}$ but in this case X_1 as a infinite set. Denote by u the third vertex of the triangle which contains the edge $[x,v]$. Then $\{u,v\}\subset X_1$ and if we denote by W_t the subset of the set X_2 containing all the vertices of X_2 adjacent to $t\in X_1$ we obtain (by the same argument as in the proof of Lemma 2): $[W_v$ is a infinite set$]\Rightarrow[W_t$ is a infinite set for each $t\in X_1$ with only one eventual exception $t\neq u]\Rightarrow$(if we replace v by $t\in\{u,v\}$ $t\in X_1$) [also W_u is infinite set].

Thus: each neighbour of x is of infinite degree and (by the same argument) each neighbour of v is of infinite degree. This proves the lemma.

COROLLARY 3. *A vertex of an infinite friendship graph is either of degree two or of infinite degree.* {*If we suppose that the vertex x of an infinite friendship graph is* *of degree* $2<d_G(x)<\infty$, *then we have* $d_G(u)=\infty=d_G(v)$ *for each edge* $[u, v]$ *belonging to a triangle which contains* x, *where* $u\neq x\neq v$ *(see Lemma 2). But then (according to Lemma 3)* x *must be of infinite degree, which is a contradiction of our supposition. Thus* x *cannot be of a finite degree greater than two*}.

The proof of the Theorem. The theorem is clearly true for finite friendship graphs. Therefore we may suppose that G is infinite. Let $\{u, v, w\}$ be the vertex-set of a triangle of G. Then according to Corollary 3—we have: $d_G(x)\in\{2,\infty\}$ for each $x\in\{u, v, w\}$ and $d_G(u)+d_G(v)+d_G(w)=\infty$ (because G is connected and has more than three vertices). According to Lemma 3 we easily obtain: $[d_G(u)=d_G(v)=\infty]=>[d_G(w)=\infty]$ and therefore the number of vertices of infinite degree in the triangle must be odd. If exactly one vertex of $\{u, v, w\}$ (say v) is of infinite degree then each neighbour of v is of degree two (otherwise u and w must be of infinite degree according to Lemma 3, which contradicts our assumption). Then v is the common vertex of an infinite set of edge disjoint triangles and G is the union of them.

This proves the theorem if there exists a triangle $\{u, v, w\}$ in which exactly one vertex of every triangle has infinite degree and it is nothing to prove.

If conversely G contains a vertex x adjacent to any other, then the validity of the theorem for this case follows from Corollary 2.

References

1. V. Chvátal, A. Kotzig, I. G. Rosenberg and Roy O. Davies, *There are* $2^{\aleph_\alpha}$ *friendship graphs of cardinal* $2^{\aleph_\alpha}$ (submitted to Canadian Mathematical Bulletin, September 1974).

2. P. Erdös, A. Rényi and V. T. Sós, *On a problem of graph theory*, Studia Math. Hungar. **1** (1966), 215–235.

3. F. Harary, *Graph theory*, Addison-Wesley, Reading, 1969.

Centre de Recherches Mathématiques,
Université de Montréal,
Québec, Canada