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A BOUND FOR THE MODULI OF THE ZEROS OF POLYNOMIALS

BY
Q. I. RAHMAN

The following theorem is due to Walsh [2]. For another proof see [1].

THEOREM A. *All the zeros of the polynomial* $p(z) = a_0+a_1z+\cdots+a_{n-1}z^{n-1}+z^n$ *lie on the disk*

$$|z+\tfrac{1}{2}a_{n-1}| \leq \tfrac{1}{2}\,|a_{n-1}|+M,$$

where $M=\sum_{j=2}^{n} |a_{n-j}|^{1/j}$.

We prove

THEOREM 1. *All the zeros of the polynomial* $p(z) = a_0+a_1z+\cdots+a_{n-1}z^{n-1}+z^n$ *lie on the disk*

$$D\colon |z+\tfrac{1}{2}a_{n-1}| \leq \tfrac{1}{2}\,|a_{n-1}|+\alpha M,$$

where

(i) $\alpha=0$ *if* $p(z)$ *is of the form* $a_{n-1}z^{n-1}+z^n$ *and*

(ii) $\alpha=\max_{2\leq j\leq n}(M^{-1}|a_{n-j}|^{1/j})^{(j-1)/j}$ *if* $p(z)$ *is not of the form* $a_{n-1}z^{n-1}+z^n$.

Proof. The part of the theorem dealing with polynomials of the form $a_{n-1}z^{n-1}+z^n$ is evident. So let us suppose that the coefficients $a_0, a_1, \ldots, a_{n-2}$ are not all zero. This implies that α is a positive number not exceeding 1. Now if

$$|z+\tfrac{1}{2}a_{n-1}|>\tfrac{1}{2}\,|a_{n-1}|+\alpha M$$

then

$$|z| > \alpha M \geq \alpha^{-1/(j-1)}|a_{n-j}|^{1/j} \quad \text{for } j = 2, 3, \ldots, n-2.$$

Hence for z lying outside the disk D and $j=2, 3, \ldots, n-2$

$$|a_{n-j}|\,|z|^{n-j} < \alpha\,|a_{n-j}|^{1/j}|z|^{n-1},$$

and

$$\left|\tfrac{1}{2}a_{n-1}z^{n-1}+\sum_{j=2}^{n} a_{n-j}z^{n-j}\right| < (\tfrac{1}{2}\,|a_{n-1}|+\alpha M)\,|z|^{n-1}$$
$$< |z+\tfrac{1}{2}a_{n-1}|\,|z|^{n-1}$$
$$= |z^n+\tfrac{1}{2}a_{n-1}z^{n-1}|.$$

Consequently, if $z\notin D$, then

$$|p(z)| = \left|z^n+a_{n-1}z^{n-1}+\sum_{j=2}^{n} a_{n-j}z^{n-j}\right|$$
$$\geq |z^n+\tfrac{1}{2}a_{n-1}z^{n-1}|-\left|\tfrac{1}{2}a_{n-1}z^{n-1}+\sum_{j=2}^{n} a_{n-j}z^{n-j}\right|$$
$$> 0,$$

i.e. $p(z)$ cannot vanish.

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References

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2. J. L. Walsh, *An inequality for the roots of an algebraic equation*, Ann. of Math. **25** (1924), 285–286.

UNIVERSITÉ DE MONTRÉAL,
MONTRÉAL, QUÉBEC

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