Ergod. Th. & Dynam. Sys., page 1 of 18

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Examples of ergodic cylindrical cascades over a two-dimensional torus

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Abstract. A cylindrical cascade on $\mathbb{T}^d \times \mathbb{R}^r$ can be seen as a deterministic random walk on $\mathbb{R}^r$ driven by an observable over the irrational toral translation on the base torus. We prove that, when the observable is the indicator function of a generic (straight) rectangle in $\mathbb{T}^2$, the cascade on $\mathbb{T}^2 \times \mathbb{R}$ is ergodic for a G_δ-dense set of translation vectors. We also provide examples of ergodic cylindrical cascades in higher dimensions with more restrictive conditions on the side lengths of the rectangles.

Key words: irrational translation, ergodic random walk, essential values
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1. *Introduction*

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the torus, parameterized by $[0, 1)$. Given an irrational vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, let $T_\alpha : \mathbb{T}^d \to \mathbb{T}^d$ denote the translation by α on $\mathbb{T}^d$ given by $x \mapsto x + \alpha \pmod 1$. Given an observable $A(\cdot)$ on $\mathbb{R}^r$, the cylindrical cascade above T_α relative to $A(\cdot)$ will be denoted by

$$\begin{aligned} W_{\alpha,A} \ : \ \mathbb{T}^d \times \mathbb{R}^r &\to \mathbb{T}^d \times \mathbb{R}^r, \\ (x, y) &\mapsto (x + \alpha, y + A(x)). \end{aligned}$$

The dynamics of $W_{\alpha,A}$ are closely related to the Birkhoff sum of A, denoted by $A_N(\cdot)$, by the expression

$$W^N_{\alpha,A}(x, y) = (T^N_\alpha x, A_N(x)),$$

where $A_N(\cdot)$ is defined as

$$A_N(x) = \begin{cases} \displaystyle\sum_{n=0}^{N-1} A(T^n_\alpha x), & N \geq 1, \\ \displaystyle -\sum_{n=1}^{-N} A(T^{-n}_\alpha x), & N \leq -1, \\ 0, & N = 0. \end{cases} \tag{1.1}$$

When α is Diophantine and A is smooth, the linear cohomological equation $A(x) - \int_{\mathbb{T}^d} A(u)\,du = -B(x+\alpha) + B(x)$ has a smooth solution B, and thus $W_{\alpha,A}$ is smoothly conjugated to the translation $W_{\alpha,\int_{\mathbb{T}^d} A}$. In this paper, we have examples where A is some indicator function and α is Liouville.

The cascade $W_{\alpha,A}$ can be seen as a random walk on the fiber $\mathbb{R}^r$ driven by the translation T_α on the base $\mathbb{T}^d$, so it is natural to study its recurrence and ergodicity. For the walk $W_{\alpha,A}$ to be *recurrent*, the *zero mean* condition $\int_{\mathbb{T}^d} A = 0$ (as vectors) is necessary, since, otherwise, A_N goes to *infinity* by the ergodic theorem. In fact, Atkinson [2] showed that zero mean is also a sufficient condition for recurrence when $r = 1$. For ergodicity, the case of one-dimensional rotations ($d = r = 1$) is well studied, and the ergodicity is often proved by the Denjoy–Koksma inequality and the *essential values* criterion (see definition below) [4, 6, 10, 14].

In higher dimensions, the difficulty comes from the lack of the Denjoy–Koksma inequality for $d \geq 2$ (see [16, Appendix 1, p. 215]). In the case where the fiber is $\mathbb{R}^r$, only recurrent and transient examples are known when the base dimension $d = 2$ and the components of A are (zero-mean) indicator functions of polytopes for $r > 1$ (see [3]). More specifically, Chevallier and Conze [3] showed that, for the walk of any base space to be recurrent, a sufficient condition is that the Birkhoff sums of A_i grow more slowly than $\mathcal{O}(n^{1/r})$ along some subsequence N_n for most starting points. By using the L^2-estimation of the ergodic deviations of A, they showed that, for almost every translation vector α, the walk $W_{\alpha,A}$ is recurrent for polytopes. For integer-valued cocyles, Conze and Fraczek [5] constructed examples of ergodic skew products for rotations of bounded type in the case of $\mathbb{T}^2 \times \mathbb{Z}^2$, where they took advantage of the recurrence of the cocycle from [3] and the boundedness of the rotations instead of the Denjoy–Koksma inequality. We refer the reader to the survey [7] for a more comprehensive discussion about the recurrence and ergodicity of cylindrical cascades.

The ergodicity of the walk $W_{\alpha,A}$ can be established if the sums A_{N_n} are increasingly well distributed over $\mathbb{R}^r$ along some subsequence N_n, while the translation $T_\alpha^{N_n}$ stays close to identity. This idea gives rise to the notion of *essential values* [15], which is the major tool for our construction of ergodic examples of $W_{\alpha,A}$.

Definition 1.1. $a \in \mathbb{R}^r$ is called an *essential value* of A if, for each measurable set $S \in \mathbb{T}^d$ of positive measure, for each $\epsilon > 0$ there exists $N \in \mathbb{Z}$ such that

$$\mu(S \cap T_\alpha^{-N} S \cap [|A_N(\cdot) - a| < \epsilon]) > 0. \tag{1.2}$$

The following lemma characterizes the set of all essential values of A, denoted by $E(A)$, and gives a criterion for ergodicity.

LEMMA 1.2. [1, 15]

(1) *$E(A)$ is a closed subgroup of $\mathbb{R}^r$.*
(2) *$E(A) = \mathbb{R}^r$ if and only if $W_{\alpha,A}$ is ergodic.*
(3) *If A is integer valued and $E(A) = \mathbb{Z}^r$, then $W_{\alpha,A}$ is ergodic in $\mathbb{T}^d \times \mathbb{Z}^r$.*

In this paper, we develop the ideas introduced by Fayad [9] to recover Denjoy–Koksma-type inequalities for Liouville rotations. By combining these with several classical results

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from number theory for Diophantine approximations, we establish the essential value criterion to obtain the ergodicity of the walk $W_{\alpha,A}$.

Given a vector $L = (l, l') \in (0, 1]^2$, denote by B_L the rectangle $[0, l] \times [0, l'] \subset \mathbb{T}^2$, and denote by $Vol(B_L)$ its volume (area) ll'. We prove the following theorem.

THEOREM 1.3. *For almost every $L = (l, l') \in (0, 1]^2$, there exists a G_δ-dense set of $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 - \mathbb{Q}^2$ such that the cylindrical cascade $W_{\alpha,A}$ constructed over the zero-mean indicator function $A(\cdot) = \chi_{B_L}(\cdot) - Vol(B_L)$ is ergodic.*

We also give examples for the case of $\mathbb{T}^2 \times \mathbb{R}^2$, with more restrictions for the second rectangle. Given two pairs of lengths (l_1, l_1'), (l_2, l_2'), define the corresponding rectangles $B_1 = [0, l_1] \times [0, l_1']$ and $B_2 = [0, l_2] \times [0, l_2']$ and the zero-mean indicator functions

$$\begin{cases} A^{(1)}(\cdot) = \chi_{B_1}(\cdot) - \mathrm{vol}(B_1), \\ A^{(2)}(\cdot) = \chi_{B_2}(\cdot) - \mathrm{vol}(B_2). \end{cases} \tag{1.3}$$

Then we have the following result for the $\mathbb{R}^2$ case.

THEOREM 1.4. *For almost every $(l_1, l_1') \in (0, 1]^2$, there exist G_δ-dense sets of $(0, 1]$ of l_2 and l_2', and a G_δ-dense set of $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 - \mathbb{Q}^2$, such that the cylindrical cascade $W_{\alpha,A}$ constructed over $A(\cdot) = (A^{(1)}(\cdot), A^{(2)}(\cdot))$ (as defined in (1.3)) is ergodic.*

It is not hard to see that by restricting the vector function $A(\cdot)$ in Theorem 1.4 to its first coordinate $A^{(1)}$ we obtain Theorem 1.3 as a corollary. In this paper, we prove Theorem 1.4 in detail and keep Theorem 1.3 as its corollary.

Using our method, higher-dimensional counterparts in $\mathbb{T}^d \times \mathbb{R}^r$ can be obtained by requiring that the lengths of additional sides of the higher-dimensional boxes are Liouville when $d > 2$ or $r > 1$, and by applying the subgroup property of the essential values (point 2 of Lemma 1.2). Ergodic examples using polytopes in $\mathbb{T}^d$, $d \geq 2$, are more delicate due to their inclined sides, and their constructions are still in progress. Although Liouville conditions are needed with our method, ergodicity of the walk is expected for typical polytopes and typical translation vectors, due to the slow divergence rate of the Birkhoff sums of A.

2. *Arithmetic notation*

Let x be a real number.

(1) Denote by $[x]$ the integer part of x.
(2) Denote by $\{x\}$ the signed distance of the x to the closest integer: that is, $\{x\} = x - n$, where n is the only integer such that $x - n \in [-1/2, 1/2)$.
(3) Denote by $\|x\| = |\{x\}|$ the distance of x to the closest integer.

3. *Lemmas*

Our construction is based on the approximation of the rectangle by small rational rectangles that tile the torus and the approximation of the translation vector by their vertices. We begin with the following observation.

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LEMMA 3.1. *Let v denote the rational vector $(p/q, p'/q')$ on $\mathbb{T}^2$, where the coordinates are in reduced forms, that is, $(p, q) = 1$ and $(p', q') = 1$. If, in addition, we have that $(q, q') = 1$, then the following identity holds for any $r \in \mathbb{Z}$: that is,*

$$\{mv \mid r \leq m \leq r + qq' - 1\} = \left\{ \left(\frac{i}{q}, \frac{j}{q'} \right) \,\middle|\, 0 \leq i \leq q-1,\ 0 \leq j \leq q'-1 \right\}. \tag{3.1}$$

Remark 3.2. This lemma states that the orbit of length qq' of the translations by v in $\mathbb{T}^2$ coincides with the vertices of the tiling of $\mathbb{T}^2$ by rectangles of side lengths $1/q$ and $1/q'$.

Proof. Because the cardinalities of the two sets are both qq', it is enough to show that, for $r \leq m, n \leq r + qq' - 1$,

$$mv = nv \text{ in } \mathbb{T}^2 \Longrightarrow m = n. \tag{3.2}$$

Note that $mv = nv$ in $\mathbb{T}^2$ is equivalent to

$$(m-n)p/q \in \mathbb{Z} \quad \text{and} \quad (m-n)p'/q' \in \mathbb{Z}. \tag{3.3}$$

Because p and q are relatively prime, q divides $m - n$, and, similarly, q' divides $m - n$. Note that q and q' are also relatively prime, so we have that qq' divides $m - n$, which is only possible when $m = n$, since $|m - n| \leq qq' - 1$. □

Our idea is to approximate the irrational translation vector α by rational vectors. First, we tile the torus $\mathbb{T}^2$ by disjoint rectangles of side lengths $1/q$ and $1/q'$: that is,

$$\mathbb{T}^2 = \bigcup_{\substack{0 \leq i \leq q-1 \\ 0 \leq j \leq q'-1}} R^{i,j}, \tag{3.4}$$

where $R^{i,j}$ is the rectangle $[i/q, (i+1)/q) \times [j/q', (j+1)/q')$.

Note that if the translation vector α is close to the rational vector $(p/q, p'/q')$, then the above lemma implies that, starting from the origin, the orbit $\{m\alpha\}$ of length qq' visits a close neighborhood of every vertex of the tiling by $R^{i,j}$ exactly once.

By approximating the rectangle $B_L = [0, l] \times [0, l']$ by a union of $R^{i,j}$, we can control precisely the number of points in the orbit $\{x + m\alpha\}$ that lie inside B_L at special times. Specifically, we can compute the Birkhoff sums for $A = \chi_{B_L} - \mathrm{Vol}(B_L)$ as follows.

Let

$$l = (b + \delta)/q, \quad l' = (b' + \delta')/q', \tag{3.5}$$

where $\delta = \{ql\}$, $\delta' = \{q'l'\}$, and let

$$\alpha_1 = (p + \eta)/q, \quad \alpha_2 = (p' + \eta')/q', \tag{3.6}$$

where $\eta = \{q\alpha_1\}$, $\eta' = \{q'\alpha_2\}$.

When η, η' is small compared with δ, δ', that is, α Liouville enough, we have the following expressions for the Birkhoff sums.

LEMMA 3.3. *Denote $N = qq'$, and suppose that $|\delta|$ and $|\delta'|$ are smaller than $1/4$. Let $\tilde{\delta} \in [|\delta|, 1/4)$ and $\tilde{\delta}' \in [|\delta'|, 1/4)$. If, for $\tilde{\delta}$ and $\tilde{\delta}'$, there exists a non-zero integer K such*

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that

$$|\eta| < \tilde{\delta}/(|K|N), \quad |\eta'| < \tilde{\delta}'/(|K|N),$$

then there exists a set $\mathcal{F} \subset \mathbb{T}^2$ *of measure* $(1 - 4\tilde{\delta})(1 - 4\tilde{\delta}')$ *such that, for every* $x \in \mathcal{F}$ *and every* k *between* 0 *and* K *(possibly negative), the Birkhoff sum* $A_{kN}(x)$ *has the form*

$$A_{kN}(x) = kA_N(x) = -k(b\delta' + b'\delta + \delta\delta'). \tag{3.7}$$

Proof. Define $\mathcal{F}$ to be the set of points that are *not* $2\tilde{\delta}$*-close* to the boundaries of any $R^{i,j}$, in the following sense: that is,

$$\mathcal{F} = \bigcup_{\substack{0\le i\le q-1\\ 0\le j\le q'-1}} \left\{ (x_1, x_2) \in \mathbb{T}^2 \;\middle|\; \begin{array}{l} (i + 2\tilde{\delta})/q \le x_1 \le (i + 1 - 2\tilde{\delta})/q, \\ (j + 2\tilde{\delta})/q' \le x_2 \le (j + 1 - 2\tilde{\delta})/q' \end{array} \right\}. \tag{3.8}$$

Remark. More precisely it should be called '$2(\tilde{\delta}, \tilde{\delta}')$-close', but, since $\tilde{\delta}$ and $\tilde{\delta}'$ always come together, we use the notation '$\tilde{\delta}$-close' to be more concise.

$\mathcal{F}$ is the disjoint union of qq' rectangles, and each of them is of measure $(1 - 4\tilde{\delta})(1 - 4\tilde{\delta})/(qq')$, so the measure of $\mathcal{F}$ is $(1 - 4\tilde{\delta})(1 - 4\tilde{\delta})$.

Note that, for $|m| \le |K|N$, $m\alpha$ is $\tilde{\delta}-$close to $m(p/q, p'/q')$: that is,

$$\begin{aligned} |m\alpha_1 - mp/q| &\le |m||\eta|/q \le \tilde{\delta}/q, \\ |m\alpha_2 - mp'/q'| &\le |m||\eta'|/q' \le \tilde{\delta}/q'. \end{aligned} \tag{3.9}$$

Starting from a point x in $\mathcal{F}$, since the points in $\mathcal{F}$ are $2\tilde{\delta}$-away from the boundaries of $R^{i,j}$, the orbit $\{x + m\alpha \mid |m| \le |K|N - 1\}$ stays $\tilde{\delta}$-away from the boundaries of $R^{i,j}$. By Lemma 3.1, for $|k| \le |K| - 1$, the orbit $\{x + m\alpha \mid kN \le m \le (k + 1)N - 1\}$ visits each $R^{i,j}$ exactly once and stays $\tilde{\delta}$-away from the boundaries of the small rectangles in the tiling.

Since $\tilde{\delta} > |\delta|$ and $\tilde{\delta}' > |\delta'|$, our target rectangle B_L contains the center parts of bb' small rectangles, and the orbit visits $B_L bb'$ times within a time range of N. Therefore, for $0 \le k \le |K|$,

$$\begin{aligned} A_{kN}(x) &= kA_N(x) \\ &= k(bb' - qq'll') \\ &= k(bb' - (b + \delta)(b' + \delta')) \\ &= -k(b\delta' + b'\delta + \delta\delta'). \end{aligned} \tag{3.10}$$

In view of the negative sign in the definition of $A_N(x)$ for $N < 0$ in (1.1), the above identity holds for $-|K| < k < 0$ with the same argument. □

If we can show that $A_N(x)$ becomes increasingly small for sequences of q and q', then, by choosing K increasingly large for Liouville translation vector α, the sequence $\{A_{kN}\}_{0\le k\le K-1}$ can be increasingly well distributed in $\mathbb{R}$. The following lemma states that this is indeed the case for generic $L = (l, l') \in \mathbb{T}^2$.

LEMMA 3.4. *For almost every* $(l, l') \in (0, 1]^2$*, there exist two sequences* $\{q_n\}_{n\in\mathbb{N}}$ *and* $\{q'_n\}_{n\in\mathbb{N}}$ *such that the following properties hold.*

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(1) $q_n, q'_n \to \infty$, *as* $n \to \infty$.
(2) q_n *and* q'_n *are relatively prime, that is* $(q_n, q'_n) = 1$, *for all* $n \in \mathbb{N}$.
(3) *We have*

$$\frac{l}{2(\ln q_n)^2} \le |b'_n(q_n l - b_n) + b_n(q'_n l' - b'_n)| \le \frac{2}{(\ln\ln\ln q_n)^{1/2}}, \tag{3.11}$$

where b_n *and* b'_n *are the closest integers to* $q_n l$ *and* $q'_n l'$.
(4) q'_n *and* q_n *are approximately of the same magnitude: that is,*

$$\frac{q_n}{(\ln q_n)^2} \le q'_n \le q_n(\ln\ln\ln q_n)^{1/2}. \tag{3.12}$$

(5) *The next two inequalities follow from the proof of (3.11). We record them here for later use.*

$$\|q_n l\| \le \frac{1}{q_n \ln\ln\ln q_n} \tag{3.13}$$

and

$$\|q'_n l'\| \le \frac{1}{q_n(\ln\ln\ln q_n)^{1/2}}. \tag{3.14}$$

Lemma 3.4 states that the Birkhoff sums for A can indeed be small along the subsequence $q_n q'_n$, while the coprime condition between q_n and q'_n is satisfied. When the fiber is of higher dimensions, $\mathbb{R}^2$ for example, we need similar bounds for the second rectangle, but, to approximate the essential values along two axes, we need the magnitude of the Birkhoff sums for the second rectangle to alternate between large and small relative to the Birkhoff sums of the first rectangle. This can be achieved by restricting the length of the second rectangle in a G_δ-dense set, as stated in the next lemma.

LEMMA 3.5. *For almost every* $l_1, l'_1 \in (0, 1]$, *there exist* $l_2, l'_2 \in (0, 1]$ *(each belonging to a* G_δ*-dense set of (0,1]) such that, for the two sequences* $\{q_n\}_{n\in\mathbb{N}}$, $\{q'_n\}_{n\in\mathbb{N}}$ *defined as in Lemma 3.4,*

$$\frac{1}{4(\ln q_{2n})^3} \le |q'_{2n}(q_{2n}l_2 - r_{2n}) + q_{2n}(q'_{2n}l'_2 - r'_{2n})| \le \frac{3}{2(\ln q_{2n})^3}, \tag{3.15}$$

and

$$\begin{aligned}\frac{1}{4(\ln\ln\ln q_{2n+1})^{1/4}} &\le |q'_{2n+1}(q_{2n+1}l_2 - r_{2n+1}) + q_{2n+1}(q'_{2n+1}l'_2 - r'_{2n+1})| \\ &\le \frac{3}{2(\ln\ln\ln q_{2n+1})^{1/4}},\end{aligned} \tag{3.16}$$

where r_n *and* r'_n *are, respectively, the closest integer to* $q_n l_2$ *and* $q'_n l'_2$. *Moreover, the inequalities above still hold if the sum in the middle is replaced by one of its summands.*

Lemma 3.4 is the key to our construction of ergodic examples on $\mathbb{T}^2 \times \mathbb{R}$, where the main difficulty is the coprime condition between q_n and q'_n. If we do not require the coprime condition, Lemma 3.4 can be easily obtained by using the classical version of Khintchine's divergence theorem for Diophantine approximations and the Dirichlet

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principle (the pigeonhole principle). We dedicate the last section of our paper to its proof, together with the proof of Lemma 3.5.

4. *Proof of the theorems using the lemmas*

For every pair of q_n, q'_n in Lemmas 3.4 and 3.5, we tile $\mathbb{T}^2$ by corresponding small rectangles

$$R_n^{i,j} = [i/q_n, (i+1)/q_n] \times [j/q'_n, (j+1)/q'_n], \tag{4.1}$$

where $0 \le i \le q_n - 1,\ 0 \le j \le q'_n - 1$.

We first prove the inequality of the essential value criterion (1.2) for special rectangles $R_n^{i,j}$, which then naturally generalizes to positive measure sets.

LEMMA 4.1. *For the irrationals l_1, l'_1, l_2, l'_2 and the sequences $\{q_n\}$, $\{q'_n\}$ in Lemma 3.5, there exists a G_δ-dense set S of α in $[0, 1)^2$ with following property.*

For every $\alpha \in S$, every $a \in \mathbb{R}$ and every $\epsilon > 0$, there exist infinitely many n, and $N = N(\alpha, a, \epsilon, n) \in \mathbb{N}^*$, such that, for every rectangle $R_n^{i,j}$ defined in (4.1),

$$\mu(R_n^{i,j} \cap T_\alpha^{-N} R_n^{i,j} \cap [|A_N(\cdot) - (a, 0)| < \epsilon]) > \frac{1}{2q_n q'_n}, \tag{4.2}$$

and the same inequality holds with a different n and N if we change $(a, 0)$ to $(0, a)$.

Proof. By choosing a subsequence of n, we can assume that

$$\ln\ln\ln q_n \ge \left(\tfrac{3}{2}\right)^4 n^2. \tag{4.3}$$

The set S and its density. Consider the subsets of $\mathbb{T}^2$ for the choices of $\alpha = (\alpha_1, \alpha_2)$ given by

$$S = \bigcap_{m\ge1}\bigcup_{n\ge m}\Bigg(\bigcup_{\substack{0\le p_n\le q_n-1\\(p_n,q_n)=1}}\left(\frac{p_n}{q_n} - \frac{1}{n(\ln q_n)^2 q_n^2 q'_n}, \frac{p_n}{q_n} + \frac{1}{n(\ln q_n)^2 q_n^2 q'_n}\right)$$
$$\times \bigcup_{\substack{0\le p'_n\le q'_n-1:\\(p'_n,q'_n)=1}}\left(\frac{p'_n}{q'_n} - \frac{1}{n(\ln q_n)^2 q_n^2 q'_n}, \frac{p'_n}{q'_n} + \frac{1}{n(\ln q_n)^2 q_n^2 q'_n}\right)\Bigg). \tag{4.4}$$

The set above is clearly a G_δ set. It remains to show that S is a dense set in $\mathbb{T}^2$. Note that $(\alpha_1, \alpha_2) \in S$ if and only if there exists a subsequence of n, denoted by n_k, and two corresponding sequences of p_{n_k} and p'_{n_k}, that are relatively prime to q_{n_k} and q'_{n_k}, respectively, such that, for every k,

$$\left|\alpha_1 - \frac{p_{n_k}}{q_{n_k}}\right| < \frac{1}{n_k(\ln q_{n_k})^2 q_{n_k}^2 q'_{n_k}},$$
$$\left|\alpha_2 - \frac{p'_{n_k}}{q'_{n_k}}\right| < \frac{1}{n_k(\ln q_{n_k})^2 q_{n_k} q'^2_{n_k}}. \tag{4.5}$$

To see that S is dense, we will show that we can approximate any point in $\mathbb{T}^2$ well enough. Note that, for any given $\epsilon > 0$, there exists M depending on ϵ such that, for every

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$q_n \geq M$ and $q'_n \geq M$ and every point (γ, γ') in $\mathbb{T}^2$, there exists p_n coprime to q_n and p'_n coprime to q'_n, with the property that

$$|p_n/q_n - \gamma| \leq \epsilon/2, \quad |p'_n/q'_n - \gamma| \leq \epsilon/2. \tag{4.6}$$

This is possible because the bound on the gaps between two consecutive totatives of a given number M is of order $o(M)$ (see [13]). Now fix n_0 and p_{n_0}, p'_{n_0} such that inequality (4.6) holds and assume that n_0 is large enough so that

$$\frac{1}{n_0(\ln q_{n_0})^2 q_{n_0}^2 q'_{n_0}} < \epsilon/2. \tag{4.7}$$

Starting with $k = 0$, by replacing ϵ with $\epsilon_k = 1/n_k(\ln q_{n_k})^2 q_{n_k}^2 q'_{n_k}$ and replacing γ with p_{n_k}/q_{n_k} in the argument for (4.6), we can inductively show the existence of n_{k+1} such that the sequence of the intervals

$$\left[\frac{p_{n_k}}{q_{n_k}} - \frac{1}{2n_k(\ln q_{n_k})^2 q_{n_k}^2 q'_{n_k}}, \frac{p_{n_k}}{q_{n_k}} + \frac{1}{2n_k(\ln q_{n_k})^2 q_{n_k}^2 q'_{n_k}}\right]$$

are nested. A similar result holds for the second coordinate. This proves the existence of $(\alpha_1, \alpha_2) \in [0, 1)^2$ such that (4.5) holds with the coprime pairs (p_{n_0}, q_{n_0}) and (p'_{n_0}, q'_{n_0}). By inequality (4.7),

$$|p_{n_0}/q_{n_0} - \alpha_1| < \epsilon/2, \quad |p'_{n_0}/q'_{n_0} - \alpha_2| < \epsilon/2.$$

Combining the inequality above with (4.6) for $n = n_0$ gives

$$|\alpha_1 - \gamma| < \epsilon, \quad |\alpha_2 - \gamma'| < \epsilon.$$

Hence, S is dense in $\mathbb{T}^2$.

Preparation work. We need to show that there are different orders of magnitude for Birkhoff sums $A_N^{(1)}$ and $A_N^{(2)}$ along rigidity sequences.

We proceed with the essential value argument. For every $\alpha \in S$, we can find a subsequence of n, and corresponding coprime numerators, such that (4.5) hold. For simplicity, we still denote the index by n.

For $M_n = \sqrt{n}(\ln q_n)^2 q_n q'_n$, and for every $0 \leq m \leq M_n$ (here we use weak inequalities to make writing easier),

$$\left|m\alpha_1 - m\frac{p_n}{q_n}\right| = m\left|\alpha_1 - \frac{p_n}{q_n}\right| \leq \frac{1}{\sqrt{n}q_n} \tag{4.8}$$

and

$$\left|m\alpha_2 - m\frac{p'_n}{q'_n}\right| = m\left|\alpha_2 - \frac{p'_n}{q'_n}\right| \leq \frac{1}{\sqrt{n}q'_n}. \tag{4.9}$$

For any $i, j \in \mathbb{N}$, denote by $\mathcal{F}_{i,j}(q_n, q'_n)$ the set of points (x_1, x_2) satisfying

$$\frac{2}{\sqrt{n}q_n} + \frac{i}{q_n} \leq x_1 \leq \frac{i+1}{q_n} - \frac{2}{\sqrt{n}q_n} \tag{4.10}$$

and

$$\frac{2}{\sqrt{n}q_n'} + \frac{j}{q_n'} \le x_2 \le \frac{j+1}{q_n'} - \frac{2}{\sqrt{n}q_n'}. \tag{4.11}$$

Then the Haar measure of the set $\mathcal{F}_{i,j}$ has the lower bound

$$\mu\{\mathcal{F}_{i,j}(q_n, q_n')\} \ge \bigg(1 - \frac{4}{\sqrt{n}}\bigg)\bigg(1 - \frac{4}{\sqrt{n}}\bigg)\frac{1}{q_n q_n'} \tag{4.12}$$

$$\ge \frac{3}{4}\frac{1}{q_n' q_n} \tag{4.13}$$

for $n \ge 32^2$.

Let

$$l_1 = \frac{b_n + \delta_n}{q_n} \quad \text{and} \quad l_1' = \frac{b_n' + \delta_n'}{q_n'}.$$

We now want to apply Lemma 3.3 to the first rectangle B_1, with $\tilde{\delta}_n = \tilde{\delta}_n' = 1/\sqrt{n}$, $K_n = \sqrt{n}(\ln q_n)^2$ and $N_n = q_n q_n'$. The inequalities (4.8) and (4.9) show that

$$|\eta_n| = \|q_n\alpha_1\| \le \frac{1}{n(\ln q_n)^2 q_n q_n'} = \frac{\tilde{\delta}_n}{K_n N_n}$$

and

$$|\eta_n'| = \|q_n'\alpha_1\| \le \frac{1}{n(\ln q_n)^2 q_n q_n'} = \frac{\tilde{\delta}_n'}{K_n N_n}.$$

By inequalities (3.13) and (3.14), and recalling the assumption (4.3) that q_n is large, we have that

$$|\delta_n| = \|q_n l_1\| \le \frac{1}{q_n(\ln\ln\ln q_n)} < \frac{1}{\sqrt{n}} = \tilde{\delta}_n \tag{4.14}$$

and

$$|\delta_n'| = \|q_n' l_1'\| \le \frac{1}{q_n(\ln\ln\ln q_n)^{1/2}} < \frac{1}{\sqrt{n}} = \tilde{\delta}_n'. \tag{4.15}$$

Hence, by applying Lemma 3.3 to $A^{(1)}$ for $x \in \mathcal{F}_{i,j}(q_n, q_n')$,

$$|A^{(1)}_{N_n}(x)| = |b_n b_n' - q_n q_n' l_1 l_1'|.$$

Thus, from (3.11),

$$\begin{aligned}
|A^{(1)}_{N_n}(x)| &= |b_n b_n' - q_n q_n' l_1 l_1'| \\
&= |b_n b_n' - (b_n + \delta_n)(b_n' + \delta_n')| \\
&\le 2|b_n\delta_n' + b_n'\delta_n| \\
&\le 2 \times \text{ right-hand side of (3.11)} \\
&= \frac{4}{(\ln\ln\ln q_n)^{1/2}}
\end{aligned}$$

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and, similarly,

$$|A^{(1)}_{N_n}(x)| \geq \frac{l_1}{4(\ln q_n)^2}.$$

Therefore, for $x \in \mathcal{F}_{i,j}(q_n, q'_n)$,

$$\frac{l_1}{4(\ln q_n)^2} \leq |A^{(1)}_{N_n}(x)| \leq \frac{4}{(\ln \ln \ln q_n)^{1/2}}. \tag{4.16}$$

Inequality (4.16) means that we can approximate the interval $[-\sqrt{2n}, \sqrt{2n}]$ with a precision of $1/(\ln \ln \ln q_n)^{1/2}$ by multiplying $A^{(1)}_{N_n}$ by a coefficient smaller than $\sqrt{2n}(\ln q_n)^2$, which is used to obtain (4.19).

For the second rectangle B_2, by (3.15) and (3.16) for individual summands (see the comment at the end of Lemma 3.5), and using the more strict bound (3.16) and the assumption (4.3) that q_n is large, we obtain similar inequalities to (4.14) and (4.15): that is,

$$\|q_n l_2\| \leq \frac{3}{2q'_n(\ln \ln \ln q_n)^{1/4}} < \frac{1}{\sqrt{n}} = \tilde{\delta}_n$$

and

$$\|q'_n l'_2\| \leq \frac{3}{2q_n(\ln \ln \ln q_n)^{1/4}} < \frac{1}{\sqrt{n}} = \tilde{\delta}'_n.$$

Therefore, we can apply Lemma (3.3) to $A^{(2)}$, and we obtain the estimation

$$|A^{(2)}_{N_{2n}}(x)| \leq \frac{3}{(\ln q_{2n})^3}, \tag{4.17}$$

$$\frac{1}{4(\ln \ln \ln q_{2n+1})^{1/4}} \leq |A^{(2)}_{N_{2n+1}}(x)| \leq \frac{3}{(\ln \ln \ln q_{2n+1})^{1/4}}. \tag{4.18}$$

Argument for the case of $(a, 0)$. For $a \in \mathbb{R}$ and $\epsilon > 0$, by (4.16) and (4.17), we can first choose n large enough, and then $K_{2n} = K_{2n}(a, \epsilon)$ with $|K_{2n}| \leq \sqrt{2n}(\ln q_{2n})^2$ such that

$$|K_{2n} A^{(1)}_{N_{2n}}(x) - a| \leq \frac{4}{(\ln \ln \ln q_{2n})^{1/2}} \leq \frac{1}{\sqrt{2}}\epsilon, \tag{4.19}$$

$$|K_{2n} A^{(2)}_{N_{2n}}(x)| \leq \frac{3\sqrt{2n}(\ln q_{2n})^2}{(\ln q_{2n})^3} \leq \frac{3\sqrt{2n}}{\ln q_{2n}} \leq \frac{3\sqrt{2n}}{2n} \leq \frac{1}{\sqrt{2}}\epsilon, \tag{4.20}$$

where we use that $\ln q_n \geq n$, which follows from assumption (4.3) at the beginning of the proof.

Finally, by combining the two inequalities above, for $N = K_{2n}N_{2n} = K_{2n}q_{2n}q'_{2n}$, $x \in \mathcal{F}_{i,j}(q_{2n}, q'_{2n}) \subset R^{i,j}_{2n}$, we get

$$A_N(x) = K_{2n}(A^{(1)}_{N_{2n}}(x), A^{(2)}_{N_{2n}}(x)),$$

which implies that

$$|A_N(x) - (a, 0)| \leq \epsilon. \tag{4.21}$$

So

$$\mathcal{F}_{i,j}(q_{2n}, q'_{2n}) \cap T_\alpha^{-N}(\mathcal{F}_{i,j}(q_{2n}, q'_{2n}) \subset T_\alpha^{-N}(R^{i,j}_{2n}) \cap R^{i,j}_{2n} \cap \{x \in [0,1)^2, |A_N(x) - (a, 0)|\} \leq \epsilon\}. \tag{4.22}$$

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Note that N is a multiple of $q_{2n}q'_{2n}$, so at time N the rational vector $N(q_{2n}, q'_{2n})$ returns to the origin. Because $N(q_{2n}, q'_{2n})$ approximates $N\alpha$ really well, by inequality (4.8) and (4.9),

$$\mu(\mathcal{F}_{i,j}(q_{2n}, q'_{2n}) \cap T_\alpha^{-N}(\mathcal{F}_{i,j}(q_{2n}, q'_{2n}))) \geq \bigg(1 - \frac{1}{\sqrt{2n}}\bigg)\bigg(1 - \frac{1}{\sqrt{2n}}\bigg)\mu(\mathcal{F}_{i,j}(q_{2n}, q'_{2n})),$$
$$\geq \frac{1}{2q_{2n}q'_{2n}} > 0.$$

Argument for the case of $(0, a)$. In the case of $(0, a)$, we can use the odd terms $2n + 1$ and repeat the argument above. Note that $A^{(2)}_{N_{2n+1}}$ is of order $(\ln \ln \ln q_{2n+1})^{1/4}$, while $A^{(1)}_{N_{2n+1}}$ is of order $(\ln \ln \ln q_{2n+1})^{1/2}$. Therefore, we can approximate any point in the interval $[-\sqrt{2n+1}, \sqrt{2n+1}]$ with a precision of $1/(\ln \ln \ln q_{2n+1})^{1/4}$ after multiplying $A^{(2)}_{N_{2n+1}}$ by a coefficient K_{2n+1} smaller than $\sqrt{2n+1}(\ln \ln \ln q_{2n+1})^{1/4}$, while $K_{2n+1}A^{(1)}_{N_{2n+1}}$ is still small and of order $1/(\ln \ln \ln q_{2n+1})^{1/4}$. This gives two inequalities similar to (4.19) and (4.20), and the same argument for positive measure at time $2n + 1$ follows through.

This finishes the proof of Lemma 4.1. □

COROLLARY 4.2. *For almost every* $l_1, l'_1 \in (0, 1]$*, a* G_δ*-dense set of* $l_2, l'_2 \in (0, 1]$ *and a* G_δ*-dense set of* $\alpha = (\alpha_1, \alpha_2) \in [0, 1)^2$*, we have* $E(A) = \mathbb{R}^2$.

Proof. By Theorem 1.2, it suffices to prove that a similar conclusion in Lemma 4.1 holds after we substitute $R^{i,j}_n$ by any set B of positive measure. In fact, for any positive measure set B, there exists n large enough, such that

$$\mu(R^{i,j}_n \cap B) > \frac{7}{8}\frac{1}{q'_n q_n} \tag{4.23}$$

for some $0 \leq i \leq q_n - 1$ and $0 \leq j \leq q'_n - 1$, and thus we can expect a slightly weaker lower bound for a general positive measure set B.

For given $a \in \mathbb{R}$, $\epsilon > 0$, choose n large enough such that inequality(4.23) holds, and denote $N = N(a, \epsilon) = K_{2n}N_{2n} = K_{2n}q'_{2n}q_{2n}$, as in Lemma 4.1. By inequality(4.21),

$$\mathcal{F}_{i,j}(q_{2n}, q'_{2n}) \subset \{x \mid |A_N(x) - (a, 0)| < \epsilon\}. \tag{4.24}$$

Now, with standard measure computations, we are ready to show the desired bound

$$\mu(\mathcal{F}_{i,j}(q_{2n}, q'_{2n}) \cap T_\alpha^{-N}(B \cap R^{i,j}_{2n}) \cap (B \cap R^{i,j}_{2n})) > 0.$$

Let $\mathcal{X} = T_\alpha^{-N}(B \cap R^{i,j}_{2n}) \cap (B \cap R^{i,j}_{2n})$. Then we have the following bound for $\mu(\mathcal{X})$: that is,

$$\begin{aligned}
\mu(\mathcal{X}) &= \mu(T_\alpha^{-N}(B \cap R^{i,j}_{2n}) \cap (B \cap R^{i,j}_{2n})) \\
&= \mu(T_\alpha^{-N}(B \cap R^{i,j}_{2n})) + \mu(B \cap R^{i,j}_{2n}) - \mu(T_\alpha^{-N}(B \cap R^{i,j}_{2n}) \cup (B \cap R^{i,j}_{2n})) \\
&\geq 2\mu(B \cap R^{i,j}_{2n}) - \mu(T_\alpha^{-N}(R^{i,j}_{2n}) \cup R^{i,j}_{2n}) \\
&\geq \tfrac{7}{4}\mu(R^{i,j}_{2n}) - \tfrac{5}{4}\mu(R^{i,j}_{2n}) \\
&= \tfrac{1}{2}\mu(R^{i,j}_{2n}),
\end{aligned}$$

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where the inequality $\mu(T_\alpha^{-N}(R_{2n}^{i,j}) \cup R_{2n}^{i,j}) \le \frac{5}{4}\mu(R_{2n}^{i,j})$ comes from the fact that T_α^{-N} is close to identity, as $q_{2n}q'_{2n}$ divides N(see (4.5)).

Denote $\mathcal{Y} = \mathcal{F}_{i,j}(q_{2n}, q'_{2n})$. Note that $\mathcal{X}$ and $\mathcal{Y}$ are both subsets of $R_{2n}^{i,j}$. By using the lower bound (4.12) of $\mu(\mathcal{Y})$,

$$\begin{aligned}\mu(\mathcal{X} \cap \mathcal{Y}) &= \mu(\mathcal{X}) + \mu(\mathcal{Y}) - \mu(\mathcal{X} \cup \mathcal{Y})\\ &\ge \tfrac{1}{2}\mu(R_{2n}^{i,j}) + \tfrac{3}{4}\mu(R_{2n}^{i,j}) - \mu(R_{2n}^{i,j})\\ &\ge \tfrac{1}{4}\mu(R_{2n}^{i,j}).\end{aligned}$$

Finally, we obtain the bound

$$\mu(T_\alpha^{-N}(B \cap R_{2n}^{i,j}) \cap (B \cap R_{2n}^{i,j}) \cap \mathcal{F}_{i,j}(q_{2n}, q'_{2n})) > \tfrac{1}{4}\mu(R_{2n}^{i,j}) > 0.$$

This proves that, for every $a \in \mathbb{R}$, $(a, 0) \in E(A)$. Similarly, we can prove that $(0, a)$ belongs to $E(A)$. By the subgroup property of essential value in Theorem 1.2, we have $E(A) = \mathbb{R}^2$. □

COROLLARY 4.3. *By restricting the above discussions to the first coordinate of $A^{(1)}$, it is easy to see that, in the case of $\mathbb{T}^2 \times \mathbb{R}$, for almost every $l_1, l'_1 \in (0, 1]$, there exists a G_δ dense set of $\alpha = (\alpha_1, \alpha_2) \in [0, 1)^2$, and we have $E(A^{(1)}) = \mathbb{R}$.*

5. *Proofs of Lemmas 3.4 and 3.5*

We can see that the *coprime* condition is essential for Lemma 3.1, and, to achieve it, we need a stronger version of Khintchine's divergence theorem that allows us to impose prime conditions for the denominators. We could also use the solved Duffin–Schaeffer conjecture [**11**], but a weaker version in their original paper suffices.

LEMMA 5.1. [**8**, Theorem 1] *If there exists a function $\psi : \mathbb{N} \to \mathbb{R}_+$ such that*

(i) $\sum \psi(q) = \infty$, *and*

(ii) *there exists a strictly positive constant, c, such that the inequality*

$$\sum_{q=1}^{n} \frac{\psi(q)\varphi(q)}{q} > c \sum_{q=1}^{n} \psi(q)$$

holds for infinitely many n, where $\varphi(q)$ is Euler's totient function,

then, for almost every $l \in \mathbb{R}$, there exist infinitely many pairs of integers (b, q) such that

$$\|ql\| = |ql - b| < \psi(q).$$

With the help of Lemma 5.1, we can restrict the approximation of $l \in \mathbb{R}$ by rationals with *prime* denominators, as shown in the next lemma.

COROLLARY 5.2. *For almost every $l \in \mathbb{R}$, there exist infinitely many pairs of integers (b, q) such that q is prime and*

$$\|ql\| = |ql - p| < \frac{1}{q \ln \ln \ln q}. \tag{5.1}$$

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Proof. Define

$$\psi(q) = \begin{cases} 1/(q \ln \ln \ln q) & \text{if } q \text{ is prime and } \ln \ln \ln q \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to check that conditions (i) and (ii) of Lemma 5.1 are satisfied for $\psi(q)$.

Note that, if q is prime, we have $\varphi(q) = q - 1$ (where φ is Euler's totient function). Then $\varphi(q)/q \geq 1/2$, and condition (ii) of Lemma 5.1 is satisfied.

For condition (i), by the inequality of the sum of the reciprocals of primes

$$\sum_{\substack{q \leq n \\ q \text{ prime}}} \frac{1}{q} \geq \ln \ln(n+1) - \ln(\pi^2/6), \tag{5.2}$$

we have

$$\sum_{\substack{q \leq n \\ q \text{ prime}}} \psi(q) \geq \frac{1}{\ln \ln \ln n} \sum_{\substack{q \leq n \\ q \text{ prime}}} \frac{1}{q}$$
$$\underset{(5.2)}{\geq} \frac{\ln \ln(n+1) - \ln(\pi^2/6)}{\ln \ln \ln n} \to \infty$$

as $n \to \infty$, and thus condition (i) is also satisfied. □

Now we proceed with the proof of Lemma 3.4.

Proof of properties (1) and (5) of Lemma 3.4. By Corollary 5.2, we have that, for almost every $l \in (0, 1]$, there exists an increasing sequence of prime numbers $\{q_n\}$ such that $q_n \to \infty$ as $n \to \infty$ and

$$\|q_n l\| \leq \frac{1}{q_n \ln \ln \ln q_n}. \tag{5.3}$$

This shows inequality (3.13) in property (5) of Lemma 3.4. By choosing a subsequence of q_n, we can assume that $q_n \geq n^2$ and $\ln \ln \ln q_n \geq 1$. By Dirichlet's principle (the pigeonhole principle), if we divide the interval $[0, 1]$ into m segments of equal length, then, for $m + 1$ points, two of them must lie in the same segment. Thus, for every $l' \in [0, 1]$, for all $n \in \mathbb{N}$, there exists $q'_n \in \mathbb{N}$ such that

$$1 \leq q'_n \leq q_n (\ln \ln \ln q_n)^{1/2} \tag{5.4}$$

and

$$\|q'_n l'\| \leq \frac{1}{q_n (\ln \ln \ln q_n)^{1/2}}. \tag{5.5}$$

This is inequality (3.14) in property (5) of Lemma 3.4. By Khintchine's convergence theorem (see, for example, [**12**, Ch. II, Theorem 4]), for almost every $l' \in (0, 1]$, there exists a constant $C(l')$ such that $\|kl'\| \geq C(l')/k^2$ for all $k \in \mathbb{N}$, and hence inequality (5.5) implies that $q'_n \to \infty$ as $n \to \infty$ for almost every $l' \in (0, 1]$. This proves property (1) of Lemma 3.4. □

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Proof of the right-hand side of (3.11). Combining (5.3) and (5.5) gives

$$q_n' \|q_n l\| + q_n \|q_n' l'\| \le \frac{2}{(\ln \ln \ln q_n)^{1/2}}.$$

Since b_n, b_n' are, respectively, the closest integers to $q_n l$ and $q_n' l'$ for $l, l' \in (0, 1]$, we have $b_n \le q_n$ and $b_n' \le q_n'$, and thus we obtain the right-hand side of (3.11) from the inequality above. □

Proof of property (2) of Lemma 3.4. Given $l \in (0, 1]$ and the sequence of prime numbers q_n as in the *proof of property (1)*, we show that, for almost every $l' \in (0, 1]$ and any infinite sequence of q_n' such that (5.5) hold, we have $(q_n, q_n') = 1$ when n is large enough. Then property (2) follows by choosing a subsequence of n.

Define

$$G_{n,k} = \left\{ l' \in [0, 1] \,\middle|\, \|kl'\| \le \frac{1}{q_n (\ln \ln \ln q_n)^{1/2}} \right\}$$

and

$$G_n = \bigcup_{k \in S_n} G_{n,k},$$

where

$$S_n = \{k \in \mathbb{N} \mid (k, q_n) \ne 1,\ k \le q_n (\ln \ln \ln n)^{1/2}\}$$

denotes the set of k's that violates the coprime condition. Note that, when l' does not belong to G_n, the corresponding q_n' that solves (5.4) and (5.5) is coprime to q_n, so it suffices to show that almost every l' belongs to at most finitely many G_n.

Since q_n is prime, if $(k, q_n) \ne 1$, then $(k, q_n) = q_n$ and $q_n \mid k$. So

$$\#S_n \le (\ln \ln \ln q_n)^{1/2}.$$

Thus,

$$\begin{aligned} \mu(G_n) &\le \sum_{k \in S_n} \mu(G_{n,k}) \\ &\le \#S_n \cdot \frac{1}{q_n (\ln \ln \ln q_n)^{1/2}} \\ &\le \frac{(\ln \ln \ln q_n)^{1/2}}{q_n (\ln \ln \ln q_n)^{1/2}} \\ &\le \frac{1}{n^2} \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \mu(G_n) \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By the Borel–Cantelli lemma, almost every $l' \in (0, 1]$ belongs to at most finitely many G_n, which proves the coprime condition. □

To prove the left-hand side of (3.11), it suffices to show that the two signed distances, $q_n l - b_n$ and $q_n' l - b_n'$, can be of the same sign infinitely many times, and that the lower

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bound holds for one of the distances. First, we introduce a lemma about the asymptotic estimation of the number of solutions to the approximation inequality.

LEMMA 5.3. [12, Ch. II, Theorem 7] *Let $\phi : \mathbb{N} \to [0, 1]$ be a decreasing function such that $\sum_{k=1}^{\infty} \phi(k)$ diverges. For each positive integer N and irrational number l, let*

$$\Phi(N) = \sum_{k=1}^{N} \phi(k),$$

and let $\lambda(l, N)$ denote the number of solutions in integers b and q of the inequalities

$$0 < ql - b < \phi(q) \quad \text{and} \quad 1 \leq q < N.$$

Then, for almost every $l \in \mathbb{R}$,

$$\lambda(l, N) = \Phi(N) + o(\Phi(N)).$$

Therefore, for almost every $l \in \mathbb{R}$, the number of solutions (b, q) for the above inequality is infinite.

Proof of the left-hand side of (3.11). By choosing a subsequence, we can assume that $q_n \geq 2q_{n-1}$, and, without loss of generality, we can also assume that $q_n l - b_n$ is positive for all n, that is,

$$0 < q_n l - p_n \leq \frac{1}{q_n \ln \ln \ln q_n}.$$

We show that, for almost every $l' \in (0, 1]$, there exist infinitely many pairs (b'_n, q'_n) such that $q'_n l' - b'_n$ are positive and smaller than $1/(q_n \ln \ln \ln q_n)$.

Define $Q_0 = q_0 = 0$, $Q_n = q_n (\ln \ln \ln q_n)^{1/2}$ for $n \geq 1$.

Define $\phi : \mathbb{N}^* \to (0, 1]$ by

$$\phi(k) = \frac{1}{Q_n} \quad \text{for } Q_{n-1} < k \leq Q_n.$$

Then $\phi(k)$ is decreasing. Note that

$$\sum_{k=1}^{Q_n} \phi(k) = \sum_{i=1}^{n} \left(1 - \frac{q_{i-1}(\ln \ln \ln q_{i-1})^{1/2}}{q_i (\ln \ln \ln q_i)^{1/2}}\right) \geq \frac{n}{2},$$

so $\sum \phi(k)$ diverges. By Lemma 5.3, for almost every $l' \in (0, 1]$, the number of pairs of solutions (b', q') such that

$$0 < q' l' - b' \leq \phi(q'), \quad 1 \leq q' < N,$$

is infinite. It is enough to note that $0 < q' l - b' < \phi(q')$ means that

$$0 < q' l' - b' \leq \frac{1}{q_n (\ln \ln \ln q_n)^{1/2}}, \quad q_{n-1}(\ln \ln \ln q_{n-1})^{1/2} < q' \leq q_n (\ln \ln \ln q_n)^{1/2}$$

for some n. These solutions can be indexed by a subsequence of n, and we omit the subindex and still denote them by (b'_n, q'_n).

This shows that there are infinitely many n's such that $q_n l - b_n$ and $q'_n l' - b'_n$ have the same sign.

It remains to show the lower bound for one of the distances. Here, we show it for $b_n|q'_n l' - b'_n|$. By Khintchine's convergence theorem, for almost every $l' \in (0, 1]$,

$$\|kl'\| \geq \frac{1}{k(\ln k)^{3/2}}$$

for all k large enough (depending on l'). By the inequality $q'_n \leq q_n(\ln \ln \ln q_n)^{1/2}$,

$$\|q'_n l'\| \geq \frac{1}{q'_n(\ln q'_n)^{3/2}} \geq \frac{1}{q_n(\ln q_n)^2} \tag{5.6}$$

for all n large enough, which gives the desired lower bound

$$b_n\|q'_n l'\| \geq \frac{b_n}{q_n}\frac{1}{(\ln q_n)^2} \geq \frac{l}{2(\ln q_n)^2}. \qquad \square$$

Proof of property (4) of Lemma 3.4. The upper bound (5.4) for q'_n gives the right-hand side of property 4. The two inequalities (5.5) and (5.6) give that

$$\frac{1}{q'_n(\ln q'_n)^{3/2}} \leq \|q'_n l'\| \leq \frac{1}{q_n(\ln \ln \ln q_n)^{1/2}}.$$

Using again the bound (5.4), we have that

$$\frac{q_n}{(\ln q_n)^2} \leq q'_n. \qquad \square$$

Proof of Lemma 3.5. Similarly to the set of (4.4), we can prove that there exists a G_δ set of l_2 (also of l'_2) such that there exists a subsequence of n where

$$\frac{1}{2(\ln q_{2n})^3} < q'_{2n}\|q_{2n}l_2\| < \frac{1}{(\ln q_{2n})^3} \tag{5.7}$$

and

$$\frac{1}{2(\ln \ln \ln q_{2n+1})^{1/4}} < q'_{2n+1}\|q_{2n+1}l_2\| < \frac{1}{(\ln \ln \ln q_{2n+1})^{1/4}}, \tag{5.8}$$

and l'_2 such that

$$q_{2n}\|q'_{2n}l'_2\| < \frac{1}{4(\ln q_{2n})^3} \tag{5.9}$$

and

$$q_{2n+1}\|q'_{2n+1}l'_2\| < \frac{1}{4(\ln \ln \ln q_{2n+1})^{1/4}}. \tag{5.10}$$

This shows that the individual terms in the sum in inequalities (3.15) and (3.16) satisfy the given bounds, and thus the final remark of Lemma 3.5 is proved. The above choice of l_2 (and l'_2) forms a dense set of (0, 1], as we can approximate any given number well enough by a rational with a denominator that is large enough. From inequalities (5.7) and (5.9), we have the existence of r_{2n} and r'_{2n} such that

$$\begin{aligned}\frac{1}{4(\ln q_{2n})^3} &\le \frac{1}{2(\ln q_{2n})^3} - \frac{1}{4(\ln q_{2n})^3} \\ &\le |q'_{2n}(q_{2n}l_2 - r_{2n}) + q_{2n}(q'_{2n}l'_2 - r'_{2n})| \\ &\le \frac{1}{(\ln q_{2n})^3} + \frac{1}{4(\ln q_{2n})^3} \\ &\le \frac{3}{2(\ln q_{2n})^3}.\end{aligned}$$

Similarly, from inequalities (5.8) and (5.10), we have the existence of r_{2n+1} and r'_{2n+1} such that

$$\begin{aligned}\frac{1}{4(\ln\ln\ln q_{2n+1})^{1/4}} &\le |q'_{2n+1}(q_{2n+1}l_2 - r_{2n+1}) + q_{2n+1}(q'_{2n+1}l'_2 - r'_{2n+1})| \\ &\le \frac{3}{2(\ln\ln\ln q_{2n+1})^{1/4}}.\end{aligned}$$ □

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