Bull. Aust. Math. Soc. **90** (2014), 113–120
doi:10.1017/S0004972714000239

THE MULTIPLIER ALGEBRA OF A BEURLING ALGEBRA

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(Received 24 August 2013; accepted 17 February 2014; first published online 15 May 2014)

Abstract

For a discrete abelian cancellative semigroup S with a weight function ω and associated multiplier semigroup $M_\omega(S)$ consisting of ω-bounded multipliers, the multiplier algebra of the Beurling algebra of (S, ω) coincides with the Beurling algebra of $M_\omega(S)$ with the induced weight.

2010 *Mathematics subject classification*: primary 46J05; secondary 20M15, 20M14.

Keywords and phrases: Banach algebra, Beurling algebra, multiplier algebra.

1. Introduction

For an abelian semigroup S, the multiplier semigroup $M(S)$ consists of all $\alpha\colon S \to S$ such that

$$\alpha(st) = s\alpha(t) = \alpha(s)t \quad (s, t \in S).$$

A *weighted semigroup* (S, ω) consists of a semigroup S with a *weight* function $\omega\colon S \to (0, \infty)$ satisfying $\omega(st) \le \omega(s)\omega(t)$ $(s, t \in S)$. A weight ω on S represents a frequency function or a norm on S. Taking (S, ω) as an intrinsic object, a study of multipliers on (S, ω) has been initiated in [2]. The subsemigroup $M_\omega(S)$ of $M(S)$ consists of multipliers α on S which are *ω-bounded* in the sense that $\omega(\alpha(s)) \le K\omega(s)$ $(s \in S)$ for some $K > 0$. The map $s \in S \mapsto \gamma_s \in M_\omega(S)$, $\gamma_s(t) = st$ $(t \in S)$, is onto if and only if S has identity; and is one to one if and only if S is faithful, that is, if $s, t \in S$ and $su = tu$ for all $u \in S$, then $s = t$. The set $\{\gamma_s : s \in S\}$ is a semigroup ideal in $M_\omega(S)$. The weight $\widetilde{\omega}$ on $M_\omega(S)$ induced by ω is

$$\widetilde{\omega}(\alpha) = \sup\left\{\frac{\omega(\alpha(s))}{\omega(s)} : s \in S\right\} \quad (\alpha \in M_\omega(S)),$$

which satisfies $\widetilde{\omega}(\gamma_s) \le \omega(s)$ $(s \in S)$.

S. J. Bhatt is thankful to NBHM, DAE, India for a Visiting Professorship. P. A. Dabhi is thankful to Sardar Patel University for SEED GRANT research support. The work has been supported by the UGC-SAP-DRS-II grant no. F.510/3/DRS/2009 provided to the Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388120, Gujarat, India.

The Beurling algebra $\ell^1(S,\omega)$ associated with (S,ω) is the convolution Banach algebra

$$\ell^1(S,\omega) = \Big\{ f : S \to \mathbb{C} : \sum_{s\in S} |f(s)|\omega(s) < \infty \Big\}$$

with the convolution product $(f \star g)(s) = \sum_{uv=s} f(u)g(v)$; $(f \star g)(s) = 0$ if $uv = s$ has no solution in S; and with the norm $\|f\|_\omega = \sum_{s\in S} |f(s)|\omega(s)$. The algebra $\ell^1(S,\omega)$ has identity if and only if S has a finite set of relative units. The Beurling algebra $\ell^1(M_\omega(S),\widetilde{\omega})$ is analogously defined. The interrelation between the Banach algebra structure of $\ell^1(S,\omega)$ and the structure of (S,ω) is a fascinating aspect of harmonic analysis [2, 5].

The *multiplier Banach algebra* $M(\mathcal{A})$ of a commutative Banach algebra $(\mathcal{A},\|\cdot\|)$ is the unital Banach algebra consisting of all $T : \mathcal{A} \to \mathcal{A}$ satisfying $T(ab) = aTb = (Ta)b$ $(a, b \in \mathcal{A})$ with the operator norm $\|T\| = \sup\{\|Ta\| : a \in \mathcal{A},\ \|a\| \le 1\}$ [13]. Multipliers, either at the level of semigroups or at the level of algebras, constitute a kind of maximal unitisation. The present paper addresses the question: when does the multiplier algebra of the Beurling algebra of a weighted semigroup coincide with the Beurling algebra of the corresponding weighted multiplier semigroup?

A semigroup S is *cancellative* if, whenever $s, t, u \in S$, $su = tu$ implies $s = t$. Cancellative semigroups are precisely the subsemigroups of groups, whereas the semigroups $(\mathbb{R}, \max)$, $(\mathbb{C}, \cdot)$ and the power set $\mathcal{P}(X)$ of a nonempty set X with the binary operation union fail to be cancellative. We prove the following theorem.

Theorem 1.1. *Let S be cancellative. Then $M(\ell^1(S,\omega))$ is homeomorphically isomorphic to $\ell^1(M_\omega(S),\widetilde{\omega})$.*

The *annihilator* S°_ω of S with a zero element 0 (that is, $0 \in S$ such that $0s = s0 = 0$ for all $s \in S$ [9]) in $M_\omega(S)$ is a semigroup ideal of $M_\omega(S)$ given by

$$S^\circ_\omega = \{\alpha \in M_\omega(S) : \alpha\gamma_s = 0 \ \text{ for all } s \in S\},$$

and it contains γ_0. Analogously, the *annihilator* $\ell^1(S,\omega)^\circ$ of $\ell^1(S,\omega)$ in $\ell^1(M_\omega(S),\widetilde{\omega})$ is a closed algebra ideal of $\ell^1(M_\omega(S),\widetilde{\omega})$ given by

$$\ell^1(S,\omega)^\circ = \{\mu \in \ell^1(M_\omega(S),\widetilde{\omega}) : \mu \star f = 0\ (f \in \ell^1(S,\omega))\}.$$

When S is a semigroup with zero element 0, $M_\omega(S)$ is also a semigroup having zero element γ_0. Also, $\alpha(0) = 0$ for all $\alpha \in M_\omega(S)$. When S has a zero element, we define

$$\ell^1(S,\omega) = \Big\{ f : S \to \mathbb{C} : f(0) = 0,\ \sum_{s\in S} |f(s)|\omega(s) < \infty \Big\}.$$

We recall the Rees quotient of S by a semigroup ideal I. The relation $\sim$ in S, defined by $s \sim t$ if either $s = t$ or both s and t are in I, is an equivalence relation in S. The equivalence classes under $\sim$ are the singleton sets $\{s\}$ with $s \in S \setminus I$ and the set I. Since I is an ideal of S, the relation $\sim$ is a congruence on S. The quotient semigroup S/I is the *Rees factor semigroup* of S modulo I [9].

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Let ω be such that $\omega_0 := \inf\{\omega(s) : s \in S\} > 0$. Consider the map $\omega_q : S/I \to (0, \infty)$ defined as $\omega_q([t]) = 1$ $(t \in I)$ and $\omega_q([t]) = \omega(t)$ $(t \notin I)$. Then ω_q is a weight on S/I. Indeed, let $s \in S$ and $t \in I$. Then $\omega_0 \leq \omega(st) \leq \omega(s)\omega(t)$. It follows that $\omega(s) \geq 1$ for all $s \in S$. Let $s, t \in S$. If $st \in I$, then $\omega_q([st]) = 1 \leq \omega_q([s])\omega_q([t])$. Let $st \notin I$. Then $\omega_q([st]) = \omega(st) \leq \omega(s)\omega(t) = \omega_q([s])\omega_q([t])$. It follows from the above arguments that $\omega_0 > 0$ if and only if $\omega \geq 1$.

Theorem 1.2. *Let S be a semigroup with zero element. Let $\widetilde{\omega}$ (in particular, ω) be bounded away from* 0. *Then $\ell^1(S, \omega)^\circ = \ell^1(S^\circ_\omega, \widetilde{\omega})$ and $\ell^1(M_\omega(S), \widetilde{\omega})/\ell^1(S^\circ_\omega, \widetilde{\omega})$ is isomorphic to the Beurling algebra $\ell^1(M_\omega(S)/S^\circ_\omega, \widetilde{\omega}_q)$.*

A weight ω on S is *semisimple* if $\lim_{n\to\infty} \omega(s^n)^{1/n} > 0$ $(s \in S)$. A semigroup S is *separating* if $s = t$ whenever $s, t \in S$ and $s^2 = t^2 = st$ [8]. By [5], the algebra $\ell^1(S, \omega)$ is semisimple if and only if S is separating and ω is semisimple. By [2], $\ell^1(S, \omega)$ is semisimple if and only if $\ell^1(M_\omega(S), \widetilde{\omega})$ is semisimple. For $\mu \in \ell^1(M_\omega(S), \widetilde{\omega})$, let $T_\mu : \ell^1(S, \omega) \to \ell^1(S, \omega)$ be

$$T_\mu(f) = \mu \star f \quad (f \in \ell^1(S, \omega)).$$

Then $T_\mu \in M(\ell^1(S, \omega))$.

Theorem 1.3. *Let S be separating and ω be semisimple, and let $\widetilde{\omega}$ be bounded away from* 0. *Then the following hold.*

(1) *The map $f \mapsto f + \ell^1(S, \omega)^\circ$ from $\ell^1(S, \omega)$ into $\ell^1(M_\omega(S), \widetilde{\omega})/\ell^1(S, \omega)^\circ$ is one to one and $\ell^1(M_\omega(S), \widetilde{\omega})/\ell^1(S, \omega)^\circ$ is semisimple.*
(2) *If $\ell^1(S, \omega)$ has a bounded approximate identity, then the map $\mu + \ell^1(S, \omega)^\circ \mapsto T_\mu$ is a homeomorphic isomorphism from $\ell^1(M_\omega(S), \widetilde{\omega})/\ell^1(S, \omega)^\circ$ onto $M(\ell^1(S, \omega))$.*

These results are inspired by [11, 12], in which the case of semigroups without weights is considered. An example in [11] shows that the condition that S is cancellative cannot be omitted.

The algebra $\ell^1(\mathbb{Z}, \omega)$ and its connection with complex analysis were noticed by the forefathers of Banach algebras [7]. Its instructive role in Fourier series was noted in [6, Example 11.15, page 41]. It provides a natural framework for theorems of Wiener, Lévy and Żelazko [1, 3]. The role of the group algebra $L^1(G, \omega)$ in abstract harmonic analysis and in Banach algebras is amply emphasised in [4, 10, 14]. For the general discrete case $\ell^1(S, \omega)$ (in particular, $\ell^1(\mathbb{Q}^+, \omega)$), it was proclaimed in 2000 in [4, page 536] that 'presumably the golden age for the study of these algebras lies in the future'. The present paper along with [2] is our response to this (see also [5]). For $S = \mathbb{Z}^+$, this gives Banach algebras of power series for which we refer the reader to [4].

2. Proofs

Lemma 2.1. *Let S be an abelian faithful semigroup. Then the natural homomorphism $s \mapsto \gamma_s$ of S into $M_\omega(S)$ induces a homomorphism of $\ell^1(S, \omega)$ into $\ell^1(M_\omega(S), \widetilde{\omega})$ which is one to one if and only if $s \mapsto \gamma_s$ is one to one and onto if and only if $s \mapsto \gamma_s$ is onto.*

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Proof. The proof is analogous to the proof of [12, Proposition 4.3]. □

Lemma 2.2. *Let ω be a weight on an abelian semigroup S and let $\mu \in \ell^1(M_\omega(S), \widetilde{\omega})$. Then the map $T_\mu : \ell^1(S, \omega) \to \ell^1(S, \omega)$ defined by $T_\mu(f) = \mu \star f$ is a multiplier of $\ell^1(S, \omega)$. The map $\mu \mapsto T_\mu$ of $\ell^1(M_\omega(S), \widetilde{\omega})$ into $M(\ell^1(S, \omega))$ is a norm-decreasing homomorphism.*

Proof. Since S is abelian, it follows that $T_\mu(f) \star g = f \star T_\mu(g)$ for all $f, g \in \ell^1(S, \omega)$. Let $\mu = \sum_{\alpha \in M_\omega(S)} \mu(\alpha)\delta_\alpha \in \ell^1(M_\omega(S), \widetilde{\omega})$ and let $f = \sum_{s \in S} f(s)\delta_{\gamma_s} \in \ell^1(S, \omega)$. Then

$$\begin{aligned}\sum_{s\in S}\sum_{\alpha\in M_\omega(S)} |f(s)||\mu(\alpha)|\omega(\alpha\gamma_s) &\leq \sum_{s\in S}\sum_{\alpha\in M_\omega(S)} |f(s)||\mu(\alpha)|\widetilde{\omega}(\alpha)\omega(s)\\ &= \Big(\sum_{\alpha\in M_\omega(S)} |\mu(\alpha)|\widetilde{\omega}(\alpha)\Big)\Big(\sum_{s\in S}|f(s)|\omega(s)\Big)\\ &= \|\mu\|_{\widetilde{\omega}}\|f\|_\omega.\end{aligned}$$

Hence, $\|T_\mu(f)\|_\omega \leq \|\mu\|_{\widetilde{\omega}}\|f\|_\omega$ $(f \in \ell^1(S, \omega))$, that is, $\|T_\mu\| \leq \|\mu\|_{\widetilde{\omega}}$. □

Lemma 2.3. *Let S be an abelian semigroup with the property: given $\alpha \in M_\omega(S)$, there exists $s_\alpha \in S$ such that for any $\beta \in M_\omega(S)$, $\alpha(s_\alpha) = \beta(s_\alpha)$ implies $\alpha = \beta$. This holds in particular when S is cancellative. Then the map $\mu \mapsto T_\mu$ from $\ell^1(M_\omega(S), \widetilde{\omega})$ to $M(\ell^1(S, \omega))$ is one to one.*

Proof. The proof is analogous to the proof of [12, Proposition 4.4 and Corollary 4.4]. □

Let S be a cancellative semigroup. Then S, $M_\omega(S)$ and $M(S)$ can be embedded in a group $Q(S)$, called *the group of the semigroup* S, which has the property that $M(S) = \{\alpha \in Q(S) : \alpha S \subset S\}$. The group $Q(S)$ is constructed as follows [4, page 15]. Let $(s, t), (u, v) \in S \times S$. We say that $(s, t) \sim (u, v)$ if $sv = tu$. Then $\sim$ is an equivalence relation on $S \times S$. Let $[s, t]$ be the equivalence class containing (s, t), that is,

$$[s, t] = \{(u, v) \in S \times S : (u, v) \sim (s, t)\}.$$

Then $Q(S) = (S \times S)/\!\sim$ is a group with the binary operation

$$[s, t][u, v] = [su, tv] \quad ([s, t], [u, v] \in Q(S)).$$

The semigroup S is embedded in $Q(S)$ via the map $s \mapsto [su, u]$.

Let ω be a weight on S. Define $\omega_Q : Q(S) \to (0, \infty)$ as

$$\omega_Q([s, t]) = \sup\Big\{\frac{\widetilde{\omega}(su)}{\widetilde{\omega}(tu)} : u \in M_\omega(S)\Big\}.$$

Let $[s, t], [u, v] \in Q(S)$. By definition, $\omega_Q([s, t]) > 0$. Let $x \in M_\omega(S)$. Then

$$\frac{\widetilde{\omega}(sux)}{\widetilde{\omega}(tvx)} = \frac{\widetilde{\omega}(sux)}{\widetilde{\omega}(tux)}\frac{\widetilde{\omega}(utx)}{\widetilde{\omega}(vtx)} \leq \omega_Q([s, t])\omega_Q([u, v]).$$

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Therefore,

$$\omega_Q([s,t][u,v]) = \omega_Q([su,tv]) \le \omega_Q([s,t])\omega_Q([u,v]).$$

Note that $\omega_Q([su,u]) = \sup\{\widetilde{\omega}(suv)/\widetilde{\omega}(uv) : v \in M_\omega(S)\} \le \widetilde{\omega}(s)$ $(s \in M_\omega(S))$. Since $\widetilde{\omega}(\gamma_s) \le \omega(s)$, it follows that $\omega_Q([su,u]) \le \omega(s)$ $(s \in S)$. Thus, given a weight ω on a cancellative semigroup S, there exists a natural weight ω_Q on $Q(S)$ whose restriction on S is dominated by ω.

Lemma 2.4. *Let (S,ω) be a cancellative, abelian weighted semigroup and let $Q(S)$ be the group of the semigroup S. Then*

$$M_\omega(S) = \{g \in Q(S) : gS \subset S,\ \omega(gs) \le K_g\omega(s)\ (s \in S)\}.$$

Proof. Let $g \in Q(S)$ be such that $gS \subset S$ and $\omega(gs) \le K_g\omega(s)$ $(s \in S)$. Then clearly the map $s \mapsto gs$ is in $M_\omega(S)$. Conversely, if $g \in M_\omega(S)$, then $gS \subset S$ and $\omega(gs) \le K_g\omega(s)$ $(s \in S)$. □

Lemma 2.5. *Let S be a cancellative, abelian semigroup. Then both $\ell^1(S,\omega)$ and $\ell^1(M_\omega(S),\widetilde{\omega})$ are subalgebras of $\ell^1(Q(S),\omega_Q)$.*

Proof. Let $f = \sum_{s\in S} f(s)\delta_{\gamma_s} \in \ell^1(S,\omega)$. For any $s \in S$, $\omega_Q([su,u]) \le \omega(s)$. Now

$$\sum_{s\in Q(S)} |f([su,u])|\omega_Q([su,u]) = \sum_{s\in S} |f(s)|\omega_Q([su,u]) \le \sum_{s\in S} |f(s)|\omega(\gamma_s).$$

A similar proof holds for $\ell^1(M_\omega(S),\widetilde{\omega})$. □

Proof of Theorem 1.1. If $\mu \in \ell^1(M_\omega(S),\widetilde{\omega})$, then the map $T_\mu : \ell^1(S,\omega) \to \ell^1(S,\omega)$ defined by $T_\mu(f) = \mu \star f$ $(f \in \ell^1(S,\omega))$ is an element of $M(\ell^1(S,\omega))$. By Lemmas 2.2 and 2.3, the map $\mu \mapsto T_\mu$ is an isomorphism of $\ell^1(M_\omega(S),\widetilde{\omega})$ to $M(\ell^1(S,\omega))$. We show that it is onto.

The algebras $\ell^1(S,\omega)$ and $\ell^1(M_\omega(S),\widetilde{\omega})$ are subalgebras of $\ell^1(Q(S),\omega_Q)$ whose elements are supported on S and $M_\omega(S)$, respectively. Let $T \in M(\ell^1(S,\omega))$. Let $f \in \ell^1(S,\omega)$ and $s \in S$. Then $T(f) \star \delta_s = f \star T(\delta_s)$. Thus, $T(f) = f \star T(\delta_s) \star \delta_{s^{-1}} \in \ell^1(S,\omega)$ for all $f \in \ell^1(S,\omega)$. We first claim that the support of $T(\delta_s) \star \delta_{s^{-1}}$ is contained in $M(S)$. Let $t \in Q(S)$, $u \in S$ be such that $tu \notin S$ and $T(\delta_s) \star \delta_{s^{-1}}(t) \ne 0$. Then $(\delta_u \star T(\delta_s) \star \delta_{s^{-1}})(tu) = (T(\delta_s) \star \delta_{s^{-1}})(t) \ne 0$ and hence $\delta_u \star T(\delta_s) \star \delta_{s^{-1}} \notin \ell^1(S,\omega)$. This contradicts the fact that $f \star T(\delta_s) \star \delta_{s^{-1}} \in \ell^1(S,\omega)$ for all $f \in \ell^1(S,\omega)$. Hence, the claim follows. Now we claim that the support of $T(\delta_s) \star \delta_{s^{-1}}$ is contained in $M_\omega(S)$. Let $\mu = T(\delta_s) \star \delta_{s^{-1}} = \sum_{\alpha\in M(S)} \mu(\alpha)\delta_\alpha$. Let $\alpha_0 \in M(S)\backslash M_\omega(S)$ be such that $\mu(\alpha_0) \ne 0$. Then there exists a sequence (s_n) in S such that $\omega(\alpha_0(s_n)) \ge n\omega(s_n)$. Let $f = \sum_{n\in\mathbb{N}}(1/n^2\omega(s_n))\delta_{\gamma_{s_n}}$. Then $f \in \ell^1(S,\omega)$. Now

$$\begin{aligned}\|\mu \star f\|_\omega &= \bigg\|\bigg(\sum_{\alpha\in M(S)} \mu(\alpha)\delta_\alpha\bigg) \star \bigg(\sum_{n\in\mathbb{N}} \frac{1}{n^2\omega(s_n)}\delta_{\gamma_{s_n}}\bigg)\bigg\|_\omega \\ &= \bigg\|\sum_{n\in\mathbb{N}}\sum_{\alpha\in M(S)} \mu(\alpha)\frac{1}{n^2\omega(s_n)}\delta_{\alpha\gamma_{s_n}}\bigg\|_\omega\end{aligned}$$

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$$= \sum_{n\in\mathbb{N}} \sum_{\alpha\in M(S)} \left|\mu(\alpha)\frac{1}{n^2\omega(s_n)}\right| \omega(\alpha\gamma_{s_n})$$
$$\geq \sum_{n\in\mathbb{N}} \left|\mu(\alpha_0)\frac{1}{n^2\omega(s_n)}\right| \omega(\alpha_0\gamma_{s_n})$$
$$= \sum_{n\in\mathbb{N}} \left|\mu(\alpha_0)\frac{1}{n^2\omega(s_n)}\right| \omega(\alpha_0(s_n))$$
$$\geq |\mu(\alpha_0)| \sum_{n\in\mathbb{N}} \frac{1}{n^2\omega(s_n)} n\omega(s_n)$$
$$= |\mu(\alpha_0)| \sum_{n\in\mathbb{N}} \frac{1}{n}.$$

This is a contradiction and proves our claim.

Since the map T is a continuous bijection between two Banach spaces, it follows from the open mapping theorem that it is a homeomorphism. □

Proof of Theorem 1.2. Let $\mu = \sum_{\alpha\in S^\circ_\omega} \mu(\alpha)\delta_\alpha \in \ell^1(S^\circ_\omega, \widetilde{\omega})$. Since $\alpha \in S^\circ_\omega$, $\alpha\gamma_s = 0$ for all s. Therefore, $\mu \star \delta_s = \sum_{\alpha\in S^\circ_\omega} \mu(\alpha)\delta_{\alpha\gamma_s} = \sum_{\alpha\in S^\circ_\omega} \mu(\alpha)\delta_0 = 0$ for all $s \in S$. Hence, $\mu \star f = 0$ for all $f \in \ell^1(S, \omega)$, that is, $\mu \in \ell^1(S, \omega)^\circ$.

Conversely, let $\mu = \sum_{\alpha\in M_\omega(S)} \mu(\alpha)\delta_\alpha \in \ell^1(S, \omega)^\circ$. Suppose that $\alpha_0 \notin S^\circ_\omega$ for some α_0 in the above expression. Then $\alpha_0\gamma_s \neq 0$ for some $s \in S$. This will give

$$0 = \|\mu \star \delta_s\|_\omega \geq |\mu(\alpha_0)|\omega(\alpha_0\gamma_s) > 0.$$

This is a contradiction. Hence, $\ell^1(S^\circ_\omega, \widetilde{\omega}) = \ell^1(S, \omega)^\circ$.

The set $\ell^1(S^\circ_\omega, \widetilde{\omega})$ consists of all functions from $\mu : M_\omega(S) \to \mathbb{C}$ which are zero outside S°_ω and $\sum_{\alpha\in M_\omega(S)} |\mu(\alpha)|\widetilde{\omega}(\alpha) < \infty$. Since S°_ω is an ideal in $M_\omega(S)$, $\ell^1(S^\circ_\omega, \widetilde{\omega})$ is a closed ideal in $\ell^1(M_\omega(S), \widetilde{\omega})$. Define $\varphi : \ell^1(M_\omega(S), \widetilde{\omega}) \to \ell^1(M_\omega(S)/S^\circ_\omega, \widetilde{\omega}_q)$ as follows. Let $\mu = \sum_{\alpha\in M_\omega(S)\setminus S^\circ_\omega} \mu(\alpha)\delta_\alpha + \sum_{\alpha\in S^\circ_\omega} \mu(\alpha)\delta_\alpha$. Then

$$\varphi(\mu) := \sum_{\alpha\in M_\omega(S)\setminus S^\circ_\omega} \mu(\alpha)\delta_\alpha.$$

Since $\widetilde{\omega}_q(\alpha) = \widetilde{\omega}(\alpha)$ for all $\alpha \in M_\omega(S)\setminus S^\circ_\omega$, φ is a continuous homomorphism with norm at most 1. Clearly, the map φ is onto. Let $\mu \in \ker\varphi$. Then $\mu = \sum_{\alpha\in S^\circ_\omega} \mu(\alpha) \delta_\alpha \in \ell^1(S^\circ_\omega, \widetilde{\omega})$. If $\mu \in \ell^1(S^\circ_\omega, \widetilde{\omega})$, then, by definition of φ, $\varphi(\mu) = 0$. Hence, $\ker\varphi = \ell^1(S^\circ_\omega, \widetilde{\omega})$. □

Proof of Theorem 1.3. (1) Let $f \in \ell^1(S, \omega)$ and let $f + \ell^1(S, \omega)^\circ = \ell^1(S, \omega)^\circ$. Then $f \star g = 0$ for all $g \in \ell^1(S, \omega)$. In particular, $f \star \delta_s = 0$ for all $s \in S$. Let $\varphi \in \Delta(\ell^1(S, \omega))$. Then $\varphi(\delta_s) \neq 0$ for some $s \in S$. But then $0 = \varphi(f \star \delta_s) = \varphi(f)\varphi(\delta_s)$ implies $\varphi(f) = 0$. Since $\ell^1(S, \omega)$ is semisimple, $f = 0$. Hence, the map $f \mapsto f + \ell^1(S, \omega)^\circ$ is one to one.

Let $\mu \in \ell^1(M_\omega(S), \widetilde{\omega})$ be such that $\mu + \ell^1(S, \omega)^\circ \neq \ell^1(S, \omega)^\circ$. Then $\mu \star f \neq 0$ for some $f \in \ell^1(S, \omega)$. Since $\ell^1(M_\omega(S), \widetilde{\omega})$ is semisimple, there exists $\varphi \in \Delta(\ell^1(M_\omega(S), \widetilde{\omega}))$

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such that $\varphi(\mu \star f) \neq 0$. Since $\mu \notin \ell^1(S,\omega)^\circ$ and $\varphi(\mu) \neq 0$, the map $\tilde{\varphi} : \ell^1(M_\omega(S),\widetilde{\omega}) \to \mathbb{C}$ defined by $\tilde{\varphi}(\nu + \ell^1(S,\omega)^\circ) = \varphi(\nu)$ is an element of $\Delta(\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S,\omega)^\circ)$. Since $\tilde{\varphi}(\mu + \ell^1(S,\omega)^\circ) = \varphi(\mu) \neq 0$, it follows that $\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S,\omega)^\circ$ is semisimple.
(2) Let $T_\mu = 0$. Then $\mu \star f = 0$ for all $f \in \ell^1(S,\omega)$, that is, $\mu + \ell^1(S,\omega)^\circ = \ell^1(S,\omega)^\circ$. Therefore, the map $\mu + \ell^1(S,\omega)^\circ \mapsto T_\mu$ is one to one. Let $\mu \in \ell^1(M_\omega(S),\widetilde{\omega})$, $f \in \ell^1(S,\omega)$ and $\mu' \in \ell^1(S,\omega)^\circ$. Then

$$\|T_\mu(f)\|_\omega = \|\mu \star f\|_\omega = \|(\mu + \mu') \star f\|_\omega \leq \|\mu + \mu'\|_{\widetilde{\omega}}\|f\|_\omega.$$

Hence, $\|T_\mu\| \leq \|\mu + \ell^1(S,\omega)^\circ\|$.

Now we show that the map $\mu + \ell^1(S,\omega)^\circ \mapsto T_\mu$ is onto. We have $\ell^1(M_\omega(S),\widetilde{\omega}) = (c_0(M_\omega(S), 1/\widetilde{\omega}))^*$. We identify the element μ of $\ell^1(M_\omega(S),\widetilde{\omega})$ with a unique element Λ_μ of $(c_0(M_\omega(S), 1/\widetilde{\omega}))^*$ given by

$$\Lambda_\mu\Big(\sum_{\alpha\in M_\omega(S)} \nu(\alpha)\delta_\alpha\Big) = \sum_{\alpha\in M_\omega(S)} \mu(\alpha)\nu(\alpha).$$

We may embed $\ell^1(S,\omega)$ in $\ell^1(M_\omega(S),\widetilde{\omega})$. Let $T \in M(\ell^1(S,\omega))$. Let (f_x) be a bounded approximate identity of $\ell^1(S,\omega)$. Then $(\Lambda_{T(f_x)})$ is a bounded net in $(c_0(M_\omega(S), 1/\widetilde{\omega}))^*$. Let $(\Lambda_{T(f_y)})$ be a subnet of $(\Lambda_{T(f_x)})$ converging to $\Lambda_\mu \in (c_0(M_\omega(S), 1/\widetilde{\omega}))^*$ in the w^*-topology, where $\mu \in \ell^1(M_\omega(S),\widetilde{\omega})$. Since $c_0(M_\omega(S), 1/\widetilde{\omega})$ is w^*-dense in $(c_0(M_\omega(S), 1/\widetilde{\omega}))^{**}$,

$$\langle\Lambda_\mu, g\rangle = \lim_y \langle\Lambda_{T(f_y)}, g\rangle$$

for every $g \in c_0(M_\omega(S), 1/\widetilde{\omega})^{**}$. Since

$$\Delta(\ell^1(M_\omega(S),\widetilde{\omega})) \subset (\ell^1(M_\omega(S),\widetilde{\omega}))^* = (c_0(M_\omega(S), 1/\widetilde{\omega}))^{**},$$

$\langle\Lambda_\mu, \varphi\rangle = \lim_y\langle\Lambda_{T(f_y)}, \varphi\rangle$ for every $\varphi \in \Delta(\ell^1(M_\omega(S),\widetilde{\omega}))$. Let $f \in \ell^1(S,\omega)$. Then $\langle\Lambda_{T(f_y)},\varphi\rangle\langle\Lambda_f,\varphi\rangle \to \langle\Lambda_\mu,\varphi\rangle\langle\Lambda_f,\varphi\rangle = \langle\Lambda_{\mu\star f},\varphi\rangle$. But $\langle\Lambda_{T(f_y)},\varphi\rangle\langle\Lambda_f,\varphi\rangle = \langle\Lambda_{f_y\star T(f)},\varphi\rangle \to \langle\Lambda_{T(f)},\varphi\rangle$. Hence, $\langle\Lambda_{T(f)},\varphi\rangle = \langle\Lambda_{\mu\star f},\varphi\rangle$. Therefore, $T(f) = \mu \star f$. Since the map $\mu + \ell^1(S,\omega)^\circ \mapsto T_\mu$ is a continuous bijection between two Banach spaces, it follows from the open mapping theorem that it is a homeomorphism. □

Acknowledgement

The referee has raised the issue of nonemptiness of S^0_ω. This requires S to contain a zero element in Theorem 1.2. The referee has also suggested the present concise title. We thank the referee very much.

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