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Flip signatures

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Abstract. A D_∞-topological Markov chain is a topological Markov chain provided with an action of the infinite dihedral group D_∞. It is defined by two zero-one square matrices A and J satisfying $AJ = JA^{\mathsf{T}}$ and $J^2 = I$. A flip signature is obtained from symmetric bilinear forms with respect to J on the eventual kernel of A. We modify Williams' decomposition theorem to prove the flip signature is a D_∞-conjugacy invariant. We introduce natural D_∞-actions on Ashley's eight-by-eight and the full two-shift. The flip signatures show that Ashley's eight-by-eight and the full two-shift equipped with the natural D_∞-actions are not D_∞-conjugate. We also discuss the notion of D_∞-shift equivalence and the Lind zeta function.

Key words: flip signatures, D_∞-topological Markov chains, D_∞-conjugacy invariants, eventual kernels, Ashley's eight-by-eight and the full two-shift
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1. *Introduction*

A *topological flip system* (X, T, F) is a topological dynamical system (X, T) consisting of a topological space X, a homeomorphism $T : X \to X$ and a topological conjugacy $F : (X, T^{-1}) \to (X, T)$ with $F^2 = \mathrm{Id}_X$. (See the survey paper [6] for the further study of flip systems.) We call the map F a *flip* for (X, T). If F is a flip for a discrete-time topological dynamical system (X, T), then the triple (X, T, F) is called a D_∞*-system* because the infinite dihedral group

$$D_\infty = \langle a, b : ab = ba^{-1} \text{ and } b^2 = 1\rangle$$

acts on X as follows:

$$(a, x) \mapsto T(x) \quad \text{and} \quad (b, x) \mapsto F(x) \quad (x \in X).$$

Two D_∞-systems (X, T, F) and (X', T', F') are said to be D_∞*-conjugate* if there is a D_∞-equivariant homeomorphism $\theta : X \to X'$. In this case, we write

$$(X, T, F) \cong (X', T', F')$$

and call the map θ a D_∞-*conjugacy* from (X, T, F) to (X', T', F').

Suppose that $\mathcal{A}$ is a finite set. A *topological Markov chain*, or TMC for short, (X_A, σ_A) over $\mathcal{A}$ is a shift space which has a zero-one $\mathcal{A} \times \mathcal{A}$ matrix A as a transition matrix:

$$\mathsf{X}_A = \{x \in \mathcal{A}^{\mathbb{Z}} : A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{Z}\}.$$

A D_∞-system (X, T, F) is said to be a D_∞-topological Markov chain, or D_∞-TMC for short, if (X, T) is a topological Markov chain.

Suppose that (X, T) is a shift space. A flip F for (X, T) is called a *one-block flip* if $x_0 = x'_0$ implies $F(x)_0 = F(x')_0$ for all x and x' in X. If F is a one-block flip for (X, T), then there is a unique map $\tau : \mathcal{A} \to \mathcal{A}$ such that

$$F(x)_i = \tau(x_{-i}) \quad \text{and} \quad \tau^2 = \mathrm{Id}_{\mathcal{A}} \quad (x \in X; i \in \mathbb{Z}).$$

The representation theorem in [4] says that if (X, T, F) is a D_∞-TMC, then there is a TMC (X', T') and a one-block flip F' for (X', T') such that (X, T, F) and (X', T', F') are D_∞-conjugate.

Suppose that $\mathcal{A}$ is a finite set and that A and J are zero-one $\mathcal{A} \times \mathcal{A}$ matrices satisfying

$$AJ = JA^{\mathsf{T}} \quad \text{and} \quad J^2 = I. \tag{1.1}$$

Since J is zero-one and $J^2 = I$, it follows that J is symmetric and that for any $a \in \mathcal{A}$, there is a unique $b \in \mathcal{A}$ such that $J(a, b) = 1$. Thus, there is a unique permutation $\tau_J : \mathcal{A} \to \mathcal{A}$ of order two satisfying

$$J(a, b) = 1 \Leftrightarrow \tau_J(a) = b \quad (a, b \in \mathcal{A}).$$

It is obvious that the map $\varphi_J : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by

$$\varphi_J(x)_i = \tau_J(x_{-i}) \quad (x \in X)$$

is a one-block flip for the full $\mathcal{A}$-shift $(\mathcal{A}^{\mathbb{Z}}, \sigma)$. Since $AJ = JA^{\mathsf{T}}$ implies

$$A(a, b) = A(\tau_J(b), \tau_J(a)) \quad (a, b \in \mathcal{A}),$$

it follows that $\varphi_J(\mathsf{X}_A) = \mathsf{X}_A$. Thus, the restriction $\varphi_{A,J}$ of φ_J to X_A becomes a one-block flip for (X_A, σ_A). A pair (A, J) of zero-one $\mathcal{A} \times \mathcal{A}$ matrices satisfying equation (1.1) will be called a *flip pair*.

The classification of shifts of finite type up to conjugacy is one of the central problems in symbolic dynamics. There is an algorithm determining whether or not two one-sided shifts of finite type ($\mathbb{N}$-SFTs) are $\mathbb{N}$-conjugate. (See §2.1 in [5].) In the case of two-sided shifts of finite type ($\mathbb{Z}$-SFTs), however, one cannot determine whether or not two systems are $\mathbb{Z}$-conjugate, even though many $\mathbb{Z}$-conjugacy invariants have been discovered. For instance, it is well known (Proposition 7.3.7 in [8]) that if two $\mathbb{Z}$-SFTs are $\mathbb{Z}$-conjugate, then their transition matrices have the same set of non-zero eigenvalues. In 1990, Ashley introduced an eight-by-eight zero-one matrix, which is called Ashley's eight-by-eight and asked whether or not it is $\mathbb{Z}$-conjugate to the full two-shift. (See Example 2.2.7 in [5] or §3 in [2].) Since the characteristic polynomial of Ashley's eight-by-eight is $t^7(t-2)$, we

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could say Ashley's eight-by-eight is very simple in terms of spectrum and it is easy to prove that Ashley's eight-by-eight is not $\mathbb{N}$-conjugate to the full two-shift. Nevertheless, this problem has not been solved yet. Meanwhile, both Ashley's eight-by-eight and the full-two shift have one-block flips. More precisely, if we set

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{1.2}$$

then A is Ashley's eight-by-eight, $\varphi_{A,J}$ is a unique one-block flip for (X_A, σ_A), B is the minimal zero-one matrix defining the full two-shift and (X_B, σ_B) has exactly two one-block flips $\varphi_{B,I}$ and $\varphi_{B,K}$. It is natural to ask whether or not $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$ or $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$. In this paper, we introduce the notion of *flip signatures* and prove

$$(\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \not\cong (\mathsf{X}_B, \sigma_B, \varphi_{B,I}), \tag{1.3}$$

$$(\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \not\cong (\mathsf{X}_B, \sigma_B, \varphi_{B,K}) \tag{1.4}$$

and

$$(\mathsf{X}_B, \sigma_B, \varphi_{B,I}) \not\cong (\mathsf{X}_B, \sigma_B, \varphi_{B,K}). \tag{1.5}$$

When (A, J) and (B, K) are flip pairs, it is clear that if θ is a D_∞-conjugacy from $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ to $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$, then θ is also a $\mathbb{Z}$-conjugacy from (X_A, σ_A) to (X_B, σ_B). However, equation (1.5) says that the converse is not true.

We first introduce analogues of elementary equivalence (EE), strong shift equivalence (SSE) and Williams' decomposition theorem for D_∞-TMCs. Let us recall the notions of EE and SSE. (See [8, 9] for the details.) Suppose that A and B are zero-one square matrices. A pair (D, E) of zero-one matrices satisfying

$$A = DE \quad \text{and} \quad B = ED$$

is said to be an EE *from* A *to* B and we write $(D, E) : A \approx B$. If $(D, E) : A \approx B$, then there is a $\mathbb{Z}$-conjugacy $\gamma_{D,E}$ from (X_A, σ_A) to (X_B, σ_B) satisfying

$$\gamma_{D,E}(x) = y \Leftrightarrow \text{for all } i \in \mathbb{Z}, \quad D(x_i, y_i) = E(y_i, x_{i+1}) = 1. \tag{1.6}$$

The map $\gamma_{D,E}$ is called an *elementary conjugacy*.

An SSE *of lag* l *from* A *to* B is a sequence of l elementary equivalences

$$(D_1, E_1) : A \approx A_1, \quad (D_2, E_2) : A_1 \approx A_2, \ldots, \quad (D_l, E_l) : A_l \approx B.$$

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It is evident that if A and B are strong shift equivalent, then (X_A, σ_A) and (X_B, σ_B) are $\mathbb{Z}$-conjugate. Williams' decomposition theorem, found in [9], says that every $\mathbb{Z}$-conjugacy between two $\mathbb{Z}$-TMCs can be decomposed into the composition of a finite number of elementary conjugacies.

To establish analogues of EE, SSE and Williams' decomposition theorem for D_∞-TMCs, we first observe some properties of a D_∞-system. If (X, T, F) is a D_∞-system, then $(X, T, T^n \circ F)$ are also D_∞-systems for all integers n. It is obvious that T^n are D_∞-conjugacies from (X, T, F) to $(X, T, T^{2n} \circ F)$ and from $(X, T, T \circ F)$ to $(X, T, T^{2n+1} \circ F)$ for all integers n. For one's information, we will see that (X, T, F) is not D_∞-conjugate to $(X, T, T \circ F)$ in Proposition 6.1.

Let (A, J) and (B, K) be flip pairs. A pair (D, E) of zero-one matrices satisfying

$$A = DE, \quad B = ED \quad \text{and} \quad E = KD^{\mathsf{T}}J$$

is said to be a *D_∞-half elementary equivalence* (D_∞-HEE) from (A, J) to (B, K) and write $(D, E) : (A, J) \approxeq (B, K)$. In Proposition 2.1, we will see that if $(D, E) : (A, J) \approxeq (B, K)$, then the elementary conjugacy $\gamma_{D,E}$ from equation (1.6) becomes a D_∞-conjugacy from $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ to $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{B,K})$. We call the map $\gamma_{D,E}$ a *D_∞-half elementary conjugacy from $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ to $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{B,K})$*.

A sequence of l D_∞-half elementary equivalences

$$(D_1, E_1) : (A, J) \approxeq (A_2, J_2), \ldots, \quad (A_l, D_l) : (A_l, D_l) \approxeq (B, K)$$

is said to be a *D_∞-strong shift equivalence* (D_∞-SSE) of lag l from (A, J) to (B, K). If there is a D_∞-SSE of lag l from (A, J) to (B, K), then $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \sigma_B^l \circ \varphi_{B,K})$. If l is an even number, then $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$, while if l is an odd number, then $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{B,K})$. In §4, we will see that Williams' decomposition theorem can be modified as follows.

PROPOSITION A. *Suppose that (A, J) and (B, K) are flip pairs.*

(1) *Two D_∞-TMCs $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ are D_∞-conjugate if and only if there is a D_∞-SSE of lag $2l$ between (A, J) and (B, K) for some positive integer l.*

(2) *Two D_∞-TMCs $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{B,K})$ are D_∞-conjugate if and only if there is a D_∞-SSE of lag $2l - 1$ between (A, J) and (B, K) for some positive integer l.*

To introduce the notion of flip signatures, we discuss some properties of D_∞-TMCs. We first indicate notation. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite sets and M is an $\mathcal{A}_1 \times \mathcal{A}_2$ zero-one matrix, then for each $a \in \mathcal{A}_1$, we set

$$\mathcal{F}_M(a) = \{b \in \mathcal{A}_2 : M(a, b) = 1\}$$

and for each $b \in \mathcal{A}_2$, we set

$$\mathcal{P}_M(b) = \{a \in \mathcal{A}_1 : M(a, b) = 1\}.$$

When (X, T) is a TMC, we denote the set of all n-blocks occurring in points in X by $\mathcal{B}_n(X)$ for all non-negative integers n.

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Suppose that (A, J) and (B, K) are flip pairs and that (D, E) is a D_∞-HEE from (A, J) to (B, K). Since B is zero-one and $B = ED$, it follows that

$$\begin{aligned}\mathcal{F}_D(a_1) \cap \mathcal{F}_D(a_2) \neq \varnothing \Rightarrow \mathcal{P}_E(a_1) \cap \mathcal{P}_E(a_2) = \varnothing,\\ \mathcal{P}_E(a_1) \cap \mathcal{P}_E(a_2) \neq \varnothing \Rightarrow \mathcal{F}_D(a_1) \cap \mathcal{F}_D(a_2) = \varnothing,\end{aligned} \tag{1.7}$$

for all $a_1, a_2 \in \mathcal{B}_1(\mathsf{X}_A)$. Suppose that u and v are real-valued functions defined on $\mathcal{B}_1(\mathsf{X}_A)$ and $\mathcal{B}_1(\mathsf{X}_B)$, respectively. If $|\mathcal{B}_1(\mathsf{X}_A)| = m$ and $|\mathcal{B}_1(\mathsf{X}_B)| = n$, then u and v can be regarded as vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If u and v satisfy

$$\text{for all } a \in \mathcal{B}_1(\mathsf{X}_A) \quad u(a) = \sum_{b \in \mathcal{F}_D(a)} v(b), \tag{1.8}$$

then for each $a \in \mathcal{B}_1(\mathsf{X}_A)$, we have

$$u(\tau_J(a))u(a) = \sum_{b \in \mathcal{P}_E(a)} v(\tau_K(b)) \sum_{b \in \mathcal{F}_D(a)} v(b)$$

by $E = KD^\top J$ and equation (1.7) leads to

$$\sum_{a \in \mathcal{B}_1(\mathsf{X}_A)} u(\tau_J(a))u(a) = \sum_{b \in \mathcal{B}_1(\mathsf{X}_B)} \sum_{d \in \mathcal{P}_B(b)} v(\tau_K(d))v(b).$$

Since J and K are symmetric, this formula can be expressed in terms of symmetric bilinear forms with respect to J and K. If we write $\langle u, u\rangle_J = u^\top J u$ and $\langle Bv, v\rangle_K = (Bv)^\top K v$, then we have

$$\langle u, u\rangle_J = \langle Bv, v\rangle_K.$$

We note that if both A and B have λ as their real eigenvalues and v is an eigenvector of B corresponding to λ, then u satisfying equation (1.8) is an eigenvector of A corresponding to λ. We consider the case where A and B have 0 as their eigenvalues and find out some relationships between the symmetric bilinear forms $\langle\ ,\ \rangle_J$ and $\langle\ ,\ \rangle_K$ on the generalized eigenvectors of A and B corresponding to 0 when (A, J) and (B, K) are D_∞-half elementary equivalent.

We call the subspace $\mathcal{K}(A)$ of $u \in \mathbb{R}^m$ such that $A^p u = 0$ for some $p \in \mathbb{N}$ the *eventual kernel* of A:

$$\mathcal{K}(A) = \{u \in \mathbb{R}^m : A^p u = 0 \text{ for some } p \in \mathbb{N}\}.$$

If $u \in \mathcal{K}(A) \setminus \{0\}$ and p is the smallest positive integer for which $A^p u = 0$, then the ordered set

$$\alpha = \{A^{p-1}u, \ldots, Au, u\}$$

is called a *cycle of generalized eigenvectors of* A *corresponding to* 0. In this paper, we sometimes call α a *cycle in* $\mathcal{K}(A)$ for simplicity. The vectors $A^{p-1}u$ and u are called the *initial vector* and the *terminal vector* of α, respectively, and we write

$$\mathrm{Ini}(\alpha) = A^{p-1}u \quad \text{and} \quad \mathrm{Ter}(\alpha) = u.$$

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We say that the length of α is p and write $|\alpha| = p$. It is well known [3] that there is a basis for $\mathcal{K}(A)$ consisting of a union of disjoint cycles of generalized eigenvectors of A corresponding to 0. The set of bases for $\mathcal{K}(A)$ consisting of a union of disjoint cycles of generalized eigenvectors of A corresponding to 0 is denoted by $\mathcal{B}as(\mathcal{K}(A))$. We will prove the following proposition in §3.

PROPOSITION B. *Suppose that* $(D, E) : (A, J) \approx (B, K)$. *Then there exist bases* $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ *and* $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ *such that if* $p > 1$ *and* $\alpha = \{u_1, u_2, \ldots, u_p\}$ *is a cycle in* $\mathcal{E}(A)$*, then one of the following holds.*

(1) *There is a cycle* $\beta = \{v_1, v_2, \ldots, v_{p+1}\}$ *in* $\mathcal{E}(B)$ *such that*

$$Dv_{k+1} = u_k \quad \text{and} \quad Eu_k = v_k \quad (k = 1, \ldots, p).$$

(2) *There is a cycle* $\beta = \{v_1, v_2, \ldots, v_{p-1}\}$ *in* $\mathcal{E}(B)$ *such that*

$$Dv_k = u_k \quad \text{and} \quad Eu_{k+1} = v_k \quad (k = 1, \ldots, p-1).$$

In either case, we have

$$\langle \mathrm{Ini}(\alpha), \mathrm{Ter}(\alpha)\rangle_J = \langle \mathrm{Ini}(\beta), \mathrm{Ter}(\beta)\rangle_K. \tag{1.9}$$

In Lemma 3.3, we will show that there is a basis $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ such that for every cycle α in $\mathcal{E}(A)$, the restriction of symmetric bilinear form $\langle\ ,\ \rangle_J$ to $\mathrm{span}(\alpha)$ is non-degenerate and in Lemma 3.2, we will see that the restriction of symmetric bilinear form $\langle\ ,\ \rangle_J$ to $\mathrm{span}(\alpha)$ is non-degenerate if and only if the left-hand side of equation (1.9) is not 0 for a cycle α in $\mathcal{E}(A)$. In this case, we define the sign of a cycle $\alpha = \{u_1, u_2, \ldots, u_p\}$ in $\mathcal{E}(A)$ by

$$\mathrm{sgn}(\alpha) = \begin{cases} +1 & \text{if } \langle \mathrm{Ini}(\alpha), \mathrm{Ter}(\alpha)\rangle_J > 0, \\ -1 & \text{if } \langle \mathrm{Ini}(\alpha), \mathrm{Ter}(\alpha)\rangle_J < 0. \end{cases}$$

We denote the set of $|\alpha|$ such that α is a cycle in $\mathcal{E}(A)$ by $\mathcal{I}nd(\mathcal{K}(A))$. It is clear that $\mathcal{I}nd(\mathcal{K}(A))$ is independent of the choice of basis for $\mathcal{K}(A)$. We denote the union of the cycles α of length p in $\mathcal{E}(A)$ by $\mathcal{E}_p(A)$ for each $p \in \mathcal{I}nd(\mathcal{K}(A))$ and define the sign of $\mathcal{E}_p(A)$ by

$$\mathrm{sgn}(\mathcal{E}_p(A)) = \prod_{\{\alpha:\alpha \text{ is a cycle in } \mathcal{E}_p(A)\}} \mathrm{sgn}(\alpha).$$

In §3, we will prove the sign of $\mathcal{E}_p(A)$ is also independent of the choice of basis for $\mathcal{K}(A)$ if the restriction of $\langle\ ,\ \rangle_J$ to $\mathrm{span}(\alpha)$ is non-degenerate for every cycle α in $\mathcal{E}_p(A)$.

PROPOSITION C. *Suppose that* $\mathcal{E}(A)$ *and* $\mathcal{E}'(A)$ *are two distinct bases in* $\mathcal{B}as(\mathcal{K}(A))$ *such that for every cycle* α *in* $\mathcal{E}(A)$ *or* $\mathcal{E}'(A)$*, the restriction of* $\langle\ ,\ \rangle_J$ *to* $\mathrm{span}(\alpha)$ *is non-degenerate. Then for each* $p \in \mathcal{I}nd(\mathcal{K}(A))$*, we have*

$$\mathrm{sgn}(\mathcal{E}_p(A)) = \mathrm{sgn}(\mathcal{E}'_p(A)).$$

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Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and that the restriction of $\langle\ ,\ \rangle_J$ to $\mathrm{span}(\alpha)$ is non-degenerate for every cycle in $\mathcal{E}(A)$. We arrange the elements of $\mathcal{I}nd(\mathcal{K}(A)) = \{p_1, p_2, \ldots, p_A\}$ to satisfy

$$p_1 < p_2 < \ldots p_A$$

and write

$$\varepsilon_p = \mathrm{sgn}(\mathcal{E}_p(A)).$$

If $|\mathcal{I}nd(\mathcal{K}(A))| = k$, then the k-tuple $(\varepsilon_{p_1}, \varepsilon_{p_2}, \ldots, \varepsilon_{p_A})$ is called the *flip signature of* (A, J) and ε_{p_A} is called the *leading signature of* (A, J). The flip signature of (A, J) is denoted by

$$\mathrm{F.Sig}(A, J) = (\varepsilon_{p_1}, \varepsilon_{p_2}, \ldots, \varepsilon_{p_A}).$$

The following is the main result of this paper.

THEOREM D. *Suppose that* (A, J) *and* (B, K) *are flip pairs and that* $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ *and* $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ *are* D_∞*-conjugate. If*

$$F.Sig(A, J) = (\varepsilon_{p_1}, \varepsilon_{p_2}, \ldots, \varepsilon_{p_A})$$

and

$$F.Sig(B, K) = (\varepsilon_{q_1}, \varepsilon_{q_2}, \ldots, \varepsilon_{q_B}),$$

then $F.Sig(A, J)$ *and* $F.Sig(B, K)$ *have the same number of* -1*s and the leading signatures* ε_{p_A} *and* ε_{q_B} *coincide:*

$$\varepsilon_{p_A} = \varepsilon_{q_B}.$$

In §7, we will compute the flip signatures of (A, J), (B, I) and (B, K), where A, J, B, I and K are as in equation (1.2) and prove equations (1.3), (1.4) and (1.5). Actually, we can obtain equations (1.3), (1.4) and (1.5) from the Lind zeta functions. In [4], an explicit formula for the Lind zeta function for a D_∞-TMC was established, which can be expressed in terms of matrices from flip pairs. From its formula (see also §6), it is obvious that the Lind zeta function is a D_∞-conjugacy invariant. In Example 7.1, we will see that the Lind zeta functions of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$, $(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ are all different. In §6, we introduce the notion of D_∞-shift equivalence (D_∞-SE) which is an analogue of shift equivalence and prove that D_∞-SE is a D_∞-conjugacy invariant. In Example 7.2, we will see that there are D_∞-SEs between (A, J), (B, I) and (B, K) pairwise. So the existence of D_∞-shift equivalence between two flip pairs does not imply that the corresponding $\mathbb{Z}$-TMCs share the same Lind zeta functions. This is a contrast to the fact that the existence of shift equivalence between two defining matrices A and B implies the coincidence of the Artin–Mazur zeta functions [1] of the $\mathbb{Z}$-TMCs (X_A, σ_A) and (X_B, σ_B). Meanwhile, Example 7.5 says that the coincidence of the Lind zeta functions of two D_∞-TMCs does not guarantee the existence of D_∞-shift equivalence between their flip pairs. This is analogous to the case of $\mathbb{Z}$-TMCs because the coincidence of the Artin–Mazur zeta functions of two $\mathbb{Z}$-TMCs does not guarantee the existence of SE between their defining matrices. (See §7.4 in [8].)

When (A, J) is a flip pair with $|\mathcal{B}_1(X)| = m$, the matrix A defines a linear transformation $A : \mathbb{R}^m \to \mathbb{R}^m$. The largest subspace $\mathcal{R}(A)$ of $\mathbb{R}^m$ on which A is invertible is the called the *eventual range* of A:

$$\mathcal{R}(A) = \bigcap_{k=1}^{\infty} A^k \mathbb{R}^m.$$

Similarly, the eventual kernel $\mathcal{K}(A)$ of A is the largest subspace of $\mathbb{R}^m$ on which A is nilpotent:

$$\mathcal{K}(A) = \bigcup_{k=1}^{\infty} \ker(A^k).$$

With this notation, we can write $\mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{K}(A)$. (See §7.4 in [8].) The flip signature of (A, J) is completely determined by $\mathcal{K}(A)$, while the Lind zeta functions and the existence of D_∞-shift equivalence between two flip pairs depend on the eventual ranges of transition matrices. In other words, two flip signatures which have the same number of -1s and share the same leading signature have nothing to do with the coincidence of the Lind zeta functions or the existence of D_∞-shift equivalence. As a result, flip signatures cannot be a complete D_∞-conjugacy invariant. This will be clear in Example 7.4.

This paper is organized as follows. In §2, we introduce the notions of D_∞-half elementary equivalence and D_∞-strong shift equivalence. In §3, we investigate symmetric bilinear forms with respect to J and K on the eventual kernels of A and B when two flip pairs (A, J) and (B, K) are D_∞-half elementary equivalent. In the same section, we prove Propositions B and C. Proposition A and Theorem D will be proved in §§4 and 5, respectively. In §6, we discuss the notion of D_∞-shift equivalence and the Lind zeta function. Section 7 consists of examples.

2. *D_∞-strong shift equivalence*

Let (A, J) and (B, K) be flip pairs. A pair (D, E) of zero-one matrices satisfying

$$A = DE, \quad B = ED \quad \text{and} \quad E = KD^{\mathsf{T}}J$$

is said to be a *D_∞-half elementary equivalence from (A, J) to (B, K)*. If there is a D_∞-half elementary equivalence from (A, J) to (B, K), then we write $(D, E) : (A, J) \approx (B, K)$. We note that symmetricities of J and K imply

$$E = KD^{\mathsf{T}}J \Leftrightarrow D = JE^{\mathsf{T}}K.$$

PROPOSITION 2.1. *If $(D, E) : (A, J) \approx (B, K)$, then $(\mathsf{X}_A, \sigma_A, \varphi_{J,A})$ is D_∞-conjugate to $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{K,B})$.*

Proof. Since D and E are zero-one and $A = DE$, it follows that for all $a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A)$, there is a unique $b \in \mathcal{B}_1(\mathsf{X}_B)$ such that

$$D(a_1, b) = E(b, a_2) = 1.$$

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We denote the block map which sends $a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A)$ to $b \in \mathcal{B}_1(\mathsf{X}_B)$ by $\Gamma_{D,E}$. If we define the map $\gamma_{D,E} : (\mathsf{X}_A, \sigma_A) \to (\mathsf{X}_B, \sigma_B)$ by

$$\gamma_{D,E}(x)_i = \Gamma_{D,E}(x_i x_{i+1}) \quad (x \in \mathsf{X}_A; i \in \mathbb{Z}),$$

then we have $\gamma_{D,E} \circ \sigma_A = \sigma_B \circ \gamma_{D,E}$.

Since $(E, D) : (B, K) \approx (A, J)$, we can define the block map $\Gamma_{E,D} : \mathcal{B}_2(\mathsf{X}_B) \to \mathcal{B}_1(\mathsf{X}_A)$ and the map $\gamma_{E,D} : (\mathsf{X}_B, \sigma_B) \to (\mathsf{X}_A, \sigma_A)$ in the same way. Since $\gamma_{E,D} \circ \gamma_{D,E} = \mathrm{Id}_{\mathsf{X}_A}$ and $\gamma_{D,E} \circ \gamma_{E,D} = \mathrm{Id}_{\mathsf{X}_B}$, it follows that $\gamma_{D,E}$ is one-to-one and onto.

It remains to show that

$$\gamma_{D,E} \circ \varphi_{A,J} = (\sigma_B \circ \varphi_{B,K}) \circ \gamma_{D,E}. \tag{2.1}$$

Since $E = KD^{\mathsf{T}}J$, it follows that

$$E(b, a) = 1 \Leftrightarrow D(\tau_J(a), \tau_K(b)) = 1 \quad (a \in \mathcal{B}_1(\mathsf{X}_A), b \in \mathcal{B}_1(\mathsf{X}_B)).$$

This is equivalent to

$$D(a, b) = 1 \Leftrightarrow E(\tau_K(b), \tau_J(a)) = 1 \quad (a \in \mathcal{B}_1(\mathsf{X}_A), b \in \mathcal{B}_1(\mathsf{X}_B)).$$

Thus, we obtain

$$\Gamma_{D,E}(a_1a_2) = b \Leftrightarrow \Gamma_{D,E}(\tau_J(a_2)\tau_J(a_1)) = \tau_K(b) \quad (a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A)). \tag{2.2}$$

By equation (2.2), we have

$$\begin{aligned}\gamma_{D,E} \circ \varphi_{J,A}(x)_i &= \Gamma_{D,E}(\tau_J(x_{-i})\tau_J(x_{-i-1})) = \tau_K(\Gamma_{D,E}(x_{-i-1}x_{-i})) \\ &= \varphi_{B,K} \circ \gamma_{D,E}(x)_{i+1} = (\sigma_B \circ \varphi_{B,K}) \circ \gamma_{D,E}(x)_i\end{aligned}$$

for all $x \in \mathsf{X}_A$ and $i \in \mathbb{Z}$ and this proves equation (2.1). □

Let (A, J) and (B, K) be flip pairs. A sequence of l half elementary equivalences

$$(D_1, E_1) : (A, J) \approx (A_2, J_2),$$

$$(D_2, E_2) : (A_2, J_2) \approx (A_3, J_3),$$

$$\vdots$$

$$(D_l, E_l) : (A_l, J_l) \approx (B, K)$$

is said to be a *D_∞-SSE of lag l from (A, J) to (B, K)*. If there is a D_∞-SSE of lag l from (A, J) to (B, K), then we say that (A, J) is D_∞-strong shift equivalent to (B, K) and write $(A, J) \approx (B, K)$ (lag l).

By Proposition 2.1, we have

$$(A, J) \approx (B, K) \text{ (lag } l) \Rightarrow (\mathsf{X}_A, \sigma_A, \varphi_{J,A}) \cong (\mathsf{X}_B, \sigma_B, {\sigma_B}^l \circ \varphi_{K,B}). \tag{2.3}$$

Because ${\sigma_B}^l$ is a conjugacy from $(\mathsf{X}_B, \sigma_B, \varphi_{K,B})$ to $(\mathsf{X}_B, \sigma_B, {\sigma_B}^{2l} \circ \varphi_{K,B})$, the implication in equation (2.3) can be rewritten as follows:

$$(A, J) \approx (B, K) \text{ (lag } 2l) \Rightarrow (\mathsf{X}_A, \sigma_A, \varphi_{J,A}) \cong (\mathsf{X}_B, \sigma_B, \varphi_{K,B}) \tag{2.4}$$

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and

$$(A, J) \approx (B, K) \text{ (lag } 2l-1) \Rightarrow (\mathsf{X}_A, \sigma_A, \varphi_{J,A}) \cong (\mathsf{X}_B, \sigma_B, \sigma_B \circ \varphi_{K,B}). \tag{2.5}$$

In §4, we will prove Proposition A which says that the converses of equations (2.4) and (2.5) are also true.

3. *Symmetric bilinear forms*

Suppose that (A, J) is a flip pair and that $|\mathcal{B}_1(\mathsf{X}_A)| = m$. Let V be an m-dimensional vector space over the field $\mathbb{C}$ of complex numbers. Let $\langle u, v\rangle_J$ denote the bilinear form $V \times V \to \mathbb{C}$ defined by

$$(u, v) \mapsto u^{\mathsf{T}} J \bar{v} \quad (u, v \in V).$$

Since J is a non-singular symmetric matrix, it follows that the bilinear form $\langle \ , \ \rangle_J$ is symmetric and non-degenerate. If $u, v \in V$ and $\langle u, v\rangle_J = 0$, then u and v are said to be *orthogonal with respect to* J and we write $u \perp_J v$. From $AJ = JA^{\mathsf{T}}$, we see that A itself is the adjoint of A in the following sense:

$$\langle Au, v\rangle_J = \langle u, Av\rangle_J. \tag{3.1}$$

If λ is an eigenvalue of A and u is an eigenvector of A corresponding to λ, then for any $v \in V$, we have

$$\lambda\langle u, v\rangle_J = \langle \lambda u, v\rangle_J = \langle Au, v\rangle_J = \langle u, Av\rangle_J. \tag{3.2}$$

Let $\operatorname{sp}(A)$ denote the set of eigenvalues of A. For each $\lambda \in \operatorname{sp}(A)$, let $\mathcal{K}_\lambda(A)$ denote the set of $u \in V$ such that $(A - \lambda I)^p u = 0$ for some $p \in \mathbb{N}$:

$$\mathcal{K}_\lambda(A) = \{u \in V : \text{there exists } p \in \mathbb{N} \text{ such that } (A - \lambda I)^p u = 0\}.$$

If $u \in \mathcal{K}_\lambda(A) \setminus \{0\}$ and p is the smallest positive integer for which $(A - \lambda I)^p u = 0$, then the ordered set

$$\alpha = \{(A - \lambda I)^{p-1} u, \ldots, (A - \lambda I)u, u\}$$

is called a *cycle of generalized eigenvectors of* A *corresponding to* λ. The vectors $(A - \lambda I)^{p-1} u$ and u are called the *initial vector* and the *terminal vector* of α, respectively, and we write

$$\operatorname{Ini}(\alpha) = (A - \lambda I)^{p-1} u \quad \text{and} \quad \operatorname{Ter}(\alpha) = u.$$

We say that the length of α is p and write $|\alpha| = p$. It is well known [3] that there is a basis for $\mathcal{K}_\lambda(A)$ consisting of a union of disjoint cycles of generalized eigenvectors of A corresponding to λ. From here on, when we say $\alpha = \{u_1, \ldots, u_p\}$ is a cycle in $\mathcal{K}_\lambda(A)$, it means α is a cycle of generalized eigenvectors of A corresponding to λ, $\operatorname{Ini}(\alpha) = u_1$, $\operatorname{Ter}(\alpha) = u_p$ and $|\alpha| = p$.

Suppose that $\mathcal{U}(A)$ is a basis for V consisting of generalized eigenvectors of A, A has 0 as its eigenvalue and that $\mathcal{E}(A)$ is the subset of $\mathcal{U}(A)$ consisting of the generalized eigenvectors of A corresponding to 0. Non-degeneracy of $\langle \ , \ \rangle_J$ says that for each

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$u \in \mathcal{E}(A)$, there is a $v \in \mathcal{U}(A)$ such that $\langle u, v\rangle_J \neq 0$. The following lemma says that the vector v must be in $\mathcal{E}(A)$.

LEMMA 3.1. *Suppose that $\lambda, \mu \in \mathrm{sp}(A)$. If λ is distinct from the complex conjugate $\bar{\mu}$ of μ, then $\mathcal{K}_\lambda(A) \perp_J \mathcal{K}_\mu(A)$.*

Proof. Suppose that

$$\alpha = \{u_1, \ldots, u_p\} \quad \text{and} \quad \beta = \{v_1, \ldots, v_q\}$$

are cycles in $\mathcal{K}_\lambda(A)$ and $\mathcal{K}_\mu(A)$, respectively. Since equation (3.2) implies

$$\lambda\langle u_1, v_1\rangle_J = \langle u_1, Av_1\rangle_J = \bar{\mu}\langle u_1, v_1\rangle_J,$$

it follows that

$$\langle u_1, v_1\rangle_J = 0.$$

Using equation (3.2) again, we have

$$\lambda\langle u_1, v_{j+1}\rangle_J = \langle u_1, \mu v_{j+1} + v_j\rangle_J = \bar{\mu}\langle u_1, v_{j+1}\rangle_J + \langle u_1, v_j\rangle_J$$

for each $j = 1, \ldots, q-1$. By mathematical induction on j, we see that

$$\langle u_1, v_j\rangle_J = 0 \quad (j = 1, \ldots, q).$$

Applying the same process to each $u_2, \ldots u_p$, we obtain

$$\text{for all } i = 1, \ldots, p, \quad \text{for all } j = 1, \ldots, q, \quad \langle u_i, v_j\rangle_J = 0.$$

□

Remark. Non-degeneracy of $\langle\ ,\ \rangle_J$ and Lemma 3.1 imply that the restriction of $\langle\ ,\ \rangle_J$ to $\mathcal{K}_0(A)$ is non-degenerate.

From here on, we restrict our attention to the zero eigenvalue and the generalized eigenvectors corresponding to 0. For notational simplicity, the smallest subspace of V containing all generalized eigenvectors of A corresponding to 0 is denoted by $\mathcal{K}(A)$ and we call the subspace $\mathcal{K}(A)$ of V the *eventual kernel* of A. We may assume that the eventual kernel of A is a real vector space. The set of bases for $\mathcal{K}(A)$ consisting of a union of disjoint cycles of generalized eigenvectors of A corresponding to 0 is denoted by $\mathcal{B}as(\mathcal{K}(A))$. If $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$, the set of $|\alpha|$ such that α is a cycle in $\mathcal{E}(A)$ is denoted by $\mathcal{I}nd(\mathcal{K}(A))$ and we call $\mathcal{I}nd(\mathcal{K}(A))$ the *index set for the eventual kernel of* A. It is clear that $\mathcal{I}nd(\mathcal{K}(A))$ is independent of the choice of $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$. When $p \in \mathcal{I}nd(\mathcal{K}(A))$, we denote the union of the cycles of length p in $\mathcal{E}(A)$ by $\mathcal{E}_p(A)$.

LEMMA 3.2. *Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and that $p \in \mathcal{I}nd(\mathcal{K}(A))$.*

(1) *Suppose that α is a cycle in $\mathcal{E}_p(A)$. The restriction of $\langle\ ,\ \rangle_J$ to* span(α) *is non-degenerate if and only if*

$$\langle \mathrm{Ini}(\alpha), \mathrm{Ter}(\alpha)\rangle_J \neq 0.$$

(2) *The restriction of $\langle\ ,\ \rangle_J$ to $\mathcal{E}_p(A)$ is non-degenerate.*

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Proof. Suppose that $\alpha = \{u_1, \ldots, u_p\}$ is a cycle in $\mathcal{E}_p(A)$. By equation (3.1), we have

$$\langle u_1, u_i\rangle_J = \langle u_1, Au_{i+1}\rangle_J = \langle Au_1, u_{i+1}\rangle_J = 0$$

and

$$\langle u_{i+1}, u_j\rangle_J - \langle u_i, u_{j+1}\rangle_J = \langle u_{i+1}, Au_{j+1}\rangle_J - \langle u_i, u_{j+1}\rangle_J = 0 \tag{3.3}$$

for each $i, j = 1, \ldots, p-1$. Suppose that T_p is the $m \times p$ matrix whose ith column is u_i for each $i = 1, \ldots, p$. If we set $\langle u_i, u_p\rangle_J = b_i$ for each $i = 1, 2, \ldots, p$, then $T_p^{\top} J T_p$ is of the form

$$T_p^{\top} J T_p = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & b_1 & b_2 \\ 0 & 0 & 0 & \cdots & b_1 & b_2 & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_{p-2} & b_{p-1} & b_p \end{bmatrix}.$$

This proves item (1).

To prove item (2), we only consider the case where $\mathcal{I}nd(\mathcal{K}(A)) = \{p, q\}(p < q)$ and both $\mathcal{E}_p(A)$ and $\mathcal{E}_q(A)$ have one cycles. Suppose that $\alpha = \{u_1, \ldots, u_p\}$ and $\beta = \{v_1, \ldots, v_q\}$ are cycles in $\mathcal{E}_p(A)$ and $\mathcal{E}_q(A)$, respectively. When T_p is as above, we will prove $T_p^{\top} J T_p$ is non-singular. We let T_q be the $m \times q$ matrix whose ith column is v_i for each $i = 1, \ldots, q$. If T is the $m \times (p+q)$ matrix defined by

$$T = \begin{bmatrix} T_p & T_q \end{bmatrix},$$

then

$$T^{\top} J T = \begin{bmatrix} T_p^{\top} J T_p & T_p^{\top} J T_q \\ T_q^{\top} J T_p & T_q^{\top} J T_q \end{bmatrix}$$

is non-singular by remark of Lemma 3.1. By arguments in the proof of item (1), we can put

$$T_p^{\top} J T_p = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & b_1 & b_2 \\ 0 & 0 & 0 & \cdots & b_1 & b_2 & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_{p-2} & b_{p-1} & b_p \end{bmatrix}$$

and

$$T_q^{\top} J T_q = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & d_1 \\ 0 & 0 & 0 & \cdots & 0 & d_1 & d_2 \\ 0 & 0 & 0 & \cdots & d_1 & d_2 & d_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ d_1 & d_2 & d_3 & \cdots & d_{q-2} & d_{q-1} & d_q \end{bmatrix}.$$

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Now we consider $T_p^{\top} J T_q$. By equation (3.1), we have

$$\begin{aligned}\langle u_1, v_k\rangle_J &= 0 \quad (k = 1, \ldots, q-1),\\ \langle u_2, v_k\rangle_J &= 0 \quad (k = 1, \ldots, q-2),\\ &\vdots\\ \langle u_p, v_k\rangle_J &= 0 \quad (k = 1, \ldots, q-p).\end{aligned}$$

If we set $\langle u_i, v_q\rangle_J = c_i$ for each $i = 1, 2, \ldots, p$, then the argument in equation (3.3) shows that $T_p^{\top} J T_q$ is of the form

$$T_p^{\top} J T_q = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & c_1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & c_1 & c_2 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & c_1 & c_2 & c_3 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & c_1 & c_2 & c_3 & \cdots & c_{p-2} & c_{p-1} & c_p \end{bmatrix}.$$

Finally, $T_q^{\top} J T_p$ is the transpose of $T_p^{\top} J T_q$. Hence, b_1 and d_1 must be non-zero and we have $\mathrm{Rank}(T_p^{\top} J T_p) = p$ and $\mathrm{Rank}(T_q^{\top} J T_q) = q$. □

The aim of this section is to find out a relationship between $\langle\ ,\ \rangle_J$ and $\langle\ ,\ \rangle_K$ on bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ when $(D, E) : (A, J) \approxeq (B, K)$. The following lemma will provide us good bases to handle.

LEMMA 3.3. *Suppose that A has the zero eigenvalue. There is a basis $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ having the following properties.*

(1) *If α is a cycle in $\mathcal{E}(A)$, then the restriction of $\langle\ ,\ \rangle_J$ to* span(α) *is non-degenerate, that is,*

$$\langle \mathrm{Ini}(\alpha), \mathrm{Ter}(\alpha)\rangle_J \neq 0.$$

(2) *Suppose that α is a cycle in $\mathcal{E}(A)$ with Ter$(\alpha) = u$ and $|\alpha| = p$. For each $k = 0, 1, \ldots p-1$, $v = A^{p-1-k}u$ is the unique vector in α such that $\langle A^k u, v\rangle_J \neq 0$.*

(3) *If α and β are distinct cycles in $\mathcal{E}(A)$, then*

$$\mathrm{span}(\alpha) \perp_J \mathrm{span}(\beta).$$

Proof. (1) Lemma 3.2 proves the case where $\mathcal{E}(A)$ has only one cycle. Suppose that $\mathcal{E}(A)$ is the union of disjoint cycles $\alpha_1, \ldots, \alpha_r$ of generalized eigenvectors of A corresponding to 0 for some $r > 1$ and that $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_r|$. Assuming

$$\langle \mathrm{Ini}(\alpha_j), \mathrm{Ter}(\alpha_j)\rangle_J \neq 0 \quad (j = 1, \ldots, r-1),$$

we will construct a cycle β of generalized eigenvectors of A corresponding to 0 such that the union of the cycles $\alpha_1, \ldots, \alpha_{r-1}, \beta$ forms a basis for $\mathcal{K}(A)$ and that $\langle \mathrm{Ini}(\beta), \mathrm{Ter}(\beta)\rangle_J \neq 0$.

By Lemma 3.2, we have

$$|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_{r-1}| < |\alpha_r| \Rightarrow \langle \mathrm{Ini}(\alpha_r), \mathrm{Ter}(\alpha_r)\rangle_J \neq 0.$$

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Thus, we only consider the case where there are other cycles in $\mathcal{E}(A)$ whose length is the same as $|\alpha_r|$. If $\alpha_r = \{w_1, \ldots, w_q\}$ and $\langle w_1, w_q\rangle_J \neq 0$, there is nothing to do. So we assume $\langle w_1, w_q\rangle_J = 0$. By non-degeneracy of $\langle \ , \ \rangle_J$ and Lemma 3.1, there is a vector $v \in \mathcal{E}(A)$ such that $\langle w_1, v\rangle_J \neq 0$. Since $\langle w_1, v\rangle_J = \langle w_q, A^{q-1}v\rangle_J$, it follows that v must be the terminal vector of a cycle in $\mathcal{E}(A)$ of length q by the maximality of q. We put $v_1 = A^{q-1}v$ and $v_q = v$ and find a number $k \in \mathbb{R} \setminus \{0\}$ such that $\langle w_1 - kv_1, w_q - kv_q\rangle_J \neq 0$. We denote the cycle whose terminal vector is $w_q - kv_q$ by β. It is obvious that the length of β is q and that the union of the cycles $\alpha_1, \ldots, \alpha_{r-1}, \beta$ forms a basis of $\mathcal{K}(A)$.

(2) We assume that $\mathcal{E}(A)$ has property (1) and that $\alpha = \{u_1, \ldots, u_p\}$ is a cycle in $\mathcal{E}(A)$. The proof of Lemma 3.2(1) says that if T_α is the $m \times p$ matrix whose ith column is u_i, then $T_\alpha^\top J T_\alpha$ is of the form

$$T_\alpha^\top J T_\alpha = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & b_1 & b_2 \\ 0 & 0 & 0 & \cdots & b_1 & b_2 & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_{p-2} & b_{p-1} & b_p \end{bmatrix}.$$

We note that b_1 must be non-zero. Now, there are unique real numbers $k_1, \ldots, k_p$ such that if we set

$$K = \begin{bmatrix} k_p & k_{p-1} & \cdots & k_1 \\ 0 & k_p & \cdots & k_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & k_p \end{bmatrix},$$

then $K^\top T_\alpha^\top J T_\alpha K$ becomes

$$K^\top T_\alpha^\top J T_\alpha K = \begin{bmatrix} 0 & 0 & \cdots & 0 & b_1 \\ 0 & 0 & \cdots & b_1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & b_1 & \cdots & 0 & 0 \\ b_1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

If α' is a cycle in $\mathcal{K}(A)$ whose terminal vector is $w = \sum_{i=1}^{p} k_i u_i$, then we have $|\alpha'| = p$ and

$$\langle A^i w, A^j w\rangle_J = \begin{cases} b_1 & \text{if } j = p - 1 - i, \\ 0 & \text{otherwise,} \end{cases}$$

for each $0 \leq i, j \leq p - 1$. If we replace α with α' for each α in $\mathcal{E}(A)$, then the result follows.

(3) Suppose that $\mathcal{E}(A)$ has properties (1) and (2) and that $\mathcal{E}(A)$ is the union of disjoint cycles $\alpha_1, \ldots, \alpha_r$ of generalized eigenvectors of A corresponding to 0 for some $r > 1$ with $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_r|$. Assuming that

$$\operatorname{span}(\alpha_i) \perp_J \operatorname{span}(\alpha_j) \quad (i, j = 1, \ldots, r-1; i \neq j),$$

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we will construct a cycle β such that the union of the cycles $\alpha_1, \ldots, \alpha_{r-1}, \beta$ forms a basis for $\mathcal{K}(A)$ and that α_i is orthogonal to β with respect to J for each $i = 1, \ldots, r-1$.

Suppose that $\alpha = \{u_1, \ldots, u_p\}$ is a cycle in $\mathcal{E}(A)$ which is distinct from $\alpha_r = \{w_1, \ldots, w_q\}$. We set

$$\langle u_1, u_p \rangle_J = a \ (\neq 0), \quad \langle u_i, w_q \rangle_J = b_i \quad (i = 1, \ldots, p)$$

and

$$z = w_q - \frac{b_1}{a} u_p - \frac{b_2}{a} u_{p-1} - \cdots - \frac{b_p}{a} u_1.$$

Let β denote the cycle whose terminal vector is z.

We first show that $u_1 \perp_J \operatorname{span}(\beta)$. Direct computation yields

$$\langle u_1, z \rangle_J = 0. \tag{3.4}$$

Since $Au_1 = 0$, it follows that

$$\langle u_1, A^j z \rangle_J = 0 \quad (j = 1, \ldots, q-1)$$

by equation (3.1). Thus, $\langle u_1, A^j z \rangle_J = 0$ for all $j = 0, \ldots, q-1$.

Now, we show that $u_2 \perp_J \operatorname{span}(\beta)$. Direct computation yields

$$\langle u_2, z \rangle_J = 0.$$

From $A^2 u_2 = 0$, it follows that

$$\langle u_2, A^j z \rangle_J = 0 \quad (j = 2, \ldots, q-1).$$

It remains to show that $\langle u_2, Az \rangle_J = 0$, but this is an immediate consequence of equations (3.1) and (3.4).

Applying this process to each u_i inductively, the result follows. □

COROLLARY 3.4. *There is a basis $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ such that if u is the terminal vector of a cycle α in $\mathcal{E}(A)$ with $|\alpha| = p$, then $v = A^{p-1-k} u$ is the unique vector in $\mathcal{E}(A)$ satisfying*

$$\langle A^k u, v \rangle_J \neq 0$$

for each $k = 0, 1, \ldots p-1$.

In the rest of the section, we investigate a relationship between $\langle\ ,\ \rangle_J$ and $\langle\ ,\ \rangle_K$ on bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ when there is a D_∞-HEE between two flip pairs (A, J) and (B, K). Throughout the section, we assume (A, J) and (B, K) are flip pairs with $|\mathcal{B}_1(\mathsf{X}_A)| = m$ and $|\mathcal{B}_1(\mathsf{X}_B)| = n$ and (D, E) is a D_∞-HEE from (A, J) to (B, K).

We note that $E = KD^{\mathsf{T}} J$ implies

$$\langle u, Dv \rangle_J = \langle Eu, v \rangle_K \quad (u \in \mathbb{R}^m, v \in \mathbb{R}^n).$$

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From this, we see that $\mathrm{Ker}(E)$ and $\mathrm{Ran}(D)$ are mutually orthogonal with respect to J and that $\mathrm{Ker}(D)$ and $\mathrm{Ran}(E)$ are mutually orthogonal with respect to K, that is,

$$\mathrm{Ker}(E) \perp_J \mathrm{Ran}(D) \quad \text{and} \quad \mathrm{Ker}(D) \perp_K \mathrm{Ran}(E). \tag{3.5}$$

LEMMA 3.5. *There exist bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ having the following properties.*

(1) *Suppose that α is a cycle in $\mathcal{E}(A)$ with $|\alpha| = p$ and $u = \mathrm{Ter}(\alpha)$. Then we have*

$$u \in \mathrm{Ran}(D) \Leftrightarrow A^{p-1}u \notin \mathrm{Ker}(E). \tag{3.6}$$

(2) *Suppose that β is a cycle in $\mathcal{E}(B)$ with $|\beta| = p$ and $v = \mathrm{Ter}(\beta)$. Then we have*

$$v \in \mathrm{Ran}(E) \Leftrightarrow B^{p-1}v \notin \mathrm{Ker}(D).$$

Proof. We only prove equation (3.6). Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ has properties (1), (2) and (3) from Lemma 3.3. Since $\langle A^{p-1}u, u\rangle_J \neq 0$, it follows that

$$u \in \mathrm{Ran}(D) \Rightarrow A^{p-1}u \notin \mathrm{Ker}(E)$$

from equation (3.5).

Suppose that $u \notin \mathrm{Ran}(D)$. To draw a contradiction, we assume that $A^{p-1}u \notin \mathrm{Ker}(E)$. By non-degeneracy of $\langle\ ,\ \rangle_K$, there is a $v \in \mathcal{K}(B)$ such that $\langle EA^{p-1}u, v\rangle_K \neq 0$, or equivalently, $\langle A^{p-1}u, Dv\rangle_J \neq 0$. This is a contradiction because $\langle A^{p-1}u, u\rangle_J \neq 0$ and $\langle A^{p-1}u, w\rangle_J = 0$ for all $w \in \mathcal{E}(A) \setminus \{u\}$. □

Now we are ready to prove Proposition B. We first indicate some notation. When $p \in \mathcal{I}nd(\mathcal{K}(A))$, let $\mathcal{E}_p(A; \partial^-_{D,E})$ denote the union of cycles α in $\mathcal{E}_p(A)$ such that $\mathrm{Ter}(\alpha) \notin \mathrm{Ran}(D)$ and let $\mathcal{E}_p(A; \partial^+_{D,E})$ denote the union of cycles α in $\mathcal{E}_p(A)$ such that $\mathrm{Ter}(\alpha) \in \mathrm{Ran}(D)$. With this notation, Proposition B can be rewritten as follows.

PROPOSITION B. *If $(D, E) : (A, J) \approx (B, K)$, then there exist bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ having the following properties.*

(1) *Suppose that $p \in \mathcal{I}nd(\mathcal{K}(A))$ and α is a cycle in $\mathcal{E}_p(A; \partial^+_{D,E})$ with $Ter(\alpha) = u$. There is a cycle β in $\mathcal{E}_{p+1}(B; \partial^-_{E,D})$ such that if $Ter(\beta) = v$, then $Dv = u$. In this case, we have*

$$\langle A^{p-1}u, u\rangle_J = \langle B^p v, v\rangle_K. \tag{3.7}$$

(2) *Suppose that $p \in \mathcal{I}nd(\mathcal{K}(A))$, $p > 1$ and α is a cycle in $\mathcal{E}_p(A; \partial^-_{D,E})$ with $Ter(\alpha) = u$. There is a cycle β in $\mathcal{E}_{p-1}(B; \partial^+_{E,D})$ such that if $Ter(\beta) = v$, then $v = Eu$. In this case, we have*

$$\langle A^{p-1}u, u\rangle_J = \langle B^{p-2}v, v\rangle_K. \tag{3.8}$$

Proof. If we define zero-one matrices M and F by

$$M = \begin{bmatrix} 0 & D \\ E & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} J & 0 \\ 0 & K \end{bmatrix},$$

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then (M, F) is a flip pair. Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ have properties (1), (2) and (3) from Lemma 3.3. If we set

$$\mathcal{E}(A) \oplus 0^n = \left\{ \begin{bmatrix} u \\ 0 \end{bmatrix} : u \in \mathcal{E}(A) \text{ and } 0 \in \mathbb{R}^n \right\}$$

and

$$0^m \oplus \mathcal{E}(B) = \left\{ \begin{bmatrix} 0 \\ v \end{bmatrix} : v \in \mathcal{E}(B) \text{ and } 0 \in \mathbb{R}^m \right\},$$

then the elements in $\mathcal{E}(A) \oplus 0^n$ or $0^m \oplus \mathcal{E}(B)$ belong to $\mathcal{K}(M)$. Conversely, every vector in $\mathcal{K}(M)$ can be expressed as a linear combination of vectors in $\mathcal{E}(A) \oplus 0^n$ and $0^m \oplus \mathcal{E}(B)$. Thus, the set $\mathcal{E}(M) = \{\mathcal{E}(A) \oplus 0^n\} \cup \{0^m \oplus \mathcal{E}(B)\}$ becomes a basis for $\mathcal{K}(M)$.

If α is a cycle in $\mathcal{E}(M)$, then $|\alpha|$ is an odd number by Lemma 3.5. If $|\alpha| = 2p - 1$ for some positive integer p, then α is one of the following forms:

$$\left\{ \begin{bmatrix} A^{p-1}u \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B^{p-2}Eu \end{bmatrix}, \begin{bmatrix} A^{p-2}u \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} Au \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ Eu \end{bmatrix}, \begin{bmatrix} u \\ 0 \end{bmatrix} \right\}$$

or

$$\left\{ \begin{bmatrix} 0 \\ B^{p-1}v \end{bmatrix}, \begin{bmatrix} A^{p-2}Dv \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B^{p-2}v \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ Bv \end{bmatrix}, \begin{bmatrix} Dv \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v \end{bmatrix} \right\}.$$

The formulae (3.7) and (3.8) follow from equation (3.3). □

Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ has property (1) from Lemma 3.3. If α is a cycle in $\mathcal{E}(A)$, we define the sign of α by

$$\operatorname{sgn}(\alpha) = \begin{cases} +1 & \text{if } \langle \operatorname{Ini}(\alpha), \operatorname{Ter}(\alpha) \rangle_J > 0, \\ -1 & \text{if } \langle \operatorname{Ini}(\alpha), \operatorname{Ter}(\alpha) \rangle_J < 0. \end{cases}$$

We define the sign of $\mathcal{E}_p(A)$ for each $p \in \mathcal{I}nd(\mathcal{K}(A))$ by

$$\operatorname{sgn}(\mathcal{E}_p(A)) = \prod_{\{\alpha : \alpha \text{ is a cycle in } \mathcal{E}_p(A)\}} \operatorname{sgn}(\alpha).$$

When $(D, E) : (A, J) \approx (B, K)$, we define the signs of $\mathcal{E}_p(A; \partial^+_{D,E})$ and $\mathcal{E}_p(A; \partial^-_{D,E})$ for each $p \in \mathcal{I}nd(\mathcal{K}(A))$ in similar ways.

Proposition B says that if $(D, E) : (A, J) \approx (B, K)$, there exist bases $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ such that

$$\operatorname{sgn}(\mathcal{E}_p(A; \partial^+_{D,E})) = \operatorname{sgn}(\mathcal{E}_{p+1}(B; \partial^-_{E,D})) \quad (p \in \mathcal{I}nd(\mathcal{K}(A))),$$

and

$$\operatorname{sgn}(\mathcal{E}_p(A; \partial^-_{D,E})) = \operatorname{sgn}(\mathcal{E}_{p-1}(B; \partial^+_{E,D})) \quad (p \in \mathcal{I}nd(\mathcal{K}(A)); \ p > 1).$$

In Proposition 3.6 below, we will see that the sign of $\mathcal{E}_1(A; \partial^-_{D,E})$ is always $+1$ if $\mathcal{E}_1(A; \partial^-_{D,E})$ is non-empty. We first prove Proposition C.

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Proof of Proposition C. Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ has properties (1), (2) and (3) from Lemma 3.3 and that $p \in \mathcal{I}nd(\mathcal{K}(A))$. We denote the terminal vectors of the cycles in $\mathcal{E}_p(A)$ by $u_{(1)}, \ldots, u_{(q)}$. Suppose that P is the $m \times q$ matrix whose ith column is $u_{(i)}$ for each $i = 1, \ldots, q$. If we set $M = (A^{p-1}P)^\top JP$, then the entry of M is given by

$$M(i, j) = \begin{cases} \langle A^{p-1}u_{(i)}, u_{(i)}\rangle_J & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and the sign of $\mathcal{E}_p(A)$ is determined by the product of the diagonal entries of M, that is,

$$\operatorname{sgn}(\mathcal{E}_p(A)) = \begin{cases} +1 & \text{if } \prod_{i=1}^q M(i, i) > 0, \\ -1 & \text{if } \prod_{i=1}^q M(i, i) < 0. \end{cases}$$

Suppose that $\mathcal{E}'(A) \in \mathcal{B}as(\mathcal{K}(A))$ is another basis having property (1) from Lemma 3.3. Then obviously $\mathcal{E}'_p(A)$ is the union of q disjoint cycles. If w is the terminal vector of a cycle in $\mathcal{E}'_p(A)$, then w can be expressed as a linear combination of vectors in $\mathcal{E}(A) \cap \operatorname{Ker}(A^p)$, that is,

$$w = \sum_{\substack{c_u \in \mathbb{R} \\ u \in \mathcal{E}(A) \cap \operatorname{Ker}(A^p)}} c_u u.$$

If $u \in \mathcal{E}_k(A)$ for $k < p$, then $A^{p-1}u = 0$. If $u \in \mathcal{E}_k(A)$ for $k > p$ or $u \in \mathcal{E}_p(A)$ and u is not a terminal vector, then $\langle A^{p-1}u, u\rangle_J = 0$ by property (2) from Lemma 3.3. This means that the sign of $\mathcal{E}'_p(A)$ is not affected by vectors $u \in \mathcal{E}_k(A)$ for $k \neq p$ or $u \in \mathcal{E}_p(A) \setminus \operatorname{Ter}(\mathcal{E}_p(A))$. In other words, if we write

$$w = \sum_{i=1}^q c_i u_{(i)} + \sum_{u \in \mathcal{E}(A) \cap \operatorname{Ker}(A^p) \setminus \operatorname{Ter}(\mathcal{E}_p(A))} c_u u \quad (c_i, c_u \in \mathbb{R}),$$

then we have

$$\langle A^{p-1}w, w\rangle_J = \langle A^{p-1} \sum_{i=1}^q c_i u_{(i)}, \sum_{i=1}^q c_i u_{(i)}\rangle_J.$$

To compute the sign of $\mathcal{E}'_p(A)$, we may assume that

$$w = \sum_{i=1}^q c_i u_{(i)} \quad (c_1, \ldots, c_q \in \mathbb{R}).$$

We denote the terminal vectors of the cycles in $\mathcal{E}'(A)$ by $w_{(1)}, \ldots, w_{(q)}$ and let Q be the $m \times q$ matrix whose ith column is $w_{(i)}$ for each $i = 1, \ldots, q$. If we set $N = (A^{p-1}Q)^\top JQ$, then $\prod_{i=1}^q N(i, i) \neq 0$ since $\mathcal{E}'(A)$ has property (1) from Lemma 3.3. So we have

$$\operatorname{sgn}(\mathcal{E}'_p(A)) = \begin{cases} +1 & \text{if } \prod_{i=1}^q N(i, i) > 0, \\ -1 & \text{if } \prod_{i=1}^q N(i, i) < 0. \end{cases}$$

It is obvious that there is a non-singular matrix R such that $PR = Q$. Since $N = R^{\mathsf{T}} M R$ and M is a diagonal matrix, it follows that

$$\prod_{i=1}^{q} M(i,i) > 0 \Leftrightarrow \prod_{i=1}^{q} N(i,i) > 0$$

and

$$\prod_{i=1}^{q} M(i,i) < 0 \Leftrightarrow \prod_{i=1}^{q} N(i,i) < 0. \qquad \square$$

PROPOSITION 3.6. *Suppose that* $(D, E) : (A, J) \approx (B, K)$ *and that* $\mathcal{I}nd(\mathcal{K}(A))$ *contains* 1. *There is a basis* $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ *such that if* α *is a cycle in* $\mathcal{E}_1(A; \partial^-_{D,E})$*, then* $\mathrm{sgn}(\alpha) = +1$. *Hence, we have*

$$\mathrm{sgn}(\mathcal{E}_1(A; \partial^-_{D,E})) = +1$$

if $\mathcal{E}_1(A; \partial^-_{D,E})$ *is non-empty.*

Proof. Suppose that $\mathcal{U}$ is a basis for the subspace $\mathrm{Ker}(A)$ of $\mathcal{K}(A)$. We may assume that for each $u \in \mathcal{U}$,

$$a_1, a_2 \in \mathcal{B}_1(\mathsf{X}_A),\ u(a_1) \neq 0 \text{ and } \mathcal{P}_E(a_1) \cap \mathcal{P}_E(a_2) = \varnothing \Rightarrow u(a_2) = 0 \tag{3.9}$$

for the following reason. If $u(a_2) \neq 0$, then we define u_1 and u_2 by

$$u_1(a) = \begin{cases} u(a) & \text{if } \mathcal{P}_E(a_1) \cap \mathcal{P}_E(a) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$u_2(a) = \begin{cases} u(a) & \text{if } \mathcal{P}_E(a_2) \cap \mathcal{P}_E(a) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $\{u_1, u_2\}$ is linearly independent. We set $u_3 = u - u_1 - u_2$. If $u_3 \neq 0$, then obviously $\{u_1, u_2, u_3\}$ is also linearly independent. We set

$$\mathcal{U}' = \mathcal{U} \cup \{u_1, u_2, u_3\} \setminus \{u\}.$$

If necessary, we apply the same process to u_3 and to each $u \in \mathcal{U}$ so that every element in $\mathcal{U}'$ satisfies equation (3.9) and then we remove some elements in $\mathcal{U}'$ so that it becomes a basis for $\mathrm{Ker}(A)$.

We first show the following:

$$u \in \mathcal{U} \Rightarrow u(\tau_J(a))u(a) \geq 0 \quad \text{for all } a \in \mathcal{B}_1(\mathsf{X}_A).$$

Suppose that $u \in \mathcal{U}$, $a_0 \in \mathcal{B}_1(\mathsf{X}_A)$ and that $u(a_0) \neq 0$. If $a_0 = \tau_J(a_0)$, then $u(\tau_J(a_0))$ $u(a_0) > 0$ and we are done. When $a_0 \neq \tau_J(a_0)$ and $u(\tau_J(a_0)) = 0$, there is nothing to do. So we assume $a_0 \neq \tau_J(a_0)$ and $u(\tau_J(a_0)) \neq 0$. If there were $b \in \mathcal{P}_E(a_0) \cap \mathcal{P}_E(\tau_J(a_0))$, then we would have

$$1 \geq B(b, \tau_K(b)) \geq E(b, a_0)D(a_0, \tau_K(b)) + E(b, \tau_J(a_0))D(\tau_J(a_0), \tau_K(b)) = 2$$

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from $E = KD^{\mathsf{T}}J$. Thus, we have $\mathcal{P}_E(a_0) \cap \mathcal{P}_E(\tau_J(a_0)) = \varnothing$ and this implies $u(\tau_J(a_0)) = 0$ by assumption (3.9).

Now, we denote the intersection of $\mathcal{U}$ and $\mathcal{E}_1(A; \partial^-_{D,E})$ by $\mathcal{V}$ and assume that the elements of $\mathcal{V}$ are $u_1, \ldots, u_k$, that is,

$$\mathcal{V} = \mathcal{U} \cap \mathcal{E}_1(A; \partial^-_{D,E}) = \{u_1, \ldots, u_k\}.$$

By Lemma 3.2 and equation (3.5), for each $u \in \mathcal{V}$, there is a $v \in \mathcal{V}$ such that $\langle u, v\rangle_J \neq 0$. If $\langle u_1, u_1\rangle_J = 0$, we choose $u_i \in \mathcal{V}$ such that $\langle u_1, u_i\rangle_J \neq 0$. There are real numbers k_1, k_2 such that $\{u_1 + k_1u_i, u_1 + k_2u_i\}$ is linearly independent and that both $\langle u_1 + k_1u_i, u_1 + k_1u_i\rangle_J$ and $\langle u_1 + k_2u_i, u_1 + k_2u_i\rangle_J$ are positive. We replace u_1 and u_i with $u_1 + k_1u_i$ and $u_1 + k_2u_i$. Continuing this process, we can construct a new basis for $\mathcal{E}_1(A; \partial^-_{D,E})$ such that if α is a cycle in $\mathcal{E}_1(A; \partial^-_{D,E})$, then $\operatorname{sgn}(\alpha) = +1$. □

Suppose that $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ has property (1) from Lemma 3.3. We arrange the elements of $\mathcal{I}nd(\mathcal{K}(A)) = \{p_1, p_2, \ldots, p_A\}$ to satisfy

$$p_1 < p_2 < \cdots < p_A$$

and write

$$\varepsilon_p = \operatorname{sgn}(\mathcal{E}_p(A)).$$

If $|\mathcal{I}nd(\mathcal{K}(A))|$=k, then the k-tuple $(\varepsilon_{p_1}, \varepsilon_{p_2}, \ldots, \varepsilon_{p_A})$ is called the *flip signature of* (A, J) and ε_{p_A} is called the *leading signature of* (A, J). The flip signature of (A, J) is denoted by

$$\text{F.Sig}(A, J) = (\varepsilon_{p_1}, \varepsilon_{p_2}, \ldots, \varepsilon_{p_A}).$$

When the eventual kernel $\mathcal{K}(A)$ of A is trivial, we write

$$\mathcal{I}nd(\mathcal{K}(A)) = \{0\}$$

and define the flip signature of (A, J) by

$$\text{F.Sig}(A, J) = (+1).$$

We have seen that both the flip signature and the leading signature are independent of the choice of basis $\mathcal{E}_A \in \mathcal{B}as(\mathcal{K}(A))$ as long as $\mathcal{E}_A$ has property (1) from Lemma 3.3.

In the next section, we prove Proposition A and in §5, we prove Theorem D.

4. *Proof of Proposition A*

We start with the notion of D_∞-higher block codes. (See [5, 8] for more details about higher block codes.) We need some notation. Suppose that (X, σ_X) is a shift space over a finite set $\mathcal{A}$ and that φ_τ is a one-block flip for (X, σ_X) defined by

$$\varphi_\tau(x)_i = \tau(x_{-i}) \quad (x \in X; i \in \mathbb{Z}).$$

For each positive integer n, we define the n-initial map $i_n : \bigcup_{k=n}^{\infty} \mathcal{B}_k(X) \to \mathcal{B}_n(X)$, the n-terminal map $t_n : \bigcup_{k=n}^{\infty} \mathcal{B}_k(X) \to \mathcal{B}_n(X)$ and the mirror map $\mathcal{M}_n : \mathcal{A}^n \to \mathcal{A}^n$ by

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$$i_n(a_1a_2\ldots a_m) = a_1a_2\ldots a_n \quad (a_1\ldots a_m \in \mathcal{B}_m(X);\ m \geq n),$$

$$t_n(a_1a_2\ldots a_m) = a_{m-n+1}a_{m-n+2}\ldots a_m \quad (a_1\ldots a_m \in \mathcal{B}_m(X);\ m \geq n)$$

and

$$\mathcal{M}_n(a_1a_2\ldots a_n) = a_n\ldots a_1 \quad (a_1\ldots a_n \in \mathcal{A}^n).$$

For each positive integer n, we denote the map

$$a_1a_2\ldots a_n \mapsto \tau(a_1)\tau(a_2)\ldots\tau(a_n) \quad (a_1\ldots a_n \in \mathcal{A}^n)$$

by $\tau_n : \mathcal{A}^n \to \mathcal{A}^n$. It is obvious that the restriction of the map $\mathcal{M}_n \circ \tau_n$ to $\mathcal{B}_n(X)$ is a permutation of order 2.

For each positive integer n, we define the nth higher block code $h_n : X \to \mathcal{B}_n(X)^{\mathbb{Z}}$ by

$$h_n(x)_i = x_{[i,i+n-1]} \quad (x \in X; i \in \mathbb{Z}).$$

We denote the image of (X, σ_X) under h_n by (X_n, σ_n) and call (X_n, σ_n) the nth higher block shift of (X, σ_X). If we write $\upsilon = \mathcal{M}_n \circ \tau_n$, then the map $\varphi_\upsilon : X_n \to X_n$ defined by

$$\varphi_\upsilon(x)_i = \upsilon(x_{-i}) \quad (x \in X_n; i \in \mathbb{Z})$$

becomes a natural one-block flip for (X_n, σ_n). It is obvious that the nth higher block code h_n is a D_∞-conjugacy from $(X, \sigma_X, \varphi_\tau)$ to $(X_n, \sigma_n, (\sigma_n)^{n-1} \circ \varphi_\upsilon)$. We call the D_∞-system $(X_n, \sigma_n, \varphi_\upsilon)$ the *nth higher block D_∞-system* of $(X, \sigma_X, \varphi_\tau)$.

For notational simplicity, we drop the subscript n and write $\tau = \tau_n$ and $\mathcal{M} = \mathcal{M}_n$ if the domains of τ_n and $\mathcal{M}_n$ are clear in the context.

Suppose that (A, J) is a flip pair. Then the flip pair (A_n, J_n) for the nth higher block D_∞-system $(\mathsf{X}_n, \sigma_n, \varphi_n)$ of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ consists of $\mathcal{B}_n(\mathsf{X}_A) \times \mathcal{B}_n(\mathsf{X}_A)$ zero-one matrices A_n and J_n defined by

$$A_n(u, v) = \begin{cases} 1 & \text{if } t_{n-1}(u) = i_{n-1}(v), \\ 0 & \text{otherwise,} \end{cases} \quad (u, v \in \mathcal{B}_n(\mathsf{X}_A))$$

and

$$J_n(u, v) = \begin{cases} 1 & \text{if } v = (\mathcal{M} \circ \tau_J)(u), \\ 0 & \text{otherwise,} \end{cases} \quad (u, v \in \mathcal{B}_n(\mathsf{X}_A)).$$

In the following lemma, we prove that there is a D_∞-SSE from (A, J) to (A_n, J_n).

LEMMA 4.1. *If n is a positive integer greater than* 1, *then we have*

$$(A_1, J_1) \approx (A_n, J_n) \text{ (lag } n-1).$$

Proof. For each $k = 1, 2, \ldots, n-1$, we define a zero-one $\mathcal{B}_k(\mathsf{X}_A) \times \mathcal{B}_{k+1}(\mathsf{X}_A)$ matrix D_k and a zero-one $\mathcal{B}_{k+1}(\mathsf{X}_A) \times \mathcal{B}_k(\mathsf{X}_A)$ matrix E_k by

$$D_k(u, v) = \begin{cases} 1 & \text{if } u = i_k(v), \\ 0 & \text{otherwise,} \end{cases} \quad (u \in \mathcal{B}_k(\mathsf{X}_A), v \in \mathcal{B}_{k+1}(\mathsf{X}_A))$$

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and

$$E_k(v,u) = \begin{cases} 1 & \text{if } u = t_k(v), \\ 0 & \text{otherwise,} \end{cases} \quad (u \in \mathcal{B}_k(\mathsf{X}_A),\, v \in \mathcal{B}_{k+1}(\mathsf{X}_A)).$$

It is straightforward to see that $(D_k, E_k) : (A_k, J_k) \approxeq (A_{k+1}, J_{k+1})$ for each k. □

In the proof of Lemma 4.1, $(\mathsf{X}_{A_{k+1}}, \sigma_{A_{k+1}}, \varphi_{A_{k+1},J_{k+1}})$ is equal to the second higher block D_∞-system of $(\mathsf{X}_{A_k}, \sigma_{A_k}, \varphi_{A_k,J_k})$ by recoding of symbols and the half elementary conjugacy

$$\gamma_{D_k,E_k} : (\mathsf{X}_{A_k}, \sigma_{A_k}, \varphi_{A_k,J_k}) \to (\mathsf{X}_{A_{k+1}}, \sigma_{A_{k+1}}, \sigma_{A_{k+1}} \circ \varphi_{A_{k+1},J_{k+1}})$$

induced by (D_k, E_k) can be regarded as the second D_∞-higher block code for each $k = 1, 2, \ldots, n-1$. A D_∞-HEE $(D, E) : (A, J) \approxeq (B, K)$ is said to be *a complete D_∞-half elementary equivalence from (A, J) to (B, K)* if $\gamma_{D,E}$ is the second D_∞-higher block code.

In the rest of the section, we prove Proposition A.

Proof of Proposition A. We only prove part (1). One can prove part (2) in a similar way.

We denote the flip pairs for the nth higher block D_∞-system of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ by (A_n, J_n) for each positive integer n. If $\psi : (\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \to (\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ is a D_∞-conjugacy, then there are non-negative integers s and t and a block map $\Psi : \mathcal{B}_{s+t+1}(\mathsf{X}_A) \to \mathcal{B}_1(\mathsf{X}_B)$ such that

$$\psi(x)_i = \Psi(x_{[i-s,i+t]}) \quad (x \in \mathsf{X}_A;\, i \in \mathbb{Z}).$$

We may assume that $s+t$ is even by extending the window size if necessary. By Lemma 4.1, there is a D_∞-SSE of lag $(s+t)$ from (A, J) to (A_{s+t+1}, J_{s+t+1}). From equation (2.4), it follows that the $(s+t+1)$th D_∞-higher block code h_{s+t+1} is a D_∞-conjugacy. It is obvious that there is a D_∞-conjugacy ψ' induced by ψ satisfying $\psi = \psi' \circ h_{s+t+1}$ and

$$x, y \in h_{s+t+1}(X) \quad \text{and} \quad x_0 = y_0 \Rightarrow \psi'(x)_0 = \psi'(y)_0.$$

So we may assume $s = t = 0$ and show that there is a D_∞-SSE of lag $2l$ from (A, J) to (B, K) for some positive integer l.

If ψ^{-1} is the inverse of ψ, there is a non-negative integer m such that

$$y, y' \in \mathsf{X}_B \quad \text{and} \quad y_{[-m,m]} = y'_{[-m,m]} \Rightarrow \psi^{-1}(y)_0 = \psi^{-1}(y')_0 \tag{4.1}$$

since ψ^{-1} is uniformly continuous. For each $k = 1, \ldots, 2m+1$, we define a set $\mathcal{A}_k$ by

$$\mathcal{A}_k = \left\{ \begin{bmatrix} v \\ w \\ u \end{bmatrix} : u, v \in \mathcal{B}_i(\mathsf{X}_B),\, w \in \mathcal{B}_j(\mathsf{X}_A) \text{ and } u\Psi(w)v \in \mathcal{B}_k(\mathsf{X}_B) \right\},$$

where $i = \lfloor (k-1)/2 \rfloor$ and $j = k - 2\lfloor (k-1)/2 \rfloor$. We define $\mathcal{A}_k \times \mathcal{A}_k$ matrices M_k and F_k to be

$$M_k\left(\begin{bmatrix} v \\ w \\ u \end{bmatrix}, \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix}\right) = 1 \Leftrightarrow \begin{bmatrix} v \\ \Psi(w) \\ u \end{bmatrix}\begin{bmatrix} v' \\ \Psi(w') \\ u' \end{bmatrix} \in \mathcal{B}_2(\mathsf{X}_{B_k})$$
$$\text{and } ww' \in \mathcal{B}_2(\mathsf{X}_{A_j})$$

and

$$F_k\left(\begin{bmatrix} v \\ w \\ u \end{bmatrix}, \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix}\right) = 1 \Leftrightarrow u' = (\mathcal{M} \circ \tau_K)(v),\ \ w' = (\mathcal{M} \circ \tau_J)(w)$$
$$\text{and } v' = (\mathcal{M} \circ \tau_K)(u)$$

for all

$$\begin{bmatrix} v \\ w \\ u \end{bmatrix}, \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix} \in \mathcal{A}_k.$$

A direct computation shows that (M_k, F_k) is a flip pair for each k. Next, we define a zero-one $\mathcal{A}_k \times \mathcal{A}_{k+1}$ matrix R_k and a zero-one $\mathcal{A}_{k+1} \times \mathcal{A}_k$ matrix S_k to be

$$R_k\left(\begin{bmatrix} v \\ w \\ u \end{bmatrix}, \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix}\right) = 1 \Leftrightarrow u\Psi(w)v = i_k\big(u'\Psi(w')v'\big)$$
$$\text{and } t_1(w) = i_1(w')$$

and

$$S_k\left(\begin{bmatrix} v' \\ w' \\ u' \end{bmatrix}, \begin{bmatrix} v \\ w \\ u \end{bmatrix}\right) = 1 \Leftrightarrow t_k\left(u'\Psi(w')v'\right) = u\Psi(w)v$$
$$\text{and } t_1(w') = i_1(w),$$

for all

$$\begin{bmatrix} v \\ w \\ u \end{bmatrix} \in \mathcal{A}_k \quad \text{and} \quad \begin{bmatrix} v' \\ w' \\ u' \end{bmatrix} \in \mathcal{A}_{k+1}.$$

A direct computation shows that

$$(R_k, S_k) : (M_k, F_k) \approxeq (M_{k+1}, F_{k+1}).$$

Because $M_1 = A$ and $F_1 = J$, we obtain

$$(A, J) \approx (M_{2m+1}, F_{2m+1}) \text{ (lag } 2m). \tag{4.2}$$

Finally, equation (4.1) implies that the D_∞-TMC determined by the flip pair (M_{2m+1}, F_{2m+1}) is equal to the $(2m+1)$th higher block D_∞-system of $(\mathsf{X}_B, \sigma_B, \varphi_{K,B})$ by recoding

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of symbols. From Lemma 4.1, we have

$$(B, K) \approx (M_{2m+1}, F_{2m+1}) \text{ (lag } 2m). \tag{4.3}$$

From equations (4.2) and (4.3), it follows that

$$(A, J) \approx (B, K) \text{ (lag } 4m). \qquad \square$$

5. *Proof of Theorem D*

We start with the case where (B, K) in Theorem D is the flip pair for the nth higher block D_∞-system of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$.

LEMMA 5.1. *Suppose that (B, K) is the flip pair for the nth higher block D_∞-system of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$.*

(1) *If $p \in \mathcal{I}nd(\mathcal{K}(A))$, then there is $q \in \mathcal{I}nd(\mathcal{K}((B))$ such that $q = p + n - 1$ and that*

$$\mathrm{sgn}(\mathcal{E}_p(A)) = \mathrm{sgn}(\mathcal{E}_q(B)).$$

(2) *If $q \in \mathcal{I}nd(\mathcal{K}((B))$ and $q \geq n$, then there is $p \in \mathcal{I}nd(\mathcal{K}(A))$ such that $q = p + n - 1$ and that*

$$\mathrm{sgn}(\mathcal{E}_p(A)) = \mathrm{sgn}(\mathcal{E}_q(B)).$$

(3) *If $q \in \mathcal{I}nd(\mathcal{K}((B))$ and $q < n$, then we have*

$$\mathrm{sgn}(\mathcal{E}_q(B)) = +1.$$

Proof. We only prove the case $n = 2$. We assume $\mathcal{E}(A) \in \mathcal{B}as(\mathcal{K}(A))$ and $\mathcal{E}(B) \in \mathcal{B}as(\mathcal{K}(B))$ are bases having properties from Proposition B. Suppose that α is a cycle in $\mathcal{E}_p(A)$ for some $p \in \mathcal{I}nd(\mathcal{K}(A))$ and that u is the initial vector of α. For any $a_1a_2 \in \mathcal{B}_2(\mathsf{X}_A)$, we have

$$Eu\left(\begin{bmatrix} a_2 \\ a_1 \end{bmatrix}\right) = u(a_2)$$

and this implies that Eu is not identically zero. By Lemma 3.5, α is a cycle in $\mathcal{E}_p(A; \partial^+_{D,E})$. Under the assumption that $\mathcal{E}(A)$ and $\mathcal{E}(B)$ have properties from Proposition B, we can find a cycle β in $\mathcal{E}(B)$ such that the initial vector of β is Eu. Thus, we obtain

$$\mathcal{E}_p(A; \partial^-_{D,E}) = \varnothing \quad \text{and} \quad \mathcal{E}_{p+1}(B; \partial^+_{E,D}) = \varnothing, \tag{5.1}$$

$$p \in \mathcal{I}nd(\mathcal{K}(A)) \Leftrightarrow p + 1 \in \mathcal{I}nd(\mathcal{K}(B)) \quad (p \geq 1)$$

and

$$\mathrm{sgn}(\mathcal{E}_p(A)) = \mathrm{sgn}(\mathcal{E}_{p+1}(B)) \quad (p \in \mathcal{I}nd(\mathcal{K}(A))).$$

If $\mathcal{E}_1(B) \neq \varnothing$, then $\mathcal{E}_1(B) = \mathcal{E}_1(B; \partial^-_{E,D})$ by equation (5.1) and we have

$$\mathrm{sgn}(\mathcal{E}_1(B)) = +1$$

by Propositions 3.6 and C. $\square$

Remark. If two D_∞-TMCs are finite, then we can directly determine whether or not they are D_∞-conjugate. In this paper, we do not consider D_∞-TMCs that have finite cardinalities. Hence, when (B, K) is the flip pair for the nth higher block D_∞-system of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ for some positive integer $n > 1$, B must have zero as its eigenvalue.

Proof of Theorem D. Suppose that (A, J) and (B, K) are flip pairs and that $\psi : (\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \to (\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ is a D_∞-conjugacy. As we can see in the proof of Proposition A, there is a D_∞-SSE from (A, J) to (B, K) consisting of the even number of complete D_∞-half elementary equivalences and $(R_k, S_k) : (M_k, F_k) \approx (M_{k+1}, F_{k+1})(k = 1, \ldots, 2m)$. In Lemma 5.1, we have already seen that Theorem D is true in the case of complete D_∞-half elementary equivalences. So it remains to compare the flip signatures of (M_k, F_k) and (M_{k+1}, F_{k+1}) for each $k = 1, \ldots, 2m$. Throughout the proof, we assume $\mathcal{A}_k$ and $(R_k, S_k) : (M_k, F_k) \approx (M_{k+1}, F_{k+1})$ are as in the proof of Proposition A.

We only discuss the following two cases:

(1) $(R_2, S_2) : (M_2, F_2) \approx (M_3, F_3)$;

(2) $(R_3, S_3) : (M_3, F_3) \approx (M_4, F_4)$.

When $k = 1$, (R_1, S_1) is a complete D_∞-half elementary conjugacy from (A, J) to (A_2, J_2). For each $k = 4, 5 \ldots, 2m$, one can apply the arguments used in cases (1) and (2) to $(R_k, S_k) : (M_k, F_k) \approx (M_{k+1}, F_{k+1})$. More precisely, when k is an even number, the argument used in case (1) can be applied and when k is an odd number, the argument used in case (2) can be applied.

(1) Suppose that (B_2, K_2) is the flip pair for the second higher block D_∞-system of $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$. We first compare the flip signatures of (B_2, K_2) and (M_3, F_3). We define a zero-one $\mathcal{B}_2(\mathsf{X}_B) \times \mathcal{A}_3$ matrix U_2 and a zero-one $\mathcal{A}_3 \times \mathcal{B}_2(\mathsf{X}_B)$ matrix V_2 by

$$U_2\left(\begin{bmatrix} b_2 \\ b_1 \end{bmatrix}, \begin{bmatrix} d_3 \\ a_2 \\ d_1 \end{bmatrix}\right) = \begin{cases} 1 & \text{if } b_1 = d_1 \text{ and } \Psi(a_2) = b_2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$V_2\left(\begin{bmatrix} d_3 \\ a_2 \\ d_1 \end{bmatrix}, \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}\right) = \begin{cases} 1 & \text{if } b_2 = d_3 \text{ and } \Psi(a_2) = b_1, \\ 0 & \text{otherwise,} \end{cases}$$

for all

$$\begin{bmatrix} b_2 \\ b_1 \end{bmatrix} \in \mathcal{B}_2(\mathsf{X}_B) \quad \text{and} \quad \begin{bmatrix} d_3 \\ a_2 \\ d_1 \end{bmatrix} \in \mathcal{A}_3.$$

A direct computation shows that

$$(U_2, V_2) : (B_2, K_2) \approx (M_3, F_3).$$

Remark of Lemma 5.1 says that $\mathcal{K}(B_2)$ is not trivial. So there is a basis $\mathcal{E}(B_2) \in \mathcal{B}as(\mathcal{K}(B_2))$ for the eventual kernel of B_2 having property (1) from Lemma 3.3. Suppose

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that $\gamma = \{w_1, \ldots, w_p\}$ is a cycle in $\mathcal{E}(B_2)$. Since

$$V_2 w_1 \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \right) = w_1 \left(\begin{bmatrix} b_3 \\ \Psi(a_2) \end{bmatrix} \right) \quad \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \in \mathcal{A}_3 \right),$$

it follows that $w_1 \notin \operatorname{Ker}(V_2)$. By Lemma 3.5, γ is a cycle in $\mathcal{E}_p(B_2; \partial^+_{U_2,V_2})$. Suppose that $\mathcal{E}(M_3) \in \mathcal{B}as(\mathcal{K}(M_3))$ is a basis for the eventual kernel of M_3 having property (1) from Lemma 3.3. Then it is obvious that for each $p \in \mathcal{I}nd(\mathcal{K}(B_2))$, we have

$$\mathcal{E}_p(B_2; \partial^-_{U_2,V_2}) = \varnothing \quad \text{and} \quad \mathcal{E}_{p+1}(M_3; \partial^+_{V_2,U_2}) = \varnothing. \tag{5.2}$$

Hence,

$$p \in \mathcal{I}nd(\mathcal{K}(B_2)) \Leftrightarrow p + 1 \in \mathcal{I}nd(\mathcal{K}(M_3)) \quad (p \geq 1)$$

and

$$\operatorname{sgn}(\mathcal{E}_p(B_2)) = \operatorname{sgn}(\mathcal{E}_{p+1}(M_3)) \quad (p \in \mathcal{I}nd(\mathcal{K}(B_2)))$$

by Proposition C. If $\mathcal{E}_1(M_3) \neq \varnothing$, then $\mathcal{E}_1(M_3) = \mathcal{E}_1(M_3; \partial^-_{V_2,U_2})$ by equation (5.2) and we have

$$\operatorname{sgn}(\mathcal{E}_1(M_3)) = +1 \tag{5.3}$$

by Propositions 3.6 and C.

Now, we compare the flip signatures of (M_2, F_2) and (M_3, F_3). Let $\beta = \{v_1, \ldots, v_{p+1}\}$ be a cycle in $\mathcal{E}(M_3)$ for some $p \geq 1$. If $b_1b_2b_3 \in \mathcal{B}_3(\mathsf{X}_B)$ and $a_2, a_2' \in \Psi^{-1}(b_2)$, then from $M_3 v_2 = v_1$, it follows that

$$v_1 \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \right) = \sum_{a_3 \in \Psi^{-1}(b_3)} \sum_{b_4 \in \mathcal{F}_B(b_3)} v_2 \left(\begin{bmatrix} b_4 \\ a_3 \\ b_2 \end{bmatrix} \right)$$

and this implies that

$$v_1 \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \right) = v_1 \left(\begin{bmatrix} b_3 \\ a_2' \\ b_1 \end{bmatrix} \right).$$

Since v_1 is a non-zero vector, there is a block $b_1b_2b_3 \in \mathcal{B}_3(\mathsf{X}_B)$ and a non-zero real number k such that

$$v_1 \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \right) = k \quad \text{for all } a_2 \in \Psi^{-1}(b_2).$$

Since $M_3 v_1 = 0$, it follows that

$$\sum_{a_2 \in \Psi^{-1}(b_2)} \sum_{b_3 \in \mathcal{F}_B(b_2)} v_1 \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \right) = k \sum_{b_3 \in \mathcal{F}_B(b_2)} v_1 \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \right) = 0.$$

From this, we see that

$$R_2 v_1 \left(\begin{bmatrix} a_2 \\ a_1 \end{bmatrix} \right) = \sum_{b_3 \in \mathcal{F}_B(b_2)} v_1 \left(\begin{bmatrix} b_3 \\ a_2 \\ b_1 \end{bmatrix} \right) = 0$$

for any $a_1 \in \Psi^{-1}(b_1)$ and $a_1 a_2 \in \mathcal{B}_2(\mathsf{X}_A)$. Hence, $v_1 \in \mathrm{Ker}(R_2)$ and β is a cycle in $\mathcal{E}_{p+1}(M_3; \partial^-_{S_2,R_2})$ by Lemma 3.5. From this, we see that

$$p+1 \in \mathcal{I}nd(\mathcal{K}(M_3)) \Leftrightarrow p \in \mathcal{I}nd(\mathcal{K}(M_2)) \quad (p \geq 2)$$

and

$$2 \in \mathcal{I}nd(\mathcal{K}(M_3)) \Leftrightarrow 1 \in \mathcal{I}nd(\mathcal{K}(M_2; \partial^+_{R_2,S_2})).$$

Suppose that $\mathcal{E}(M_2) \in \mathcal{B}as(\mathcal{K}(M_2))$ is a basis for the eventual kernel of M_2 having property (1) from Lemma 3.3. If $1 \in \mathcal{I}nd(\mathcal{K}(M_2))$ and $\mathcal{E}_1(M_2; \partial^-_{R_2,S_2})$ is non-empty, then we have

$$\mathrm{sgn}(\mathcal{E}_1(M_2; \partial^-_{R_2,S_2})) = +1$$

by Propositions 3.6, C and equation (3.5). Thus, we have

$$\mathrm{sgn}(\mathcal{E}_{p+1}(M_3)) = \mathrm{sgn}(\mathcal{E}_p(M_2)) \quad (p+1 \in \mathcal{I}nd(\mathcal{K}(M_3));\ p \geq 1).$$

If $1 \in \mathcal{I}nd(\mathcal{K}(M_3))$ and $\mathcal{E}_1(M_3; \partial^+_{S_2,R_2})$ is non-empty, then we have

$$\mathrm{sgn}(\mathcal{E}_1(M_3; \partial^+_{S_2,R_2})) = +1$$

and if $\mathcal{E}_1(M_3; \partial^-_{S_2,R_2})$ is non-empty, then we have

$$\mathrm{sgn}(\mathcal{E}_1(M_3; \partial^-_{S_2,R_2})) = +1$$

by equations (3.5), (5.3) and Propositions 3.6, C. As a consequence, the flip signatures of (M_2, F_2) and (M_3, F_3) have the same number of -1s and their leading signatures coincide.

(2) Suppose that α is a cycle in $\mathcal{K}(M_3)$ and that u is the initial vector of α. Since

$$S_3 u \left(\begin{bmatrix} b_4 \\ a_3 \\ a_2 \\ b_1 \end{bmatrix} \right) = u \begin{bmatrix} b_4 \\ a_3 \\ \Psi(a_2) \end{bmatrix} \quad \left(\begin{bmatrix} b_4 \\ a_3 \\ a_2 \\ b_1 \end{bmatrix} \in \mathcal{A}_4 \right),$$

it follows that $S_3 u$ is not identically zero. The argument used in the proof of Lemma 5.1 completes the proof. □

6. *D_∞-shift equivalence and the Lind zeta functions*

We first introduce the notion of D_∞-shift equivalence which is an analogue of shift equivalence. Let (A, J) and (B, K) be flip pairs and let l be a positive integer. A *D_∞-shift equivalence* (D_∞-SE) *of lag l from* (A, J) *to* (B, K) is a pair (D, E) of non-negative integral matrices satisfying

$$A^l = DE, \quad B^l = ED, \quad AD = DB \quad \text{and} \quad E = KD^\mathsf{T} J.$$

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We observe that $AD = DB$, $E = KD^{\top}J$ and the fact that (A, J) and (B, K) are flip pairs imply $EA = BE$. If there is a D_∞-SE of lag l from (A, J) to (B, K), then we say that (A, J) is D_∞-shift equivalent to (B, K) and write

$$(A, J) \sim (B, K) \text{ (lag } l).$$

Suppose that

$$(D_1, E_1), (D_2, E_2), \ldots, (D_l, E_l)$$

is a D_∞-SSE of lag l from (A, J) to (B, K). If we set

$$D = D_1 D_2 \ldots D_l \quad \text{and} \quad E = E_l \ldots E_2 E_1,$$

then (D, E) is a D_∞-SE of lag l from (A, J) to (B, K). Hence, we have

$$(A, J) \approx (B, K) \quad (\text{lag } l) \Rightarrow (A, J) \sim (B, K) \text{ (lag } l).$$

In the rest of the section, we review the Lind zeta function of a D_∞-TMC. In [4], an explicit formula for the Lind zeta function of a D_∞-system was established. In the case of a D_∞-TMC, the Lind zeta function can be expressed in terms of matrices from flip pairs. We briefly discuss the formula.

Suppose that G is a group and that α is a G-action on a set X. Let $\mathcal{F}$ denote the set of finite index subgroups of G. For each $H \in \mathcal{F}$, we set

$$p_H(\alpha) = |\{x \in X : \text{for all } h \in H \ \alpha(h, x) = x\}|.$$

The Lind zeta function ζ_α of the action α is defined by

$$\zeta_\alpha(t) = \exp\left(\sum_{H \in \mathcal{F}} \frac{p_H(\alpha)}{|G/H|} t^{|G/H|}\right). \tag{6.1}$$

It is clear that if $\alpha : \mathbb{Z} \times X \to X$ is given by $\alpha(n, x) = T^n(x)$, then the Lind zeta function ζ_α becomes the Artin–Mazur zeta function ζ_T of a topological dynamical system (X, T). The formula for the Artin–Mazur zeta function can be found in [1]. Lind defined the function (6.1) in [7] for the case $G = \mathbb{Z}^d$.

Every finite index subgroup of the infinite dihedral group $D_\infty = \langle a, b : ab = ba^{-1} \text{ and } b^2 = 1\rangle$ can be written in one and only one of the following forms:

$$\langle a^m \rangle \quad \text{or} \quad \langle a^m, a^k b\rangle \quad (m = 1, 2, \ldots; k = 1, \ldots, m-1)$$

and the index is given by

$$|G_2/\langle a^m \rangle| = 2m \quad \text{or} \quad |G_2/\langle a^m, a^k b\rangle| = m.$$

Suppose that (X, T, F) is a D_∞-system. If m is a positive integer, then the number of periodic points in X of period m will be denoted by $p_m(T)$:

$$p_m(T) = |\{x \in X : T^m(x) = x\}|.$$

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If m is a positive integer and n is an integer, then $p_{m,n}(T, F)$ will denote the number of points in X fixed by T^m and $T^n \circ F$:

$$p_{m,n}(T, F) = |\{x \in X : T^m(x) = T^n \circ F(x) = x\}|.$$

Thus, the Lind zeta function $\zeta_{T,F}$ of a D_∞-system (X, T, F) is given by

$$\zeta_{T,F}(t) = \exp\left(\sum_{m=1}^{\infty} \frac{p_m(T)}{2m} t^{2m} + \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \frac{p_{m,k}(T, F)}{m} t^m\right). \tag{6.2}$$

It is evident if two D_∞-systems (X, T, F) and (X', T', F') are D_∞-conjugate, then

$$p_m(T) = p_m(T') \quad \text{and} \quad p_{m,n}(T, F) = p_{m,n}(T', F')$$

for all positive integers m and integers n. As a consequence, the Lind zeta function is a D_∞-conjugacy invariant.

The formula (6.2) can be simplified as follows. Since $T \circ F = F \circ T^{-1}$ and $F^2 = \mathrm{Id}_X$, it follows that

$$p_{m,n}(T, F) = p_{m,n+m}(T, F) = p_{m,n+2}(T, F)$$

and this implies that

$$\begin{aligned} p_{m,n}(T, F) &= p_{m,0}(T, F) && \text{if } m \text{ is odd,} \\ p_{m,n}(T, F) &= p_{m,0}(T, F) && \text{if } m \text{ and } n \text{ are even,} \\ p_{m,n}(T, F) &= p_{m,1}(T, F) && \text{if } m \text{ is even and } n \text{ is odd.} \end{aligned} \tag{6.3}$$

Hence, we obtain

$$\sum_{k=0}^{m-1} \frac{p_{m,n}(T, F)}{m} = \begin{cases} p_{m,0}(T, F) & \text{if } m \text{ is odd,} \\ \dfrac{p_{m,0}(T, F) + p_{m,1}(T, F)}{2} & \text{if } m \text{ is even.} \end{cases}$$

Using this, equation (6.2) becomes

$$\zeta_\alpha(t) = \zeta_T(t^2)^{1/2} \exp\left(G_{T,F}(t)\right),$$

where ζ_T is the Artin–Mazur zeta function of (X, T) and $G_{T,F}$ is given by

$$G_{T,F}(t) = \sum_{m=1}^{\infty} \left(p_{2m-1,0}(T, F)\, t^{2m-1} + \frac{p_{2m,0}(T, F) + p_{2m,1}(T, F)}{2}\, t^{2m}\right).$$

If there is a D_∞-SSE of lag $2l$ between two flip pairs (A, J) and (B, K) for some positive integer l, then $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ have the same Lind zeta function by item (1) in Proposition A. The following proposition says that the Lind zeta function is actually an invariant for D_∞-SSE.

PROPOSITION 6.1. *If (X, T, F) is a D_∞-system, then*

$$p_{2m-1,0}(T, F) = p_{2m-1,0}(T, T \circ F),$$

$$p_{2m,0}(T, F) = p_{2m,1}(T, T \circ F),$$

$$p_{2m,1}(T, F) = p_{2m,0}(T, T \circ F)$$

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for all positive integers m. As a consequence, the Lind zeta functions of (X, T, F) *and* $(X, T, T \circ F)$ *are the same.*

Proof. The last equality is trivially true. To prove the first two equalities, we observe that

$$T^m(x) = F(x) = x \Leftrightarrow T^m(Tx) = T \circ (T \circ F)(Tx) = Tx$$

for all positive integers m. Thus, we have

$$p_{m,0}(T, F) = p_{m,1}(T, T \circ F) \quad (m = 1, 2, \ldots). \tag{6.4}$$

Replacing m with $2m$ yields the second equality. From equations (6.3) and (6.4), the first equality follows. □

When (A, J) is a flip pair, the numbers $p_{m,\delta}(\sigma_A, \varphi_{A,J})$ of fixed points can be expressed in terms of A and J for all positive integers m and $\delta \in \{0, 1\}$. To present it, we indicate notation. If M is a square matrix, then Δ_M will denote the column vector whose ith coordinates are identical with ith diagonal entries of M, that is,

$$\Delta_M(i) = M(i, i).$$

For instance, if I is the 2×2 identity matrix, then

$$\Delta_I = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The following proposition is proved in [4].

PROPOSITION 6.2. *If* (A, J) *is a flip pair, then*

$$p_{2m-1,0}(\sigma_A, \varphi_{A,J}) = {\Delta_J}^T (A^{m-1}) \Delta_{AJ},$$

$$p_{2m,0}(\sigma_A, \varphi_{A,J}) = {\Delta_J}^T (A^{m}) \Delta_{J},$$

$$p_{2m,1}(\sigma_A, \varphi_{A,J}) = {\Delta_{JA}}^T (A^{m-1}) \Delta_{AJ}$$

for all positive integers m.

7. *Examples*

Let A be Ashley's eight-by-eight and let B be the minimal zero-one transition matrix for the full two-shift, that is,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

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There is a unique one-block flip for (X_A, σ_A) and there are exactly two one-block flips for (X_B, σ_B). Those flips are determined by the permutation matrices

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In the following example, we calculate the Lind zeta functions of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$, $(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$.

Example 7.1. Direct computation shows that the number of fixed points of $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$, $(\mathsf{X}_B, \sigma_B, \varphi_{B,I})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ are as follows:

$$p_m(\sigma_A) = p_m(\sigma_B) = 2^m,$$

$$p_{2m-1,0}(\sigma_A, \varphi_{A,J}) = p_{2m,0}(\sigma_A, \varphi_{A,J}) = 0,$$

$$p_{2m,1}(\sigma_A, \varphi_{A,J}) = \begin{cases} 2^m & \text{if } m \neq 6, \\ 80 & \text{if } m = 6, \end{cases}$$

$$p_{2m-1,0}(\sigma_B, \varphi_{B,I}) = 2^m, \quad p_{2m,0}(\sigma_B, \varphi_{B,I}) = 2^{m+1}, \quad p_{2m,1}(\sigma_B, \varphi_{B,I}) = 2^m,$$

$$p_{2m-1,0}(\sigma_B, \varphi_{B,K}) = p_{2m,0}(\sigma_B, \varphi_{B,K}) = 0, \quad p_{2m,1}(\sigma_B, \varphi_{B,K}) = 2^m$$

for all positive integers m. Thus, the Lind zeta functions are as follows:

$$\zeta_{A,J}(t) = \frac{1}{\sqrt{1-2t^2}} \exp\left(\frac{t^2}{1-2t^2} + 8t^{12}\right),$$

$$\zeta_{B,I}(t) = \frac{1}{\sqrt{1-2t^2}} \exp\left(\frac{2t+3t^2}{1-2t^2}\right)$$

and

$$\zeta_{B,K}(t) = \frac{1}{\sqrt{1-2t^2}} \exp\left(\frac{t^2}{1-2t^2}\right).$$

As a result, we see that

$$(\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \ncong (\mathsf{X}_B, \sigma_B, \varphi_{B,I}),$$

$$(\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \ncong (\mathsf{X}_B, \sigma_B, \varphi_{B,K})$$

and

$$(\mathsf{X}_B, \sigma_B, \varphi_{B,I}) \ncong (\mathsf{X}_B, \sigma_B, \varphi_{B,K}).$$

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Example 7.2. In spite of $\zeta_{A,J} \neq \zeta_{B,I}$, $\zeta_{A,J} \neq \zeta_{B,K}$ and $\zeta_{B,I} \neq \zeta_{B,K}$, there are D_∞-SEs between (A, J), (B, I) and (B, K) pairwise. If D and E are matrices given by

$$D = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad E = 2\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

then (D, E) is a D_∞-SE of lag 6 from (A, J) to (B, K) and from (A, J) to (B, I):

$$(D, E) : (A, J) \sim (B, I) \text{ (lag 6)} \quad \text{and} \quad (D, E) : (A, J) \sim (B, K) \text{ (lag 6)}.$$

Direct computation shows that (B^l, B^l) is a D_∞-SE from (B, I) to (B, K):

$$(B^l, B^l) : (B, I) \sim (B, K) \text{ (lag } 2l)$$

for all positive integers l. This contrasts with the fact that the existence of SE between two transition matrices implies that the corresponding $\mathbb{Z}$-TMCs share the same Artin–Mazur zeta functions. (See §7 in [8].)

Example 7.3. We compare the flip signatures of (A, J), (B, I) and (B, K). Direct computation shows that the index sets for the eventual kernels of A and B are

$$\mathcal{I}nd(\mathcal{K}(A)) = \{1, 6\} \quad \text{and} \quad \mathcal{I}nd(\mathcal{K}(B)) = \{1\}$$

and the flip signatures are

$$\text{F.Sig}(A, J) = (-1, +1),$$

$$\text{F.Sig}(B, I) = (+1)$$

and

$$\text{F.Sig}(B, K) = (-1).$$

By Theorem D, we see that

$$(\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \ncong (\mathsf{X}_B, \sigma_B, \varphi_{B,I}),$$

$$(\mathsf{X}_A, \sigma_A, \varphi_{A,J}) \ncong (\mathsf{X}_B, \sigma_B, \varphi_{B,K})$$

and

$$(\mathsf{X}_B, \sigma_B, \varphi_{B,I}) \ncong (\mathsf{X}_B, \sigma_B, \varphi_{B,K}).$$

The flip signature is completely determined by the eventual kernel of a transition matrix, while the Lind zeta functions and the existence of D_∞-shift equivalence between two flip pairs rely on the eventual ranges of transition matrices. The nilpotency index of Ashley's eight-by-eight A on the eventual kernel $\mathcal{K}(A)$ is 6. In the case of (A, J) in Example 7.1,

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the number of periodic points $p_m(\sigma_A)$ is completely determined by the eventual range of A, the numbers of fixed points $p_{2m-1,0}(\sigma_A, \varphi_{A,J})$ and $p_{2m,1}(\sigma_A, \varphi_{A,J})$ are completely determined by the eventual ranges if $m \geq 7$, and $p_{2m,0}(\sigma_A, \varphi_{A,J})$ is completely determined by the eventual ranges if $m \geq 6$. In Example 7.2, (D, E) is actually the D_∞-SE from (A, J) to (B, I) and from (A, J) to (B, K) having the smallest lag, and this means that the existence of D_∞-SE from (A, J) to (B, I) and from (A, J) to (B, K) are not related to the eventual kernels of A and B at all. Similarly, the existence of D_∞-SE from (B, I) to (B, K) is not related to the eventual kernel of B at all. Therefore, the coincidence of the Lind zeta functions or the existence of D_∞-shift equivalence are not enough to guarantee the same number of -1s in the corresponding flip signatures or the coincidence of leading signatures. The following example shows that the flip signatures of two flip pairs can have the same number of -1s and share the same leading signatures even when their non-zero eigenvalues are totally different.

Example 7.4. Let A and B be the minimal zero-one transition matrices for the even shift and full two-shift, respectively:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If we set

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then (A, J) and (B, K) are flip pairs. Let $\mathrm{sp}^\times(A)$ and $\mathrm{sp}^\times(B)$ be the sets of non-zero eigenvalues of A and B, respectively:

$$\mathrm{sp}^\times(A) = \left\{ \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right\} \quad \text{and} \quad \mathrm{sp}^\times(B) = \{2\}.$$

Because $\mathrm{sp}^\times(A)$ and $\mathrm{sp}^\times(B)$ do not coincide, (X_A, σ_A) and (X_B, σ_B) are not $\mathbb{Z}$-conjugate, and hence $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,K})$ are not D_∞-conjugate. More precisely, $\mathrm{sp}^\times(A) \neq \mathrm{sp}^\times(B)$ implies that A and B are not shift-equivalent, and hence (A, J) and (B, K) are not D_∞-shift equivalent:

$$\mathrm{sp}^\times(A) \neq \mathrm{sp}^\times(B) \Rightarrow A \nsim B \Rightarrow (A, J) \nsim (B, K).$$

In addition, $\mathrm{sp}^\times(A) \neq \mathrm{sp}^\times(B)$ implies that the Artin–Mazur zeta functions $\zeta_A(t)$ and $\zeta_B(t)$ of (X_A, σ_A) and (X_B, σ_B) do not coincide (see Ch. 7 in [8]), and hence the Lind zeta functions $\zeta_{A,J}(t)$ and $\zeta_{B,K}(t)$ of (X_A, σ_A) and (X_B, σ_B) do not coincide:

$$\mathrm{sp}^\times(A) \neq \mathrm{sp}^\times(B) \Rightarrow \zeta_A(t) \neq \zeta_B(t) \Rightarrow \zeta_{A,J}(t) \neq \zeta_{B,K}(t).$$

However, the flip signatures of (A, J) and (B, K) are the same:

$$\mathrm{F.Sig}(A, J) = (+1) \quad \text{and} \quad \mathrm{F.Sig}(B, K) = (+1).$$

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In the following example, we see that the coincidence of the Lind zeta functions does not guarantee the existence of D_∞-SE between the corresponding flip pairs.

Example 7.5. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic functions χ_A and χ_B of A and B are the same:

$$\chi_A(t) = \chi_B(t) = t(t-1)^4(t^2 - 3t + 1).$$

We denote the zeros of $t^2 - 3t + 1$ by λ and μ. Direct computation shows that (A, J) and (B, J) are flip pairs and $(\mathsf{X}_A, \sigma_A, \varphi_{A,J})$ and $(\mathsf{X}_B, \sigma_B, \varphi_{B,J})$ share the same numbers of fixed points.

$$\begin{aligned} p_m &= 4 + \lambda^m + \mu^m, \\ p_{2m-1,0} &= \frac{8\lambda^m - 3\lambda^{m-1}}{11\lambda - 4} + \frac{8\mu^m - 3\mu^{m-1}}{11\mu - 4}, \\ p_{2m,0} &= \frac{\lambda^{m+1}}{11\lambda - 4} + \frac{\mu^{m+1}}{11\mu - 4}, \\ p_{2m,1} &= \frac{55\lambda^m - 21\lambda^{m-1}}{11\lambda - 4} + \frac{55\mu^m - 21\mu^{m-1}}{11\mu - 4} \quad (m = 1, 2, \ldots). \end{aligned}$$

As a result, they share the same Lind zeta functions:

$$\sqrt{\frac{1}{t^2(1-t^2)^4(1-3t^2+t^4)}} \exp\left(\frac{t + 3t^2 - t^3 - 2t^4}{1 - 3t^2 + t^4}\right).$$

If there is a D_∞-SE (D, E) from (A, J) to (B, J), then (D, E) also becomes a SE from A to B. It is well known [8] that the existence of SE from A to B implies that A and B have the same Jordan forms away from zero up to the order of Jordan blocks. The Jordan

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canonical forms of A and B are given by

$$\begin{bmatrix} \lambda & & & & & & \\ & \mu & & & & & \\ & & 1 & 1 & 0 & 0 & \\ & & 0 & 1 & 1 & 0 & \\ & & 0 & 0 & 1 & 1 & \\ & & 0 & 0 & 0 & 1 & \\ & & & & & & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \lambda & & & & & & \\ & \mu & & & & & \\ & & 1 & 1 & & & \\ & & 0 & 1 & & & \\ & & & & 1 & 1 & \\ & & & & 0 & 1 & \\ & & & & & & 0 \end{bmatrix},$$

respectively. From this, we see that (A, J) cannot be D_∞-shift equivalent to (B, J).

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