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Adding Regularized Horseshoes to the Dynamics of Latent Variable Models

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Abstract

Dynamic latent variable models generally link units' positions on a latent dimension over time *via* random walks. Theoretically, these trajectories are often expected to resemble a mixture of periods of stability interrupted by moments of change. In these cases, a prior distribution such as the regularized horseshoe— that allows for both stasis and change—can prove a better theoretical and empirical fit for the underlying construct than other priors. Replicating Reuning, Kenwick, and Fariss (2019), we find that the regularized horseshoe performs better than the standard normal and the Student's t-distribution when modeling dynamic latent variable models. Overall, the use of the regularized horseshoe results in more accurate and precise estimates. More broadly, the regularized horseshoe is a promising prior for many similar applications.

1. Introduction

When modeling dynamic trends in latent variable models, researchers typically use random walks to link observations from one time-period to the next: tomorrow's position on a latent dimension is today's position plus a random effect (see, e.g., Caughey and Warshaw 2015; König, Marbach, and Osnabrügge 2013; Martin and Quinn 2002; Schnakenberg and Fariss 2014). Often, the expected trajectory of the political constructs modeled in such a manner (such as countries' democracy scores) is a mixture of stasis and change: periods of stability are interrupted by moments of rupture (see also Reuning *et al.* 2019, 504).

In this letter, we suggest the use of the regularized horseshoe (RHS; Piironen and Vehtari 2017) as a prior distribution for dynamic latent variable models. Stemming from the Bayesian literature on sparsity-inducing priors (Carvalho, Polson, and Scott 2010; Polson and Scott 2011), draws from the regularized horseshoe distribution are shrunk towards 0 (stasis) or allow for deviations from 0 (change). The correspondence between the prior's properties and expectations about the construct makes the RHS an ideal candidate for these applications.

Below, we introduce the regularized horseshoe as a useful addition to the distributions often employed in political science[1] and highlight its utility in the domain of latent variable models. We

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[1]To the best of our knowledge, the RHS has so far not been discussed or used in political science. It's predecessor—the horseshoe—was employed by Ratkovic and Tingley (2017) and considered by Park and Yamauchi (2023).

build on Reuning *et al.* (2019, from hereon: RKF), who assessed the performance of the standard normal and the Student's t-distribution (the two distributions generally used in these applications) in a similar endeavor. We show that using the RHS leads to better results than using either of the other two distributions. Model specifications using the RHS produce more accurate and precise estimates, and better model fit than those using the other distributions.

2. Motivation

The standard normal distribution as well as the Student's t-distribution are sensible starting points for the random walks of dynamic models. Both concentrate their probability around 0 (stasis), while also allowing for deviations from 0 (change). Their empirical density functions are shown in Figure 1. The key difference between the tails of these two distributions is best visible in panels (b) and (c). As the tails of the t-distribution are determined *via* the degrees of freedom (ν), their decay can be more gradual than those of a standard normal (here, we show draws from $StudentT(4,0,1)$ in line with RKF, where the parameters of the t-distribution are ν, location, and scale).

While the two distributions differ in their tails, they are similar in their center: a gradual decline from a peak at 0 assigns a lot of probability to small deviations from the center. Within the Bayesian literature, distributions such as the horseshoe have been suggested that exhibit a pronounced peak at 0 to induce sparsity in the posterior distributions of parameters (Carvalho *et al.* 2010; Polson and Scott 2011). Essentially, the horseshoe prior is a continuous mixture of two components with one component (potentially) shrinking values towards 0 (τ), and another component allowing for large deviations from 0 (λ).[2]

More recently, Piironen and Vehtari 2017 proposed a regularized version of the horseshoe (RHS) to address the heavy tails of the horseshoe prior. By adding a hyperparameter c^2 to the horseshoe's parameterization, the extent of deviations from 0 can be regularized. Formally, the regularized horse-shoe distribution for a parameter γ is constructed in the following manner:

$$\gamma_i \sim N(0, \tau^2 \tilde{\lambda}_i^2), \quad \tilde{\lambda}_i^2 = \frac{c^2 \lambda_i^2}{c^2 + \tau^2 \lambda_i^2},$$

$$\tau \sim StudentT^+(1,0,1),$$

$$\lambda_i \sim StudentT^+(1,0,1) \quad \forall i = 1, \ldots, I,$$

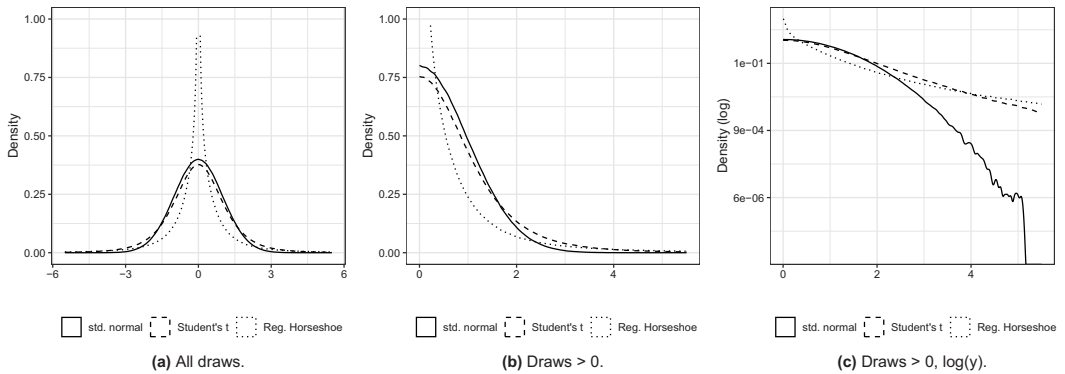**(a)** All draws.   **(b)** Draws > 0.   **(c)** Draws > 0, log(y).

Figure 1. Empirical density functions.

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[2]Here, our discussion focuses on the RHS. In the Supplementary Information (SI), we also discuss and assess the original horseshoe and discrete spike-and-slab priors (George and McCulloch 1993; Mitchell and Beauchamp 1988).

where i indexes elements of the vector γ. As discussed by Piironen and Vehtari 2017, c^2 is given a prior too:

$$c^2 \sim InvGamma(\alpha, \beta), \quad \alpha = \nu/2, \quad \beta = \nu s^2/2,$$

which corresponds to a $StudentT^+(\nu, 0, s^2)$ prior. We set $\nu = s = 1$, so that $c^2 \sim InvGamma(0.5, 0.5)$.[3] If c^2 is dropped and $\gamma_i \sim N(0, \tau^2 \lambda_i^2)$, the prior corresponds to the original horseshoe.

The distribution of the RHS is also shown in Figure 1. More probability is assigned to values at or around 0, with a far steeper decline from the spike at the center than the more gradual descent of the other distributions. Simultaneously, the tails of the RHS are fatter, exhibiting similar, and then lower decay rates than those of the t-distribution (visible in panel (c)). These two attributes of this distribution—a spike at 0 combined with fat tails—make the RHS an ideal *a priori* candidate to model the theoretically expected mixture of stasis and change in latent constructs over time. Its use should result in a posterior distribution that is both more accurate (as it better reflects the expected trajectory) and precise (as its probability is more concentrated) than the other two distributions. Both aspects are important in applications, especially if estimates are used in subsequent models that propagate measurement uncertainty (rather than focusing on point estimates only; see Tai, Hu, and Solt 2024).

3. Model Specification

All subsequent empirical analyses are based on the model specifications and priors described in this section (unless mentioned otherwise). In line with RKF, we use Bayesian inference implemented in Stan (Carpenter *et al.* 2017) to approximate the posterior distribution of our model.

Following RKF, the likelihood for a unidimensional binary item response theory (IRT) model for units $i = 1, \ldots, I$, items $k = 1, \ldots, K$, and time-points $t = 1, \ldots, T$ is

$$\mathcal{L} = \prod_{i,t=1}^{I,T} \prod_{k=1}^{K} \Lambda(\alpha_k - \beta_k \theta_{i,t})^{y_{ikt}} (1 - \Lambda(\alpha_k - \beta_k \theta_{i,t})^{1-y_{ikt}}),$$

where $\theta_{i,t}$ is a unit's ideal point on a latent dimension at a given time-point, α_k and β_k are an item's difficulty and discrimination parameters, and Λ is the logistic function. y_{ikt} is an observed binary outcome for a unit × item × time-point combination. The latent model is identified by constraining the discrimination parameters to be positive (rotational invariance), and by fixing the mean and standard deviation of the ideal points to 0 and 1 (local and scalar invariance; see Bafumi *et al.* 2005). We assign standard (half) normal distributions to the vectors $\vec{\alpha}$ and $\vec{\beta}$ of length K, and to the vector $\vec{\theta}_{t=1}$ of length I: $\vec{\alpha} \sim N(0,1)$, $\vec{\beta} \sim N^+(0,1)$, and $\vec{\theta}_{t=1} \sim N(0,1)$.

The dynamic nature of the model is ensured by specifying yesterday's $\theta_{i,t-1}$ as the expectation of today's $\theta_{i,t}$, so that:

$$\theta_{i,t} = \theta_{i,t-1} + \gamma_{i,t}\sigma \quad \forall i = 1, \ldots, I \quad \& \quad \forall t = 2, \ldots, T,$$

where $\gamma_{i,t}$ are the random effects over time and σ is a smoothing hyperparameter that governs how much the past informs the present with a standard half normal prior: $\sigma \sim N^+(0,1)$.[4]

The primary difference in the model comparisons below are the priors used for $\gamma_{i,t}$.[5] We compare the results from models using three different prior distributions for $\gamma_{i,t}$: (i) a specification using the standard

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[3]Both ν and s could be set to other values. We chose these values so that τ, λ_i, and c^2 have the same Student's t prior (equivalent to a Cauchy distribution), and to thereby allow for large deviations from 0 across all three parameters *via* their prior. Future research could explore the results' sensitivity to these choices.

[4]RKF assign (half) normal distributions with a standard deviation of 3 to α, β, and σ, while we assign a standard deviation of 1. This difference does not matter within this comparison.

[5]We omit σ when using the RHS because τ performs a similar function in the RHS prior and its inclusion would lead to identification issues.

normal distribution ($N(0,1)$), (ii) one using the Student's t-distribution (with $\nu = 4$; $StudentT(4,0,1)$), and (iii) our suggested specification with the regularized horseshoe (as detailed in Section 2 above).

4. Results

In this section, we replicate the simulation study of RKF and their re-analysis of Pemstein, Meserve, and Melton 2010, who estimated the trajectories of countries' democracy scores—a construct likely to exhibit stasis and change across time.[6] We limit ourselves to a selection of quantities of interest here (focusing on the accuracy and precision of estimates as well as model fit). We present more background and quantities of interest in the SI (Section B focuses on the simulation study, Section C on the application). The results there corroborate the results here.

4.1. Simulation Study

RKF simulate data according to the model specification in Section 3. $\theta_{i,t}$ is either $\theta_{i,t-1} + \gamma_{i,t}\sigma$ (and $\gamma_{i,t} \sim N(0,1)$) or $\theta_{i,t}$ is a new draw from $N(0,1)$. This induces breaks in the otherwise more gradual trajectory of θ over time. RKF vary the extent of smoothing over time (*via* σ) as well as the probability of a break, set $I = T = 50$, and $K = 5$, and generate 25 datasets per condition. This results in a total of 225 datasets.

The main results are shown in Figure 2. We focus on four quantities of interest: the correlation between $\hat{\theta}$ and θ (using Pearson's ρ), the root mean squared error of $\hat{\theta}$, the size of the standard errors around $\hat{\theta}$, and the expected log pointwise predictive density (ELPD) based on leave-one-out cross-validation (LOO-CV) as a measure of model fit (Vehtari *et al.* 2017). We set the RHS as the reference distribution, subtract the values of the other two specifications (indicated along the vertical axis), and calculate the median relative improvement of the RHS (shown alongside the median line). Correlation and model fit using the RHS are larger, while RMSE and standard errors are smaller. The relative improvements in RMSE and SE are larger (between 2.3%–7.3%) than in correlation or model fit (between 0.2%–2.3%). The improvements are larger when the probability of a shock is larger and σ is smaller (see Figures 8 and 9 in the SI). The RHS outperforms both the standard normal and the Student's t-distribution in the simulation study.
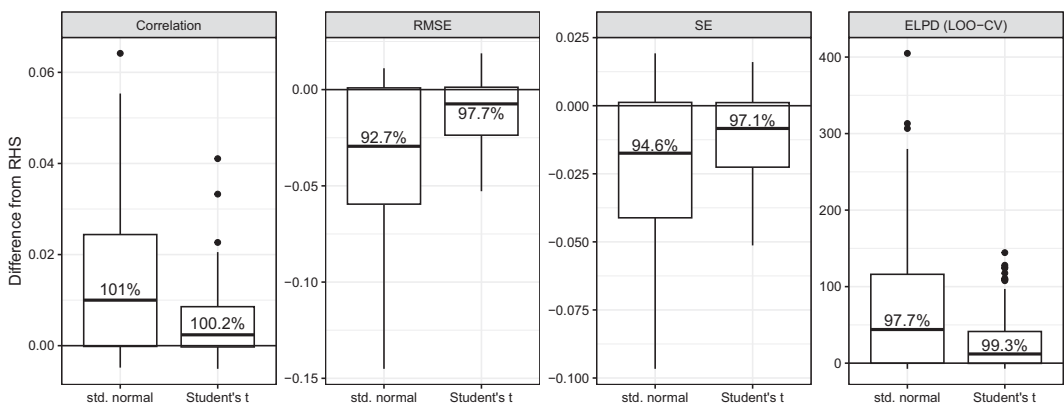

Figure 2. Simulation results. Differences shown along the vertical axis, relative improvement alongside the median lines. For example, the correlation estimated *via* the RHS is 1% larger than *via* the standard normal. Note that the difference for the ELPD is positive although the relative improvement is smaller than 100% because ELPD is measured on a negative scale.

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[6]We omit RKF's replication of Martin and Quinn 2002 for brevity and because the lack of shared items from one time-period to the next (as Supreme Court Justices are faced with new cases every session) necessitates a series of rigid constraints on parameters.

4.2. Revisiting Pemstein et al. 2010

RKF estimate a dynamic model of countries' democracy scores based on the static model of Pemstein *et al.* 2010 (*via* an ordered logit specification). Here, we first focus on model fit *via* the ELPD (based on LOO-CV). The differences between specifications are large and statistically significant. The specification using the RHS performs better than the one using the t-distribution by 1781 units (SE: 58), and better than the one using the normal distribution by 6023 (SE: 125). Using the RHS results in a better model fit than using either of the other two distributions. Second, we turn to predictive accuracy: how often do predictions equal the observed values of democracy indicators? Here, too, the RHS performs better than the other specifications. Overall, the predictive accuracy using the RHS is 62.9% (95% credible interval: 62.6%–63.2%). It is 61.7% (95% ci: 61.4%–62%) for the t-distribution and 58.5% (95% ci: 58.2%–58.8%) for the standard normal distribution. A breakdown by indicator is provided in Figure 11 in the SI. Using the RHS results in a higher predictive accuracy than using the other two priors.

Third, we revisit the two countries discussed by RKF: the Philippines and Afghanistan (Figure 3). The countries exhibit longer periods of stasis (e.g., after the Marcos regime in the Philippines) as well as rapid change (e.g., the Saur Revolution in Afghanistan). Both the RHS and the t-distribution more aptly recover abrupt large changes in latent democracy scores than the normal distribution does. In addition, the RHS appears to have narrower credible intervals than the t-distribution does in some periods of stability (e.g., before and after the Marcos regime).

What if we consider the results of all country-year observations rather than this selection? We turn to this in Figure 4 in which we compare the magnitude of year-over-year changes in countries' mean democracy scores and the credible intervals of their scores between models. We subtract the values
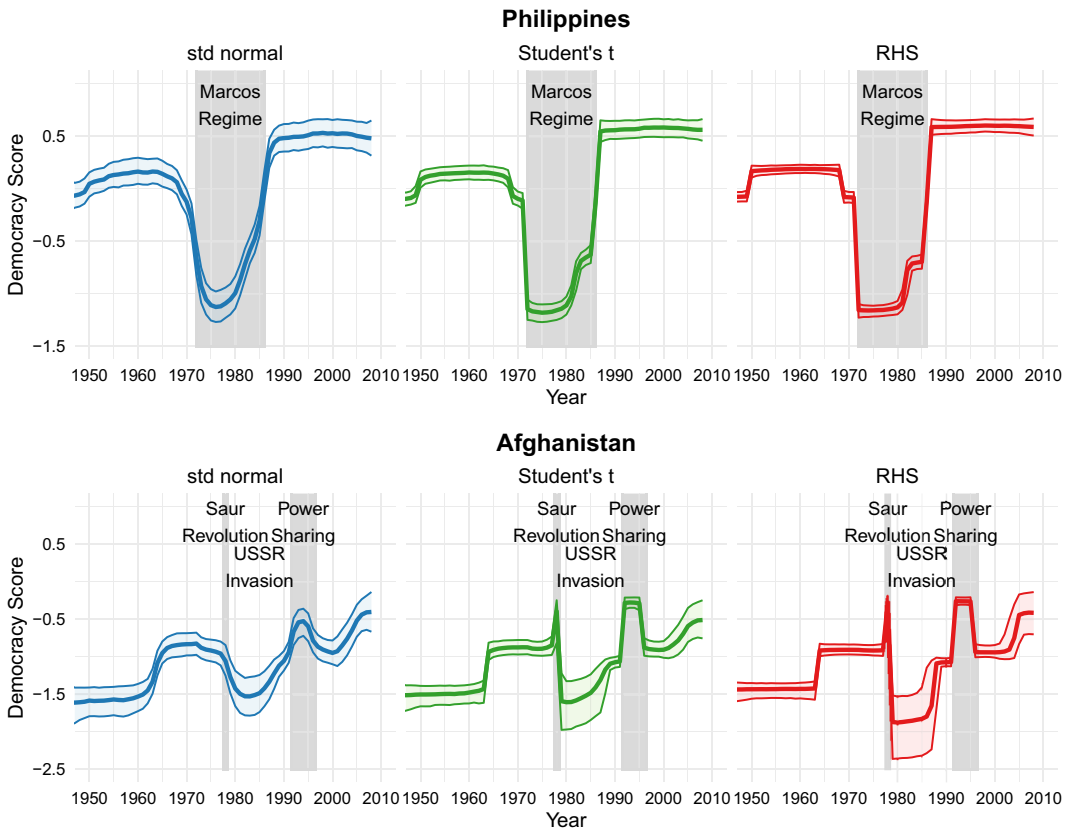

Figure 3. Trajectories of the Philippines and Afghanistan (1946–2008; mean and 95% ci).

(a) Year-over-year changes.
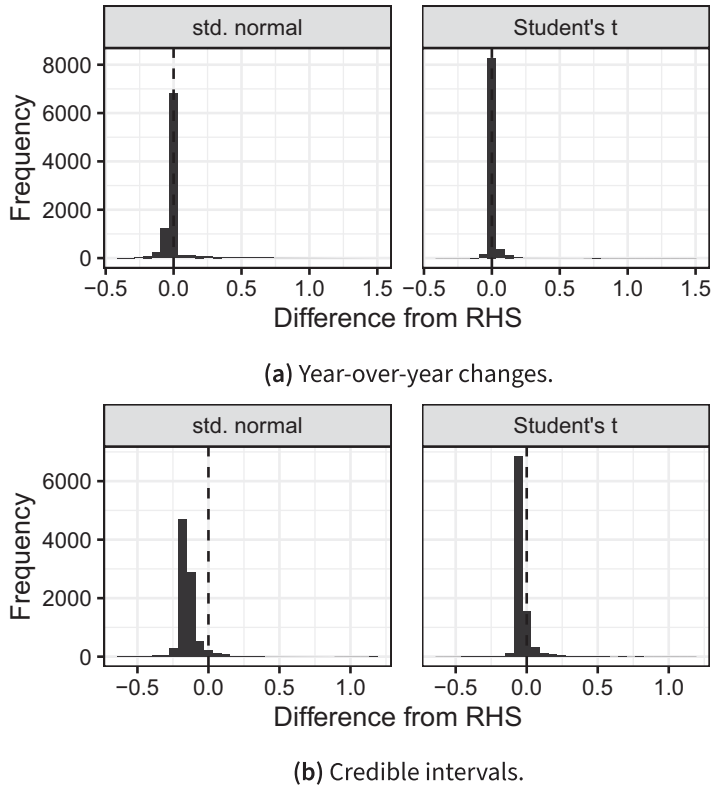

(b) Credible intervals.

Figure 4. Year-over-year changes and credible intervals by model.

from the standard normal and t-distribution specifications from those based on the RHS. There is little systematic difference in the year-over-year changes across models (panel (a)). In contrast, the credible intervals of the RHS are systematically smaller than those of either other distribution (panel (b)). On average, the intervals of the RHS are 44% the size of those using the standard normal distribution, and 73% of those based on the Student's t-distribution. If researchers use estimates from similar models in further explanatory models and take measurement uncertainty into account (as suggested by Tai *et al.* 2024), then the RHS will produce less variance and more precision to begin with.

5. Conclusion

In this letter, we introduce the regularized horseshoe as a prior to political science, and discuss its theoretical and empirical utility in the domain of dynamic latent variable models where periods of stasis can be interrupted by moments of rapid change. In our replication of RKF, the RHS prior performed better than either the standard normal or the Student's t-distribution. The use of the RHS leads to more accurate and precise estimates as well as better model fit.

Our substantive focus has been on modeling countries' democracy scores over time. It is likely that the RHS prior would perform similarly well in other areas where dynamic processes are characterized by a punctuated equilibrium such as, e.g., the trajectories of party positions, public opinion, or budgetary allocations. In addition, due to its sparsity-inducing nature, the RHS could be a candidate for applications where researchers wish to find a balance between a model's flexibility and rigidity *via* a prior distribution, such as in the case of differential item functioning (see, e.g., Binding, Koedam, and Steenbergen 2024).

Data Availability Statement. Replication code for this article is available in Binding and Koc (2024) at https://doi.org/10.7910/DVN/G2VRQH.

Supplementary Material. For supplementary material accompanying this paper, please visit https://doi.org/10.1017/pan.2024.30.

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