

## LIFTING PUZZLES IN DEGREE FOUR

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(Received 19 June 2008)

### Abstract

We identify the majority of Siegel modular eigenforms in degree four and weights up to 16 as being Duke–Imamoğlu–Ikeda or Miyawaki–Ikeda lifts. We give two examples of eigenforms that are probably also lifts but of an undiscovered type.

2000 *Mathematics subject classification*: primary 11F46, 11F60; secondary 65D20, 65-04.

*Keywords and phrases*: Siegel modular forms, Satake parameters, Miyawaki–Ikeda lifts.

### 1. Introduction

For degree four, vector spaces of Siegel modular cusp forms are known through weights  $k \leq 16$ . For weight 16 there are seven eigenforms: three quadratic pairs and one rational cusp form. From the work of Ikeda, the  $L$ -functions of two of the quadratic pairs are known. In this article, we compute the Euler 2-factors for the remaining quadratic pair and for the rational eigenform. Some of the roots of these Euler factors are not unimodular and so, presumably, there are two new lifts to be discovered. If not, the generalized Ramanujan–Petersson conjecture needs significant modification. The unexplained Euler 2-factors do factor in interesting ways, see Table 2. In this article we survey all  $L$ -functions for degree four and weights  $k \leq 16$ . We both explain how to compute Euler factors in higher degree and actually compute some. We find congruences between particular Euler factors and prove the nonvanishing of particular Miyawaki lifts.

Lifting conjectures have been motivated by the computation of Euler factors of  $L$ -functions of Siegel modular Hecke eigenforms. Let  $S_n^k$  denote the Siegel modular cusp forms of degree  $n$  and weight  $k$ . For Hecke eigenforms in  $S_2^{10}$  and  $S_2^{12}$ , Kurokawa [15] computed some Euler factors and noted their factorization into Euler products of elliptic modular cusp forms and zeta functions.

1. **Saito–Kurokawa–Maass lift.** For even weights  $k$ , each eigenform  $f \in S_1^{2k-2}$  corresponds to an eigenform  $F \in S_2^k$  such that the standard  $L$ -function of  $F$

factors

$$L(F, s, \text{st}) = \zeta(s)L(f, s + k - 1)L(f, s + k - 2).$$

This was proven by Maass, Andrianov and Zagier, see [5]. For eigenforms in  $S_3^{12}$  and  $S_3^{14}$ , Miyawaki [16] computed some Euler factors and used their factorizations to give two conjectures. We rephrase them as follows.

2. **Miyawaki Conjecture I.** For even  $k$ , each pair of eigenforms  $f \in S_1^{2k-4}$  and  $g \in S_1^k$  corresponds to an eigenform  $F \in S_3^k$  such that

$$L(F, s, \text{st}) = L(f, s + k - 2)L(f, s + k - 3)L(g, s, \text{st}).$$

3. **Miyawaki Conjecture II.** For even  $k$ , each pair of eigenforms  $f \in S_1^{2k-2}$  and  $g \in S_1^{k-2}$  corresponds to an eigenform  $F \in S_3^k$  such that

$$L(F, s, \text{st}) = L(f, s + k - 1)L(f, s + k - 2)L(g, s, \text{st}).$$

Duke and Imamoğlu studied the factorization of the Euler factors of the Schottky form in  $S_4^8$  and, in analogy with the Saito–Kurokawa lift, conjectured a lifting proven by Ikeda [9].

4. **Duke–Imamoğlu–Ikeda Lift.** For even  $k, n$  and  $(n + k)/2$ , each eigenform  $f \in S_1^k$  corresponds to an eigenform  $I_n(f) \in S_n^{(n+k)/2}$  whose standard  $L$ -function is

$$L(I_n(f), s, \text{st}) = \zeta(s) \prod_{i=1}^n L\left(f, s + \frac{n+k}{2} - i\right).$$

Ikeda has also solved Miyawaki’s first conjecture with an additional natural nonvanishing condition, see [8]. See also [12] for the resolution of a germane conjecture in [8]. In contrast, Miyawaki’s second conjecture remains open. Given a complex function  $g$  on the Siegel upper half space, define  $g^c$  by  $g^c(Z) = g(-\overline{Z})$ .

5. **Miyawaki–Ikeda Lift.** Let  $k, n + r$  and  $\hat{k} = (n + r + k)/2$  be even integers with  $r \leq n$ . Each pair of eigenforms  $f \in S_1^k$  and  $g \in S_r^{\hat{k}}$  corresponds to an element  $M_n(f, g) \in S_n^{\hat{k}}$  via  $M_n(f, g)(Z_n) = \langle I_{n+r}(f)(Z_n \oplus Z_r), g^c(Z_r) \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the Petersson inner product. If  $M_n(f, g)$  is nontrivial, then it is an eigenform for the even Hecke algebra and

$$L(M_n(f, g), s, \text{st}) = L(g, s, \text{st}) \prod_{i=r+1}^n L(f, s + \hat{k} - i).$$

A persistent theme in the articles of Kurokawa and Miyawaki is the difficulty of computing Fourier coefficients and Euler factors of eigenforms. The Fourier coefficients for our examples were computed in [18] and here we use Krieg’s matrix [14] to compute Euler factors from Hecke eigenvalues. In particular, we compute Euler 2-factors for all of the Hecke eigencuspforms in degree four where

$\dim S_4^k$  is currently known, namely  $k \leq 16$ . Two of our examples look suspiciously like unknown lifts because their Euler 2-factors possess rational factors of lower degree and because not all of their Satake parameters are unimodular. However, we cannot identify these factors with Euler factors of eigenforms of lower degree and so leave the matter as lifting puzzles.

The most interesting case is weight 16 with  $\dim S_4^{16} = 7$ , see Table 2. Here there are three pairs of eigenforms with quadratic irrationalities and one rational eigenform. Two eigenforms,  $h_1$  and  $h_2$ , are Ikeda lifts. Two eigenforms,  $h_5$  and  $h_6$ , are Miyawaki lifts and we prove this by verifying the nonvanishing of the relevant inner product. Another pair,  $h_3$  and  $h_4$ , have Euler 2-factors which factor into elliptic factors and a rational quartic factor of unknown origin. Neither do we know what to make of the rational form  $h_7$ , whose Euler 2-factor has no unimodular roots.

We summarize the results of our computations: For an elliptic modular eigenform  $f \in S_1^k$ , we may normalize  $f$  so that the Fourier expansion  $f(\tau) = \sum_{j=1}^\infty a(j; f)q^j$  satisfies  $a(1; f) = 1$ . The Hecke  $L$ -function is then defined by

$$L(f, s) = \prod_{p \text{ prime}} Q_p(f, p^{-s})^{-1} \quad \text{where } Q_p(f, X) = 1 - a(p; f)X + p^{k-1}X^2.$$

Let  $\phi_{28}^\pm(\tau) = q + (-4140 \pm 108\beta)q^2 + \dots$  with  $\beta = \sqrt{18209}$  be the two normalized eigenforms of weight 28. The standard  $L$ -function of the Ikeda lift  $I_4(\phi_{28}^\pm) \in S_4^{16}$  is

$$L(I_4(\phi_{28}^\pm), s, \text{st}) = \zeta(s)L(\phi_{28}^\pm, s + 12)L(\phi_{28}^\pm, s + 13)L(\phi_{28}^\pm, s + 14)L(\phi_{28}^\pm, s + 15).$$

The Euler 2-factor is given by

$$\begin{aligned} &(1 - X)Q_2(\phi_{28}^\pm, 2^{-12}X)Q_2(\phi_{28}^\pm, 2^{-13}X)Q_2(\phi_{28}^\pm, 2^{-14}X)Q_2(\phi_{28}^\pm, 2^{-15}X) \\ &= (1 - X)(1 - 2^{-12}(-4140 \pm 108\beta)X + 2^3X^2) \\ &\quad \times (1 - 2^{-13}(-4140 \pm 108\beta)X + 2^1X^2) \\ &\quad \times (1 - 2^{-14}(-4140 \pm 108\beta)X + 2^{-1}X^2) \\ &\quad \times (1 - 2^{-15}(-4140 \pm 108\beta)X + 2^{-3}X^2). \end{aligned}$$

These are exactly the Euler 2-factors computed for  $h_1$  and  $h_2$  in Table 2, so that  $h_1$  is the Ikeda lift of  $\phi_{28}^+$  to degree four and  $h_2$  is the lift of  $\phi_{28}^-$ .

To identify  $h_5$  and  $h_6$  as Miyawaki lifts, let  $\phi_{26}(\tau) = q - 48q^2 + \dots$  be the normalized eigenform of weight 26 and  $\phi_{30}^\pm(\tau) = q + (4320 \pm 96\gamma)q^2 + \dots$  with  $\gamma = \sqrt{51349}$  be the two normalized eigenforms of weight 30. The two Saito–Kurokawa lifts  $I_2(\phi_{30}^\pm)$  span  $S_2^{16}$ . In Section 5 we prove that  $M_4(\phi_{26}, I_2(\phi_{30}^\pm))$  is nontrivial, so that it is an eigenform with standard  $L$ -function

$$\begin{aligned} L(M_4(\phi_{26}, I_2(\phi_{30}^\pm)), s, \text{st}) &= L(I_2(\phi_{30}^\pm), s, \text{st})L(\phi_{26}, s + 12)L(\phi_{26}, s + 13) \\ &= \zeta(s)L(\phi_{30}^\pm, s + 14)L(\phi_{30}^\pm, s + 15)L(\phi_{26}, s + 12)L(\phi_{26}, s + 13). \end{aligned}$$

The Euler 2-factor is given by

$$(1 - X)(1 - 2^{-14}(4320 \pm 96\gamma)X + 2^1 X^2)(1 - 2^{-15}(4320 \pm 96\gamma)X + 2^{-1} X^2) \\ \times (1 - 2^{-12}(-48)X + 2^1 X^2)(1 - 2^{-13}(-48)X + 2^{-1} X^2).$$

From Table 2 we see that  $M_4(\phi_{26}, I_2(\phi_{30}^+))$  gives  $h_5$  and  $M_4(\phi_{26}, I_2(\phi_{30}^-))$  gives  $h_6$ .

Consider the pair  $h_3$  and  $h_4$ . From Table 2, the Euler 2-factor of  $h_3$  is given by

$$Q_2(h_3, X, st) = (1 - X)Q_2(\phi_{28}^+, 2^{-14}X)Q_2(\phi_{28}^+, 2^{-15}X) \\ \times \left(1 + \frac{9}{512}X + \frac{1601}{2048}X^2 + \frac{9}{512}X^3 + X^4\right).$$

Thus,  $h_3$  looks like a lift but we do not know the origin of the quartic factor. It is the only factor in Table 2 with unimodular roots, namely  $\eta \pm i\sqrt{1 - \eta^2}$  for  $\eta = (9 \pm \sqrt{1\,277\,521})/2048$ . Both quadratic factors,  $Q_2(\phi_{28}^+, 2^{-14}X)$  and  $Q_2(\phi_{28}^+, 2^{-15}X)$ , are shared with the Euler 2-factor of  $h_1$  and the Fourier coefficients of  $h_1$  and  $h_3$  are both algebraic integers in  $\mathbb{Q}(\sqrt{18\,209})$ . We also see that  $h_1$  and  $h_3$  are *congruence neighbors* in the following sense: their Hecke eigenvalues at  $p = 2$  are all congruent modulo certain primes in the ring of integers of  $\mathbb{Q}(\sqrt{18\,209})$ ; namely at three, at a prime above five and at a prime above seven.

Finally, consider the Euler 2-factor of the rational form  $h_7$ . If we introduce a pretend Euler factor with an integer parameter  $\ell$ ,

$$Q^\pm(X) = 1 - 2^\ell(165 \pm 3\sqrt{764242})X + 2^{21+2\ell}X^2,$$

then we have the following factorization for any choice of  $\ell$ :

$$Q_2(h_7, X, st) = (1 - X)Q^+(2^{-10-\ell}X)Q^+(2^{-11-\ell}X)Q^-(2^{-10-\ell}X)Q^-(2^{-11-\ell}X).$$

These look like Euler factors of weight  $22 + 2\ell$  with the  $s$  variable translated by  $10 + \ell$  and  $11 + \ell$ . However, none of the level-one elliptic Euler 2-factors from weights  $k = 24, 28, 30, 32, 34$  or  $38$  have coefficients that generate the quadratic field  $\mathbb{Q}(\sqrt{764242})$ .

### 2. Notation and background

In this section we give basic definitions and recall theorems of Andrianov and Satake. We define Siegel modular forms, the Hecke algebra, the spherical map and the Satake parameters. For a commutative ring  $R$  let  $M_{m \times n}(R)$  denote the  $R$ -module of  $m$ -by- $n$  matrices with coefficients in  $R$ . For  $x \in M_{m \times n}(R)$  let  $x' \in M_{n \times m}(R)$  denote the transpose. Let  $V_n(R) = \{x \in M_{n \times n}(R) \mid x' = x\}$  be the symmetric  $n$ -by- $n$  matrices over  $R$ ;  $V_n(\mathbb{R})$  is a euclidean vector space under the inner product  $\langle x, y \rangle = \text{tr}(xy)$ . For  $R \subseteq \mathbb{R}$ , an element  $x \in V_n(R)$  is called positive definite, written  $x > 0$ , when  $v'xv > 0$  for all  $v \in \mathbb{R}^n \setminus \{0\}$ ; we denote the set of these by  $\mathcal{P}_n(R)$ . The half-integral matrices are  $\mathcal{X}_n = \{T \in \mathcal{P}_n(\mathbb{Q}) \mid \forall v \in \mathbb{Z}^n, v'Tv \in \mathbb{Z}\}$ .

Let  $\text{GL}_n(R) = \{x \in M_{n \times n}(R) \mid \det(x) \text{ is a unit in } R\}$  be the general linear group and  $\text{SL}_n(R) = \{x \in \text{GL}_n(R) \mid \det(x) = 1\}$  the special linear group. For  $x \in \text{GL}_n(R)$  let

$x^*$  denote the inverse transpose. Let  $I_n \in \text{GL}_n(R)$  be the identity matrix and set  $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \text{SL}_{2n}(R)$ . The symplectic group is defined by  $\text{Sp}_n(R) = \{x \in \text{GL}_{2n}(R) \mid x'J_nx = J_n\}$ . We write  $\Gamma_n = \text{Sp}_n(\mathbb{Z})$  and for  $R \subseteq \mathbb{R}$  define the group of positive  $R$ -similitudes by  $\text{GSp}_n^+(R) = \{x \in M_{2n \times 2n}(R) \mid \exists \mu \in R^+ : g'J_n g = \mu J_n\}$ . Each  $\gamma \in \text{GSp}_n^+(R)$  has a unique  $\mu = \mu(\gamma) = \det(\gamma)^{1/n}$ .

Define the Siegel upper half space  $\mathbb{H}_n = \{\Omega \in V_n(\mathbb{C}) : \Im \Omega \in \mathcal{P}_n(\mathbb{R})\}$ . The group  $\text{GSp}_n^+(\mathbb{R})$  acts on  $\mathbb{H}_n$  as  $M \langle \Omega \rangle = (A\Omega + B)(C\Omega + D)^{-1}$  for  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ . For any function  $f : \mathbb{H}_n \rightarrow \mathbb{C}$  and any  $k \in \mathbb{Z}$  we follow Andrianov and let  $\langle n \rangle = n(n + 1)/2$  and define the group action for  $M \in \text{GSp}_n^+(R)$  by

$$(f|_k M)(\Omega) = \mu(M)^{kn - \langle n \rangle} \det(C\Omega + D)^{-k} f(M \langle \Omega \rangle).$$

The complex vector space of Siegel modular forms of degree  $n$  and weight  $k$  is denoted by  $M_n^k$  and is defined as the set of holomorphic  $f : \mathbb{H}_n \rightarrow \mathbb{C}$  such that  $f|_k M = f$  for all  $M \in \Gamma_n$  and such that for all  $Y_0 \in \mathcal{P}_n(\mathbb{R})$ ,  $f$  is bounded on  $\{\Omega \in \mathbb{H}_n \mid \Im \Omega > Y_0\}$ . For  $f \in M_n^k$  the Siegel  $\Phi$ -map is defined by  $(\Phi f)(\Omega) = \lim_{\lambda \rightarrow +\infty} f(\begin{pmatrix} i\lambda & 0 \\ 0 & \Omega \end{pmatrix})$  and the space of cusp forms is defined by  $S_n^k = \{f \in M_n^k \mid \Phi f = 0\}$ . Let  $e(z) = e^{2\pi iz}$ . By the Koecher principle, an  $f \in S_n^k$  has a Fourier expansion

$$f(\Omega) = \sum_{T \in \mathcal{X}_n} a(T; f) e(\langle T, \Omega \rangle),$$

where the  $a(T; f)$  satisfy  $a(U'TU; f) = \det(U)^k a(T; f)$  for all  $U \in \text{GL}_n(\mathbb{Z})$ . The Petersson inner product  $\langle f, g \rangle = \int_{F_n} \det(Y)^{k-n-1} f(Z) \overline{g(Z)} dZ$  makes  $S_n^k$  into an inner product space. Here the integral is over a fundamental domain  $F_n$  for the action of  $\Gamma_n$  on  $\mathbb{H}_n$  and  $Z = X + iY \in \mathbb{H}_n$  and  $dZ = \bigwedge_{1 \leq i \leq j \leq n} dX_{ij} \wedge dY_{ij}$ .

We now introduce various Hecke rings. For a group  $\Gamma$  contained in a semigroup  $S$ , the pair  $(\Gamma, S)$  is called a Hecke pair if, for all  $s \in S$ ,  $\Gamma \setminus \Gamma s \Gamma$  is finite. The free  $\mathbb{Q}$ -module on the left cosets  $\Gamma s$  with  $s \in S$  is denoted by  $L(\Gamma, S)$ . A right action of  $\Gamma$  on  $L(\Gamma, S)$  is given by  $\Gamma s \mapsto \Gamma s \gamma$  for  $\gamma \in \Gamma$  and the  $\Gamma$ -invariant subspace is denoted by  $H(\Gamma, S)$ . If we identify a double coset  $\Gamma s \Gamma = \bigsqcup_i \Gamma s_i$  with  $\sum_i \Gamma s_i \in L(\Gamma, S)$ , then  $H(\Gamma, S)$  is generated by double cosets. The binary operation on  $H(\Gamma, S) \times L(\Gamma, S)$  given by  $(\sum a_i \Gamma s_i)(\Gamma s) = \sum a_i \Gamma s_i s$  is well defined and restricts to make  $H(\Gamma, S)$  an associative ring. We define

$$\begin{aligned} \mathcal{S}_n &= \text{GSp}_n^+(\mathbb{Q}); & \mathcal{H}_n &= H(\Gamma_n, \mathcal{S}_n) = \text{the (global) Hecke algebra,} \\ \overline{\mathcal{S}}_n &= \mathcal{S}_n \cap M_{2n \times 2n}(\mathbb{Z}); & \overline{\mathcal{H}}_n &= H(\Gamma_n, \overline{\mathcal{S}}_n) = \text{the integral Hecke algebra,} \\ \mathcal{S}_{n,p} &= \mathcal{S}_n \cap \text{GL}_{2n}(\mathbb{Z}[\frac{1}{p}]); & \mathcal{H}_{n,p} &= H(\Gamma_n, \mathcal{S}_{n,p}) = \text{the } p\text{-part of } \mathcal{H}_n, \\ \overline{\mathcal{S}}_{n,p} &= \mathcal{S}_{n,p} \cap M_{2n \times 2n}(\mathbb{Z}); & \overline{\mathcal{H}}_{n,p} &= H(\Gamma_n, \overline{\mathcal{S}}_{n,p}) = \text{the } p\text{-part of } \overline{\mathcal{H}}_n. \end{aligned}$$

The Hecke algebra is a commutative ring and acts as a ring of endomorphisms on each  $M_n^k$  via

$$f \Big|_k \sum_i a_i \Gamma_n s_i = \sum_i a_i f \Big|_k s_i,$$

an action that stabilizes  $S_n^k$ . The subalgebra generated by the double cosets  $\Gamma_n s \Gamma_n$  for  $s \in \text{GSp}_n^+(\mathbb{Q})$  with  $\mu(s) \in (\mathbb{Q}^\times)^2$  is called the *even part* of the Hecke algebra. Each Hecke operator is self-adjoint with respect to the Petersson inner product and so  $M_n^k$  has a basis of simultaneous eigenforms. We know that  $\mathcal{H}_n$  is generated by  $\cup_p \mathcal{H}_{n,p}$  and that  $\mathcal{H}_{n,p}$  is generated by  $\overline{\mathcal{H}}_{n,p}$  and  $\Gamma_n(pI_{2n})^{-1}$ . The  $p$ -part of the integral Hecke algebra,  $\overline{\mathcal{H}}_{n,p}$ , is generated by the double cosets, for  $1 \leq i \leq n$ :

$$T(p) = \Gamma_n \text{diag}(I_n; pI_n)\Gamma_n \quad \text{and} \quad T_i(p^2) = \Gamma_n \text{diag}(I_{n-i}, pI_i; p^2I_{n-i}, pI_i)\Gamma_n.$$

We also define  $T_0(p^2) = \Gamma_n \text{diag}(I_n; p^2I_n)\Gamma_n$ .

It is no small matter to turn these abstract definitions into a programmable action on the Fourier coefficients. The following formulae in [2] are what we require here. Using the notation in Section 2, for

$$f(\Omega) = \sum_{T \in \mathcal{X}_n} a(T; f)e(\langle T, \Omega \rangle) \in S_n^k,$$

the  $T$ th coefficient of  $f|T(p)$  (respectively,  $f|T_i(p^2)$ ) is given by

$$a(T; f|T(p)) = \sum_{\substack{0 \leq r \leq n \\ U \in G(r, n-r, 0)}} p^{r(r-2n+2k-1)/2} a\left(\frac{1}{p}T[UD(r, n-r, 0)]; f\right)$$

and

$$a(T; f|T_i(p^2)) = \sum_{\substack{0 \leq r, s \leq n \\ r+s \leq n \\ U \in G(r, s, n-r-s)}} p^{(2r+s)(k-n+r-1)-r(r-1)} e(T, U) \\ \times a\left(\frac{1}{p^2}T[UD(r, s, n-r-s)]; f\right).$$

In the above formulae, use the following notation. Define the exponential sum

$$e(T, U) = \sum_{\substack{M \bmod p \\ \text{rank}_p(M) = s - (n-i)}} e\left(\frac{1}{p}\text{tr}(T[U] \text{diag}(0^r, M^s, 0^{n-r-s}))\right)$$

where the notation  $M^s$  indicates that  $M \in M_{s \times s}(\mathbb{Q})$  and 0 indicates the zero matrix. The summation is only over symmetric  $M$ .

The representatives of double cosets are found by enumerating the set  $G(r, s, t)$  for  $r + s + t = n$ , which is the union of the following (extend the notation to let  $M^{r,s}$  denote an element of  $M_{r \times s}(\mathbb{Q})$ ):

$$\left\{ \begin{pmatrix} x & 1 \\ U & 0 \end{pmatrix} \middle| U^{(n-1)} \in G(r, s, t-1), x = (pa, 0), a^{1,r} \bmod p \right\} \\ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix} \middle| U^{n-1} \in G(r-1, s, t), x = (0, a, b)', a^{1,s} \bmod p, b^{1,t} \bmod p^2 \right\}$$

and

$$\left\{ \begin{aligned} & \left( \begin{array}{ccc} 0 & 1 & 0 \\ U_1 & 0 & U_2 \end{array} \right) \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & I \end{array} \right) \Big| (U_1^{n-1,r}, U_2^{n-1,n-r-1}) \in G(r, s-1, t), \\ & x = (0, a)', a^{1,t} \pmod p \end{aligned} \right\}$$

where it is understood that  $G(r, s, t) = \emptyset$  if any of  $r, s$  or  $t$  is negative. The only comment we add to these formulae of Breulmann and Kuss is that, in the summation for  $e(T, U)$ , a zero-dimensional matrix of rank zero must be allowed; that is, we take  $e(T, U) = 1$  when  $i = n$  and  $s = 0$ . These formulae were used to compute the Hecke eigenvalues in Table 1 from the Fourier coefficients at [19].

By a fundamental result of Satake, the  $p$ -part of the integral Hecke algebra is isomorphic to the invariant ring  $\mathbb{Q}[x_0, \dots, x_n]^{W_n}$ , where  $W_n$  is the group generated by the permutations of  $x_1, \dots, x_n$  and by the automorphisms

$$\tau_i(x_0) = x_0 x_i, \quad \tau_i(x_i) = x_i^{-1}, \quad \tau_j(x_i) = x_i \quad \text{for } i \neq j.$$

To define this isomorphism, write left cosets from  $\Gamma_n \setminus \mathcal{S}_{n,p}$  in the form  $\Gamma_n M$  with

$$\mu(M) = p^\delta \quad \text{and} \quad M = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} p^{\delta_1} & & & \\ & p^{\delta_2} & & * \\ & 0 & \ddots & \\ & & & p^{\delta_n} \end{pmatrix}.$$

The terms  $p^\delta$  and  $\text{GL}_n(\mathbb{Z})D$  depend only on the left coset  $\Gamma_n M$  and the sequence of elementary divisors  $p^{\delta_i}$  is determined by  $\text{GL}_n(\mathbb{Z})D$ . Letting

$$\Omega(\Gamma_n M) = x_0^\delta \prod_{i=1}^n \left( \frac{x_i}{p^i} \right)^{\delta_i}$$

defines an algebra homomorphism  $\Omega : L(\Gamma_n, \mathcal{S}_{n,p}) \rightarrow \mathbb{Q}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$  that induces the Satake isomorphism  $\Omega : \overline{\mathcal{H}}_{n,p} \rightarrow \mathbb{Q}[x_0, \dots, x_n]^{W_n}$ . The map  $\Omega$  is a universal object for complex one-dimensional representations of  $\mathcal{H}_{n,p}$ , see [1, p. 165].

**THEOREM 2.1.** *Let  $\lambda \in \text{Hom}_{\mathbb{Q}}(\mathcal{H}_{n,p}, \mathbb{C})$  be a nontrivial homomorphism. There is an  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{C}^{n+1}$  such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{H}_{n,p} & & \\ \Omega \downarrow & \searrow \lambda & \\ \mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W_n} & \xrightarrow{x:\alpha} & \mathbb{C} \end{array}$$

In particular, given an eigenform  $F \in S_n^k$  of the Hecke algebra  $\mathcal{H}_n$ , we may define a homomorphism  $\lambda_F : \mathcal{H}_n \rightarrow \mathbb{C}$  by  $\lambda_F(T)F = F|_k T$ . For each prime  $p$ , we have

TABLE 1. Hecke eigenvalues for all seven eigenforms in  $S_4^{16}$  as given in [18] where  $\beta = \sqrt{18\,209}$  and  $\gamma = \sqrt{51\,349}$ .

$h_1$	$T(2)$	$12\,960(67\,989 + 443\beta)$
	$T_0(4)$	$276\,480(2\,124\,076\,963\,591 + 2\,858\,223\,465\beta)$
	$T_1(4)$	$353\,894\,400(3\,255\,937 + 7\,494\,543\beta)$
	$T_2(4)$	$-42\,278\,584\,320(-1\,232\,945 + 801\beta)$
	$T_3(4)$	$515\,396\,075\,520(-523 + 27\beta)$
$h_2$	$T(2)$	$-12\,960(-67\,989 + 443\beta)$
	$T_0(4)$	$-276\,480(-2\,124\,076\,963\,591 + 2\,858\,223\,465\beta)$
	$T_1(4)$	$-353\,894\,400(-3\,255\,937 + 7\,494\,543\beta)$
	$T_2(4)$	$42\,278\,584\,320(1\,232\,945 + 801\beta)$
	$T_3(4)$	$-515\,396\,075\,520(523 + 27\beta)$
$h_3$	$T(2)$	$-230\,400(1703 + 9\beta)$
	$T_0(4)$	$283\,115\,520(3\,556\,625 + 644\,607\beta)$
	$T_1(4)$	$1\,698\,693\,120(12\,749\,123 + 67\,821\beta)$
	$T_2(4)$	$43\,486\,543\,872(183\,579 + 533\beta)$
	$T_3(4)$	$206\,158\,430\,208(221 + 27\beta)$
$h_4$	$T(2)$	$230\,400(-1703 + 9\beta)$
	$T_0(4)$	$-283\,115\,520(-3\,556\,625 + 644\,607\beta)$
	$T_1(4)$	$-1\,698\,693\,120(-12\,749\,123 + 67\,821\beta)$
	$T_2(4)$	$-43\,486\,543\,872(-183\,579 + 533\beta)$
	$T_3(4)$	$-206\,158\,430\,208(-221 + 27\beta)$
$h_5$	$T(2)$	$1\,175\,040(557 + \gamma)$
	$T_0(4)$	$70\,778\,880(1\,331\,260\,505 + 3\,483\,163\gamma)$
	$T_1(4)$	$18\,119\,393\,280(2\,397\,057 + 11\,974\gamma)$
	$T_2(4)$	$38\,654\,705\,664(317\,997 + 529\gamma)$
	$T_3(4)$	$824\,633\,720\,832(449 + 3\gamma)$
$h_6$	$T(2)$	$-1\,175\,040(-557 + \gamma)$
	$T_0(4)$	$-70\,778\,880(-1\,331\,260\,505 + 3\,483\,163\gamma)$
	$T_1(4)$	$-18\,119\,393\,280(-2\,397\,057 + 11\,974\gamma)$
	$T_2(4)$	$-38\,654\,705\,664(-317\,997 + 529\gamma)$
	$T_3(4)$	$-824\,633\,720\,832(-449 + 3\gamma)$
$h_7$	$T(2)$	$230\,400\,000$
	$T_0(4)$	$163\,381\,183\,072\,174\,080$
	$T_1(4)$	$-29\,115\,328\,285\,900\,800$
	$T_2(4)$	$-7\,821\,199\,870\,525\,440$
	$T_3(4)$	$399\,947\,354\,603\,520$



TABLE 2. Euler 2-factors  $Q_2(h_i, X, \text{st})/(1 - X)$  for the eigenforms  $h_i$  in Table 1, where  $\beta = \sqrt{18\,209}$  and  $\gamma = \sqrt{51\,349}$ .

$h_1$	$2^{-46}(-1024 + (-1035 + 27\beta)x - 8192x^2)$ $(-2048 + (-1035 + 27\beta)x - 4096x^2)$ $(-4096 + (-1035 + 27\beta)x - 2048x^2)$ $(-8192 + (-1035 + 27\beta)x - 1024x^2)$
$h_2$	$2^{-46}(1024 + (1035 + 27\beta)x + 8192x^2)$ $(2048 + (1035 + 27\beta)x + 4096x^2)$ $(4096 + (1035 + 27\beta)x + 2048x^2)$ $(8192 + (1035 + 27\beta)x + 1024x^2)$
$h_3$	$2^{-34}(-2048 + (-1035 + 27\beta)x - 4096x^2)$ $(-4096 + (-1035 + 27\beta)x - 2048x^2)$ $(2048 + 36x + 1601x^2 + 36x^3 + 2048x^4)$
$h_4$	$2^{-34}(2048 + (1035 + 27\beta)x + 4096x^2)$ $(4096 + (1035 + 27\beta)x + 2048x^2)$ $(2048 + 36x + 1601x^2 + 36x^3 + 2048x^4)$
$h_5$	$2^{-36}(-512 + (135 + 3\gamma)x - 1024x^2)$ $(-1024 + (135 + 3\gamma)x - 512x^2)$ $(512 + 3x + 256x^2)$ $(256 + 3x + 512x^2)$
$h_6$	$2^{-36}(512 + (-135 + 3\gamma)x + 1024x^2)$ $(1024 + (-135 + 3\gamma)x + 512x^2)$ $(512 + 3x + 256x^2)$ $(256 + 3x + 512x^2)$
$h_7$	$2^{-28}(32768 - 5280x - 20\,755x^2 - 2640x^3 + 8192x^4)$ $(8192 - 2640x - 20\,755x^2 - 5280x^3 + 32\,768x^4)$

the restriction  $\lambda_F : \mathcal{H}_{n,p} \rightarrow \mathbb{C}$  and the  $(\alpha_{0,p}, \dots, \alpha_{n,p})$  promised in Theorem 2.1 are called the *Satake  $p$ -parameters of  $F$*  and satisfy  $\Omega(T)|_{\mathfrak{X}:=\alpha} = \lambda_F(T)$  for  $T \in \mathcal{H}_{n,p}$ . The Satake parameters of  $F$  are used in the definition of the  $L$ -functions attached to  $F$ . In particular, the standard  $L$ -function is given by

$$L(F, s, \text{st}) = \prod_{p \text{ prime}} Q_p(\alpha_p, p^{-s})^{-1}$$

where

$$Q_p(F, X, \text{st}) = Q_p(\alpha_p, X) = (1 - X) \prod_{i=1}^n (1 - \alpha_{i,p}X)(1 - \alpha_{i,p}^{-1}X).$$

The spinor  $L$ -function is given by

$$L(F, s, \text{spin}) = \prod_{p \text{ prime}} Q_p(F, p^{-s}, \text{spin})^{-1}$$

where

$$Q_p(F, X, \text{spin}) = (1 - \alpha_{0,p}X) \prod_{r=1}^n \prod_{1 \leq i_1 < \dots < i_r \leq n} \binom{n}{r} (1 - \alpha_{0,p} \alpha_{i_1,p} \dots \alpha_{i_r,p} X).$$

### 3. Computing Euler factors

In recent work the first and third authors have computed bases for the spaces of degree four Hecke eigenforms in weights up to 16. In what follows, we show how to compute the Hecke eigenvalues of these forms and from this deduce their Satake parameters at the prime two. In all cases, the standard  $L$ -functions factor, suggesting that all of the forms in [18] are lifts. All but three of the computed forms we identify as either Ikeda lifts or Miyawaki lifts, leaving us to wonder whether there are two previously unidentified lifts into genus four.

In what follows, we make use of the following notation. For a polynomial  $f \in \mathbb{Q}[x_0, \dots, x_n]$  we set  $(\sigma \cdot f)(x_0, \dots, x_n) = f(\sigma(x_0), \dots, \sigma(x_n))$  and set

$$[f] = \sum_{\sigma \in W_n / \text{Stab}(f)} \sigma \cdot f.$$

Also, if  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$ , by  $x^{\mathbf{b}}$  we mean the monomial  $x_1^{b_1} \dots x_n^{b_n}$ . A similar meaning is implied by  $\alpha^{\mathbf{b}}$ .

Parallel to the generators for  $\overline{\mathcal{H}}_{n,p}$ , the ring  $\mathbb{Q}[x_0, x_1, \dots, x_n]^{W_n}$  is generated by  $x_0(1 + x_1) \dots (1 + x_n)$  and  $[x_0^2 x^{(1,1,\dots,1)}], [x_0^2 x^{(2,1,\dots,1)}], \dots, [x_0^2 x^{(2,2,\dots,2,1)}]$ . Note that, if there are  $\ell$  twos in the superscript  $(2, \dots, 2, 1, \dots, 1)$ , then

$$[x^{(2,\dots,2,1,\dots,1)}] = x_0^2 x_1 \dots x_n V_\ell(\mathbf{x}) \quad \text{where} \\ V_\ell(\mathbf{x}) = \sum_{e_1, \dots, e_n \in \{1,0,-1\}: |e_1| + \dots + |e_n| = \ell} x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}.$$

We note that  $V_\ell(x)$  is just the  $\ell$ th elementary symmetric polynomial in the variables  $x_i + x_i^{-1}$  and we set

$$V(\mathbf{x}) = (V_0(\mathbf{x}), V_1(\mathbf{x}), \dots, V_n(\mathbf{x})).$$

The generators of  $\overline{\mathcal{H}}_{n,p}$  are related to the generators of  $\mathbb{Q}[x_0, x_1, \dots, x_n]^{W_n}$  in the following Theorem [6], see [21] for a more general statement.

**THEOREM 3.1.** *Let  $n$  be a positive integer and  $p$  a prime. Then*

$$\Omega(T(p)) = x_0(1 + x_1) \dots (1 + x_n).$$

Also, there exists an upper triangular matrix  $K_n(p^2) \in M_{(n+1) \times (n+1)}(\mathbb{Z}[1/p])$  with positive coefficients on and above the diagonal such that

$$\begin{aligned} &\Omega(T_n(p^2), \dots, T_0(p^2)) \\ &= ([x_0^2 x^{(1,1,\dots,1)}], [x_0^2 x^{(2,1,\dots,1)}], \dots, [x_0^2 x^{(2,2,\dots,2,2)}]) K_n(p^2). \end{aligned}$$

The key computational point is that, for  $n \geq 2$ , the entries of the matrix  $K_n(p^2)$  have been given by Krieg [14]. For  $n = 4$  the Krieg matrix is given by

$$\begin{pmatrix} \frac{1}{p^{10}} & \frac{p^4-1}{p^{10}} & \frac{4p^{17}-p^{16}-2p^{14}-p^{12}}{p^{20}} & \frac{(p^2-1)(4p^{17}-p^{16}-2p^{14}-p^{12})}{p^{20}} & \frac{6p^4-8p^3+2}{p^4} \\ 0 & \frac{1}{p^6} & \frac{p^3-1}{p^6} & \frac{3p^{11}-p^8(1+p+p^2)}{p^{12}} & \frac{(p-1)(3p^3-p^2-p-1)}{p^4} \\ 0 & 0 & \frac{1}{p^3} & \frac{p^2-1}{p^3} & \frac{2(p-1)}{p} \\ 0 & 0 & 0 & \frac{1}{p} & \frac{p-1}{p} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For general  $n$ , a Maple program that computes  $K_n(p^2)$  may be found at [20].

**DEFINITION 3.2.** For  $n \in \mathbb{Z}^+$ , define  $P_n \in M_{(n+1) \times (n+1)}(\mathbb{Z})$  by  $(P_n)_{ij} = 0$  if  $i > j$  or  $i + j$  is odd and by  $(P_n)_{ij} = \binom{n+1-i}{(j-i)/2}$  if  $i + j$  is even.

For example, in  $n = 4$ ,

$$P_4 = \begin{pmatrix} 1 & 0 & 4 & 0 & 6 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**THEOREM 3.3.** Let  $n \geq 2$  and  $k$  be positive integers and  $p$  a prime. Let  $F \in M_n^k$  be a simultaneous eigenform of  $\mathcal{H}_{n,p}$  with Satake  $p$ -parameters  $(\alpha_0, \dots, \alpha_n)$ . Write the standard Euler  $p$ -factor as

$$Q_p(\mathbf{x}, X) = (1 - X) \sum_{j=0}^{2n} (-1)^j k_j(\mathbf{x}) X^j \quad \text{with } k_j(\mathbf{x}) = k_{2n-j}(\mathbf{x}).$$

If we write

$$\begin{aligned} k(\mathbf{x}) &= (k_0(\mathbf{x}), k_1(\mathbf{x}), \dots, k_n(\mathbf{x})), \\ \lambda_F(p^2) &= \lambda_F(T_n(p^2), T_{n-1}(p^2), \dots, T_0(p^2)), \end{aligned}$$

then the following relations give  $k(\alpha)$  as a linear function of  $\lambda_F(p^2)$ :

$$k(\alpha) = V(\alpha) P_n \quad \text{and} \quad \lambda_F(p^2) = p^{nk-(n)} V(\alpha) K_n(p^2).$$

**PROOF.** We first note the effect of the slash normalization on the Satake parameters. For any modular form  $F \in M_n^k$ ,

$$F|_k T_n(p^2) = F|_k p\Gamma_n = (p^2)^{nk-(n)} \det(pI_n)^{-k} F = p^{nk-2(n)} F.$$

Thus, the equality  $\lambda_F(T_n(p^2)) = p^{nk-2(n)}$  is independent of  $F$ . Since  $T_n(p^2) = p\Gamma_n$  consists of one coset, we may compute  $\Omega(T_n(p^2)) = p^{-(n)} x_0^2 x_1 \dots x_n$  directly from the definition. Therefore, the Satake parameters of an eigenform  $F$  satisfy  $\alpha_0^2 \alpha_1 \dots \alpha_n = p^{nk-(n)}$ .

In the defining relation of the Krieg matrix,  $K_n(p^2)$ ,

$$\Omega(T_n(p^2), \dots, T_0(p^2)) = ([x_0^2 x^{(1,1,\dots,1)}], [x_0^2 x^{(2,1,\dots,1)}], \dots, [x_0^2 x^{(2,2,\dots,2,2)}]) \times K_n(p^2),$$

we may substitute the Satake parameters  $\mathbf{x} := \alpha$  to obtain

$$\lambda_F(p^2) = p^{nk-(n)} V(\alpha) K_n(p^2).$$

The other relation,  $k(\mathbf{x}) = V(\mathbf{x}) P_n$ , is combinatorial; for example, when  $\ell$  is even, the coefficient of  $X^\ell$  in  $\prod_{i=1}^n (1 - (x_i + x_i^{-1})X + X^2)$  is

$$V_\ell(\mathbf{x}) + \binom{n - (\ell - 2)}{1} V_{\ell-2}(\mathbf{x}) + \binom{n - (\ell - 4)}{2} V_{\ell-4}(\mathbf{x}) + \dots \quad \square$$

With these formulae from the Krieg matrix we may compute the Euler 2-factors in Table 2 from the Hecke eigenvalues in Table 1. As a check on computations, one may use the following relation.

**THEOREM 3.4.** For  $n \geq 2$ , the following relation holds in  $\overline{\mathcal{H}}_{n,p}$ :

$$T(p)^2 = T_0(p^2) + (1 + p)T_1(p^2) + (1 + p)(1 + p^2)T_2(p^2) + \dots + (1 + p) \dots (1 + p^n)T_n(p^2).$$

**PROOF.** This is Proposition 5.1 in Hafner–Walling [7], rewritten in Andrianov’s notation. □

### 4. Computing Duke–Imamoglu–Ikeda lifts

Let  $n, k$  and  $(n + k)/2$  be even. Let  $\phi \in S_1^k$  be an eigenform. The Ikeda lift of  $\phi$  to  $I_n(\phi) \in S_n^{(n+k)/2}$  is described in [9]. Let  $\psi \in S_1^{(k+1)/2}(\Gamma_0(4))^+$  be a form corresponding to  $\phi$  under the Shimura correspondence. (In this paper we always normalize  $\psi$  so that the leading coefficient of its Fourier expansion is one and thereby give meaning to equations such as  $I_4(\phi_{16}) = -168 G_{10}$ . This somewhat arbitrary choice aids the reproduction of our results and the debugging of code.)

For a fixed  $T \in \mathcal{X}_n$ , we now describe how to calculate the Fourier coefficient  $a(T; I_n(\phi))$ . For each prime  $p$  that divides  $\det(2T)$ , along with the prime two, calculate the genus symbol  $G_p$  of  $T$ , see [3]. Using the recursion worked out by Katsurada [10], the polynomial  $F_p$  corresponding to this genus symbol is computed. We refer the reader to [9] or [10] for the definition of the  $F_p$  polynomial. We thank O. King for giving us his LISP program that computes  $F_p$  when the genus symbol  $G_p$  is given as input, see [13].

Use the rational function

$$\tilde{F}_p(x) = x^{-(\deg F_p)/2} F_p(p^{-(n+1)/2}x)$$

with the property that  $\tilde{F}_p(x^{-1}) = \tilde{F}_p(x)$ . Define

$$b_p = p^{-(k-1)/2} \left( a(p; \phi) + \sqrt{a(p; \phi)^2 - 4p^{k-1}} \right) / 2$$

so that we have  $(1 - p^{(k-1)/2}b_p x)(1 - p^{(k-1)/2}b_p^{-1}x) = 1 - a(p; \phi)x + p^{k-1}x^2$ . Let

$$c_T = \prod_{\text{primes } p: p|\det(2T) \text{ or } p=2} \tilde{F}_p(b_p).$$

Define a function  $\delta : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{Z} \setminus \{0\}$  by

$$\delta(y) = \begin{cases} y & \text{if } y \text{ is square free and } y \equiv 1 \pmod{4}, \\ 4y & \text{if } y \text{ is square free and } y \not\equiv 1 \pmod{4}, \\ \delta(\bar{y}) & \text{where } \bar{y} \text{ is square free and } y = \ell^2 \bar{y} \text{ for some } \ell \in \mathbb{Z}. \end{cases}$$

The Ikeda Fourier coefficient at  $T$  is

$$a(T; I_n(\phi)) = a(|\delta((-1)^{n/2} \det(2T))|; \psi) \left( \frac{\det(2T)}{|\delta((-1)^{n/2} \det(2T))|} \right)^{(k-1)/4} c_T.$$

### 5. Computing Miyawaki–Ikeda lifts

The Witt maps are given by restriction to the reducible locus, see [22].

**PROPOSITION 5.1.** *For  $i + j = n$  there are homomorphisms  $\psi_{ij}^* : S_n^k \rightarrow S_i^k \otimes S_j^k$  for  $\psi_{ij} : \mathbb{H}_i \times \mathbb{H}_j \rightarrow \mathbb{H}_n$  defined by  $\psi_{ij}(\Omega_1, \Omega_2) = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$ . Furthermore, if  $n = 2i$  we have  $\psi_{ii}^* : S_n^k \rightarrow \text{Sym}(S_i^k \otimes S_i^k)$ .*

The following is essentially due to Ozeki, see [17].

**PROPOSITION 5.2.** *For all  $f \in S_n^k$ ,*

$$a(T_1 \otimes T_2; \psi_{ij}^* f) = \sum_{T \in \mathcal{X}_n : \pi_{i \times i}^{\text{upper}}(T) = T_1, \pi_{j \times j}^{\text{lower}}(T) = T_2} a(T; f).$$

The Fourier coefficients of Witt images are typically calculated by the enumeration of  $T$  with fixed  $T_1$  and  $T_2$ ; see [18, p. 210] for a specific example.

In the next proposition we use the generators of the ring  $M_2 = \mathbb{C}[E_4, E_6, \chi_{10}, \psi_{12}]$  given by Igusa. We refer to [18] for the definition of the various eigenforms  $f_i$  and for the normalization of Fourier coefficients as well. We let root lattices  $A_4, D_4$ , and so on, stand for any Gram matrix of the corresponding lattice, see [3].

**PROPOSITION 5.3.** *In  $S_4^{14} \otimes S_2^{14}$  we have*

$$\psi_{42}^* I_6(\phi_{22}) = 1848 f_9 \otimes \chi_{10} E_4.$$

Here  $f_9$  is the element of a Hecke eigenbasis that is not an Ikeda lift, see [18].

**PROOF.** The space  $S_2^{14}$  is one dimensional, spanned by  $\chi_{10} E_4$  and determined by the single Fourier coefficient  $a(\frac{1}{2} A_2)$ . The space  $S_4^{14}$  is three dimensional, spanned by  $f_9$  and two Ikeda lifts  $I_4(\phi_{24}^\pm)$  and determined by the three Fourier coefficients  $a(\frac{1}{2} D_4)$ ,  $a(\frac{1}{2} A_4)$  and  $a(\frac{1}{2}(A_1 \oplus A_3))$ , see [18, p. 214]. Hence, the element  $\psi_{42}^* I_6(\phi_{22})$  of  $S_4^{14} \otimes S_2^{14}$  is determined by the first three of the following four Fourier coefficients. The fourth coefficient is redundant but serves as a check. Computing as in Proposition 5.2 we have

$$\begin{aligned} & a(\tfrac{1}{2} D_4 \otimes \tfrac{1}{2} A_2; \psi_{42}^* I_6(\phi_{22})) \\ &= a(\tfrac{1}{2} D_4 \oplus \tfrac{1}{2} A_2; I_6(\phi_{22})) + 72a(\tfrac{1}{2} D_6; I_6(\phi_{22})) + 192a(\tfrac{1}{2} E_6; I_6(\phi_{22})) \\ &= 2784 + 72(10) + 192(1) = 3696 = 1848(2), \\ & a(\tfrac{1}{2} A_4 \otimes \tfrac{1}{2} A_2; \psi_{42}^* I_6(\phi_{22})) \\ &= a(\tfrac{1}{2} A_4 \oplus \tfrac{1}{2} A_2; I_6(\phi_{22})) + 120a(\tfrac{1}{2} D_6; I_6(\phi_{22})) + 240a(\tfrac{1}{2} E_6; I_6(\phi_{22})) \\ &\quad + 30a(\tfrac{1}{2} A_6; I_6(\phi_{22})) \\ &= -8040 + 120(10) + 240(1) + 30(-88) = -9240 = 1848(-5), \\ & a(\tfrac{1}{2}(A_2 \oplus A_2) \otimes \tfrac{1}{2} A_2; \psi_{42}^* I_6(\phi_{22})) \\ &= a(\tfrac{1}{2}(A_2 \oplus A_2 \oplus A_2); I_6(\phi_{22})) + 432a(\tfrac{1}{2} D_6; I_6(\phi_{22})) + 252a(\tfrac{1}{2} E_6; I_6(\phi_{22})) \\ &\quad + 216a(\tfrac{1}{2} A_6; I_6(\phi_{22})) + 24a(\tfrac{1}{2} D_4 \oplus \tfrac{1}{2} A_2; I_6(\phi_{22})) \\ &\quad + 36a(\tfrac{1}{2} A_4 \oplus \tfrac{1}{2} A_2; I_6(\phi_{22})) \\ &= -128\,844 + 432(10) + 252(1) + 216(-88) \\ &\quad + 24(2784) + 36(-8040) = -365\,904 = 1848(-198). \end{aligned}$$

$$\begin{aligned}
 & a\left(\frac{1}{2}(A_1 \oplus A_3) \otimes \frac{1}{2}A_2; \psi_{42}^* I_6(\phi_{22})\right) \\
 &= a\left(\frac{1}{2}(A_1 \oplus A_3 \oplus A_2); I_6(\phi_{22})\right) + 264a\left(\frac{1}{2}D_6; I_6(\phi_{22})\right) + 288a\left(\frac{1}{2}E_6; I_6(\phi_{22})\right) \\
 &\quad + 96a\left(\frac{1}{2}A_6; I_6(\phi_{22})\right) + 90a\left(\frac{1}{2}D_5 \oplus \frac{1}{2}A_1; I_6(\phi_{22})\right) \\
 &\quad + 24a\left(\frac{1}{2}A_5 \oplus \frac{1}{2}A_1; I_6(\phi_{22})\right) + 6a\left(\frac{1}{2}A_3 \oplus \frac{1}{2}A_3; I_6(\phi_{22})\right) \\
 &= -54 \cdot 120 + 264(10) + 288(1) + 96(-88) + 90(-132) \\
 &\quad + 24(736) + 6(17600) \\
 &= 51\,744 = 1848(28).
 \end{aligned}$$

The Fourier coefficients  $a(\cdot; I_6(\phi_{22}))$  are computed according to Section 4. Since  $a(\frac{1}{2}D_4; f_9) = 2$ ,  $a(\frac{1}{2}A_4; f_9) = -5$ ,  $a(\frac{1}{2}(A_2 \oplus A_2); f_9) = -198$  and  $a(\frac{1}{2}(A_1 \oplus A_3); f_9) = 28$ , see [18, Table 2, p. 218], and  $a(\frac{1}{2}A_2; \chi_{10}E_4) = 1$  we have the result.  $\square$

**COROLLARY 5.4.** *The eigenform  $f_9 \in S_4^{14}$  is a Miyawaki lift, namely*

$$M_4(\phi_{22}, \chi_{10}E_4) = 1848\langle \chi_{10}E_4, \chi_{10}E_4 \rangle f_9.$$

**PROOF.** By definition, we have  $M_4(\phi_{22}, \chi_{10}E_4) = \langle \psi_{42}^* I_6(\phi_{22}), (\chi_{10}E_4)^c \rangle$ . Since  $\chi_{10}E_4$  has Fourier coefficients in  $\mathbb{Q}$ , a totally real field, we have  $(\chi_{10}E_4)^c = \chi_{10}E_4$ . By Proposition 5.3, we conclude  $\langle \psi_{42}^* I_6(\phi_{22}), (\chi_{10}E_4)^c \rangle = \langle 1848f_9 \otimes \chi_{10}E_4, \chi_{10}E_4 \rangle = 1848\langle \chi_{10}E_4, \chi_{10}E_4 \rangle f_9$ .  $\square$

A similar but more involved computation gives the following proposition and corollary. Since  $S_2^{16}$  is two dimensional, we expect to obtain two Miyawaki lifts from the Ikeda lift  $I_6(\phi_{26}) \in S_6^{16}$ .

**PROPOSITION 5.5.** *Let  $\omega^\pm = 48(1\,643\,168 \pm 2141\gamma)/200\,620\,543$  for  $\gamma = \sqrt{51\,349}$ . In  $S_4^{16} \otimes S_2^{16}$  we have*

$$\psi_{42}^* I_6(\phi_{26}) = \omega^- h_5 \otimes I_2(\phi_{30}^+) + \omega^+ h_6 \otimes I_2(\phi_{30}^-).$$

**COROLLARY 5.6.** *The eigenforms  $h_5, h_6 \in S_4^{16}$  are Miyawaki lifts,  $M_4(\phi_{26}, I_2(\phi_{30}^+)) = \omega^- \langle I_2(\phi_{30}^+), I_2(\phi_{30}^+) \rangle h_5$  and  $M_4(\phi_{26}, I_2(\phi_{30}^-)) = \omega^+ \langle I_2(\phi_{30}^-), I_2(\phi_{30}^-) \rangle h_6$ .*

### 6. Final remarks

Hecke eigenbases of  $S_4^k$  for even  $k \leq 16$  were given in [18]. For odd  $k \leq 16$ , the  $S_4^k$  are trivial, see [4] for some of these results. We survey these standard  $L$ -functions in degree four. The Schottky form  $J_8$  spans  $S_4^8$  and  $I_4(\Delta_{12}) = -120J_8$  so that

$$L(J_8, s, \text{st}) = \zeta(s)L(\Delta_{12}, s + 4)L(\Delta_{12}, s + 5)L(\Delta_{12}, s + 6)L(\Delta_{12}, s + 7).$$

The form  $G_{10}$  spans  $S_4^{10}$  and  $I_4(\phi_{16}) = -168G_{10}$  so that

$$L(G_{10}, s, \text{st}) = \zeta(s)L(\phi_{16}, s + 6)L(\phi_{16}, s + 7)L(\phi_{16}, s + 8)L(\phi_{16}, s + 9).$$

The space  $S_4^{12} = \mathbb{C}f_5 + \mathbb{C}f_6$  has  $M_4(\phi_{18}, I_2(\phi_{22}))/\langle I_2(\phi_{22}), I_2(\phi_{22}) \rangle = 144f_6$  and  $I_4(\phi_{20}) = 360f_5$ . As given by Ikeda [8]:

$$L(f_5, s, st) = \zeta(s)L(\phi_{20}, s + 8)L(\phi_{20}, s + 9)L(\phi_{20}, s + 10)L(\phi_{20}, s + 11),$$

$$L(f_6, s, st) = \zeta(s)L(\phi_{18}, s + 8)L(\phi_{18}, s + 9)L(\phi_{22}, s + 10)L(\phi_{22}, s + 11).$$

The space  $S_4^{14} = \mathbb{C}f_7 + \mathbb{C}f_8 + \mathbb{C}f_9$  has  $M_4(\phi_{22}, I_2(\phi_{26}))/\langle I_2(\phi_{26}), I_2(\phi_{26}) \rangle = 1848f_9$ , and the two Ikeda lifts  $I_4(\phi_{24}^+) = 12f_7$  and  $I_4(\phi_{24}^-) = 12f_8$ :

$$L(f_7, s, st) = \zeta(s)L(\phi_{24}^+, s + 10)L(\phi_{24}^+, s + 11)L(\phi_{24}^+, s + 12)L(\phi_{24}^+, s + 13),$$

$$L(f_8, s, st) = \zeta(s)L(\phi_{24}^-, s + 10)L(\phi_{24}^-, s + 11)L(\phi_{24}^-, s + 12)L(\phi_{24}^-, s + 13),$$

$$L(f_9, s, st) = \zeta(s)L(\phi_{22}, s + 10)L(\phi_{22}, s + 11)L(\phi_{26}, s + 12)L(\phi_{26}, s + 13).$$

The  $L$ -functions for weight 16, insofar as they are known, were given in the introduction. Recall that the Euler 2-factor of  $h_3$  is given by

$$(1 - X)Q_2(\phi_{28}^+, 2^{-14}X)Q_2(\phi_{28}^+, 2^{-15}X)\left(1 + \frac{9}{512}X + \frac{1601}{2048}X^2 + \frac{9}{512}X^3 + X^4\right).$$

In the hope of eventually locating this quartic among Euler factors of lower degree, we also give the spinor Euler factor. If

$$(1 - X)\left(1 + \frac{9}{512}X + \frac{1601}{2048}X^2 + \frac{9}{512}X^3 + X^4\right)$$

were the standard Euler 2-factors of a degree-two eigenform of weight  $k$ , then the spinor Euler 2-factor would be

$$1 \pm 2^{k-7} \cdot 3 \cdot 5^2 X + 2^{2k-12} \cdot 5 \cdot 7 \cdot 29 X^2 \pm 2^{3k-10} \cdot 3 \cdot 5^2 X^3 + 2^{4k-6} X^4$$

$$= (1 \pm 2^{k-8} \cdot 3(5^2 \pm \sqrt{641})X + 2^{2k-3} X^2)$$

$$\times (1 \pm 2^{k-8} \cdot 3(5^2 \mp \sqrt{641})X + 2^{2k-3} X^2).$$

Finally, Katsurada has shown that Ikeda lifts have non-Ikeda lifts as congruence neighbors when numerators of certain normalized values of  $L$ -functions are divisible by large primes. In our examples, we may verify the following: Let  $\mathcal{O}$  be the ring of integers of  $\mathbb{Q}(\beta)$ . The eigenvalues of the Ikeda lift  $h_1$  and the non-Ikeda lift  $h_3$  have differences  $\lambda_{h_3}(T) - \lambda_{h_1}(T) \in 3\mathcal{O} \cap a_5\mathcal{O} \cap a_7\mathcal{O}$  for  $T = T(2), T_0(4), T_1(4), T_2(4)$  and  $T_3(4)$ , where  $a_5$  of norm 5 and  $a_7$  of norm 7 are given, respectively, by 3 564 298 180 345 653 559 251 811 701 533 100 061 949 380 892 601 299 566 199 385 611 273 529 273 539 155 883 991 421 + 26 413 805 826 710 358 011 546 969 364 051 514 900 136 663 884 028 934 590 117 770 380 976 171 799 102 958 519 902 $\beta$  and 3 270 594 572 157 720 143 694 411 010 173 434 026 798 776 896 584 037 998 807 223 353 400 758 612 485 308 343 317 508 716 + 24 237 267 926 469 934 357 153 909 329 439 375 246 810 596 424 836 893 715 023 944 129 121 867 151 461 567 374 501 869 $\beta$ .



## Acknowledgements

We thank S. Hayashida for helpful discussions. We thank B. Heim for pointing out that Miyawaki lifts are currently only known to be eigenforms for the even part of the Hecke algebra. We thank O. King for giving us a nice LISP program to compute the  $F_p$  polynomials via Katsurada's recursion formulae. We thank H. Katsurada for bringing his work [11] to our attention.

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