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BOCHNER'S THEOREM AND THE HAUSDORFF MOMENT THEOREM ON FOUNDATION TOPOLOGICAL SEMIGROUPS

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Introduction. One of the most basic theorems in harmonic analysis on locally compact commutative groups is Bochner's theorem (see [**16**, p. 19]). This theorem characterizes the positive definite functions. In 1971, R. Lindhal and P. H. Maserick proved a version of Bochner's theorem for discrete commutative semigroups with identity and with an involution * (see [**13**]). Later, in 1980, C. Berg and P. H. Maserick in [**6**] generalized this theorem for exponentially bounded positive definite functions on discrete commutative semigroups with identity and with an involution *. In this work we develop these results, and also the Hausdorff moment theorem, for an extensive class of topological semigroups, the so-called "foundation topological semigroups". We shall give examples to show that these theorems do not extend in the obvious way to general locally compact topological semigroups.

In Section 2 of this paper, we prove a version of Bochner's theorem for the w-bounded, continuous, positive definite functions on a foundation topological semigroup with a continuous involution and a Borel measurable weight function w. Next, in Section 3, we study the spectral theory of operator-valued transformations on weighted foundation topological semigroups. We devote Section 4 to a brief discussion on the semisimplicity of some certain commutative weighted measure algebras. Finally, in Section 5, we develop the Hausdorff moment theorem on foundation topological semigroups.

1. Preliminaries.

1.1. *Foundation topological semigroups.* On a locally compact semigroup S, the collection $\tilde{L}(S)$ of all $\mu \in M(S)$ (the algebra of the bounded Radon measures on S), for which the translations $x \to \bar{x}*|\mu|$ and $x \to |\mu|*\bar{x}$ of $|\mu|$ are weakly continuous, form an interesting analogue of $L^1(G)$ of a locally compact group G (c.f. [**2**, **3**, **4**]). A topological semigroup S is said to be *foundation* if S coincides with the closure of

$$\cup\{\mathrm{supp}(\mu):\mu \in \tilde{L}(S)\}.$$

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As well as from topological groups and discrete semigroups there are many examples of foundation semigroups, for example, S_1, the semigroup with underlying space the subset $[1, 3] \times [1, 3]$, of $\mathbf{R}^2$ and multiplication defined as follows:

$$(a, b)(c, d) := (\min(ac, 3), \min(ad + b, 3))$$

for all $a, b, c, d \in [1, 3]$, and, S_2 with the underlying space also $[1, 3] \times [1, 3]$, but multiplication defined by

$$(a, b)(c, d) := (\min(ac, 3), \min(bc + d, 3))$$

for all $a, b, c, d \in [1, 3]$, whenever both S_1 and S_2 are endowed with restriction topology of $\mathbf{R}^2$ are foundation semigroups. For more details see [**18**]. It is also easy to see that $S_3 = [0, 1]$ with the restriction topology of $\mathbf{R}$ and multiplication defined by $xy := \min(x + y, 1)$ for all $x, y \in [0, 1]$ is a foundation semigroup. For further examples we refer to [**18**] and the appendix B of [**17**].

1.2. *Weighted measure algebras* $M(S, w)$ *and* $\widetilde{L}(S, w)$. A real-valued function w on a semigroup S is said to be a *weight function* if $w(x) \geqq 0$ and $w(xy) \leqq w(x)w(y)$, for all $x, y \in S$. A complex-valued function f on S is said to be *w-bounded* if there exists a positive constant k such that

$$|f(x)| \leqq kw(x) \quad \text{for all } x \in S.$$

In Theorem 4.6 of our earlier paper [**12**] we proved that for a locally compact semigroup S with a Borel measurable weight function w such that $w(x) > 0$ for all $x \in S$, w and $1/w$ are locally bounded (that is; w and $1/w$ are bounded on each compact subset of S), $M(S, w)$ the space of all regular Borel measures μ on S such that $w|\mu| \in M(S)$ with the norm,

$$\|\mu\|_w = \int_S w d|\mu|,$$

can be identified with the dual of the Banach space $\mathscr{C}_0(S, w)$ the space of all Borel measurable functions f on S such that $f/w \in \mathscr{C}_0(S)$ (see Definition 1.3) with the norm,

$$\|f\| = \sup\{ |f/w(x)| : x \in S\},$$

via pairing

$$\langle \mu, f \rangle = \mu(f) = \int_S f(x) d\mu(x)$$

for every $\mu \in M(S, w)$ and $f \in \mathscr{C}_0(S, w)$. It was also proved that $M(S, w)$ with the convolution product * given by

$$\int_S f(x) d(\mu * \nu)(x) = \int_S \int_S f(xy) d\mu(x) d\nu(y) \qquad (\mu, \nu \in M(S, w), f \in \mathscr{C}_0(S, w))$$

becomes a Banach algebra. Furthermore, for every w-bounded Borel measurable function f and every μ, ν in $M(S, w)$ we have

$$(1) \qquad \int_S f(x)d(\mu * \nu)(x) = \int_S \int_S f(xy)d\mu(x)d\nu(y).$$

Moreover, $\widetilde{L}(S, w)$, the space of all measures μ in $M(S, w)$ such that $w|\mu| \in \widetilde{L}(S)$, is a two-sided ideal of $M(S, w)$ which is also solid in $M(S, w)$, in the sense that; given $\mu \in \widetilde{L}(S, w)$ and $\nu \in M(S, w)$ such that ν is locally absolutely continuous with respect to μ (that is; $|\nu|(F) = 0$ for every compact subset F of S with $|\mu|(F) = 0$), then $\nu \in \widetilde{L}(S, w)$. The algebra $M_K(S)$, the space of all measures in $M(S)$ with compact supports, and $\widetilde{L}_K(S)(= M_K(S) \cap \widetilde{L}(S))$ are w norm ($\|\circ\|_w$) dense in $M(S, w)$ and $\widetilde{L}(S, w)$, respectively. We also recall from Lemma 4.8 of [**12**] that if S is foundation and f is a w-bounded continuous function on S such that

$$\int_S f(x)d\mu(x) = 0 \quad \text{for all } \mu \in \widetilde{L}(S, w),$$

then f vanishes identically on S.

Throughout this paper we shall assume that w is a Borel measurable weight function such that $w(x) > 0$ for all $x \in S$, and that w and $1/w$ are locally bounded.

Definition 1.3. A (not necessarily bounded) complex-valued function χ on a topological semigroup S is said to be a *semicharacter* if

$$\chi(xy) = \chi(x)\chi(y) \quad \text{for every } x, y \in S.$$

If S has an involution $*$ (a map $*: S \to S$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$), and χ is a semicharacter on S such that $\chi(x^*) = \overline{\chi(x)}$ for all $x \in S$, then χ is said to be a *$*$-semicharacter*. We denote the set of all bounded and continuous semicharacters on S by $\hat{S}$. It is clear that if $\chi \in \hat{S}$ then

$$|\chi(x)| \leqq 1 \quad \text{for all } x \in S.$$

The space of all $*$-semicharacters in $\hat{S}$ will be denoted by S^*. A semicharacter χ on S is said to be *positive* if $\chi(x) \geqq 0$ for all $x \in S$. We denote by $\hat{S}_+$ the space of all positive semicharacters in $\hat{S}$. If S has a weight function w, then the spaces of all w-bounded and continuous semicharacters on S will be denoted by Γ_w. It is evident that if $\chi \in \Gamma_w$ then

$$|\chi(x)| \leqq w(x) \quad \text{for all } x \in S.$$

If S has an involution $*$, then we denote by Γ_w^* the space of all $*$-semicharacters in Γ_w. Finally, for a locally compact Hausdorff space X, we denote by $C(X)$, $C_0(X)$, and $C_{00}(X)$, the algebras of all continuous complex-valued functions on X that are bounded, vanish at infinity, or have compact support, respectively.

2. The Bochner theorem for w-bounded positive definite functions on foundation semigroups with a weight w. In this section, among other things, we shall prove a version of Bochner's theorem for the w-bounded continuous positive definite functions on a commutative locally compact foundation semigroup S with identity and with a continuous involution $*$ and with a Borel measurable weight function w such that $w(x^*) = w(x)$ for all $x \in S$.

Definition 2.1. Let S be a semigroup with an involution $*$. A complex-valued function ϕ on S is said to be *positive definite* if and only if

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \phi(x_i x_j^*) \geqq 0$$

for every finite subset $\{x_1, \dots, x_n\}$ of S and $\{c_1, \dots, c_n\}$ of $\mathbf{C}$.

The proof of next result is standard.

LEMMA 2.2. *Let S be a semigroup with identity* 1 *and with an involution $*$. Then each positive definite function ϕ on S satisfies the following properties*

(i) $\phi(s^*) = \overline{\phi(s)}$, *and* $\phi(ss^*) \geqq 0$, *for all* $s \in S$,

(ii) $|\phi(st^*)|^2 \leqq \phi(ss^*)\phi(tt^*)$, *for all* $s, t \in S$.

In particular

$$|\phi(s)|^2 \leqq \phi(1)\phi(ss^*), \quad \textit{for all } s \in S,$$

which shows that ϕ is identically zero whenever $\phi(1) = 0$.

We now turn to a central lemma concerning the w-bounded continuous positive definite functions on a topological semigroup with a weight function w.

LEMMA 2.3. *Let S be a topological semigroup with identity and with a continuous involution $*$. Suppose that w is a weight function on S such that $w(x^*) = w(x)$ for all $x \in S$. If ϕ is a w-bounded, continuous, and positive definite function on S, then the formula*

$$T_\phi(\mu) = \int_S \phi(x) d\mu(x) \quad (\mu \in M(S, w)),$$

defines a bounded positive functional T_ϕ on the Banach $*$-algebra $M(S, w)$ for which

(i) $T_\phi(\mu^*) = \overline{T_\phi(\mu)} (\mu \in M(S, w))$, *and*

(ii) $|T_\phi(\mu)|^2 \leqq k T_\phi(\mu * \mu^*) \quad (\mu \in M(S, w))$,

for some positive constant k.

Proof. Without loss of generality we may assume that

$$|\phi(x)| \leqq w(x) \quad \text{for all } x \in S.$$

Then

$$|T_\phi(\mu)| \leqq \|\mu\|_w \quad \text{for every } \mu \text{ in } M(S, w).$$

Thus, T_ϕ defines a bounded linear functional on $M(S, w)$. By the use of (i) of Lemma 2.2, we infer that

$$T_\phi(\mu^*) = \overline{T_\phi(\mu)} \quad \text{for every } \mu \in M(S, w).$$

To prove that T_ϕ is a positive functional, that is;

$$T_\phi(\nu * \nu^*) \geqq 0 \quad \text{for all } \nu \in M(S, w),$$

we only need to establish this inequality for all $\nu \in M(S, w)$ which are of the form $f \cdot \mu$ where $f \in C_{00}(S)$ and μ is a positive measure in $M(S, w)$, by the w-norm density of $M_K(S)$ in $M(S, w)$. Therefore we consider $\nu \in M(S, w)$ with $\nu = f \cdot \mu$ where $f \in C_{00}(S)$ and μ is a positive measure in $M(S, w)$. Since ϕ is w-bounded, from (1) it follows that

$$T_\phi(\nu * \nu^*) = \int_S \int_S \phi(xy^*) f(x) \overline{f(y)} d\mu(x) d\mu(y).$$

For simplicity we denote the function $(x, y) \to \phi(xy^*) f(x) \overline{f(y)}$ by h. We then have

(2) $\quad h(x, y) = \overline{h(y, x)}, \quad (x, y \in S).$

Let K denote the support of f. Then h is continuous on $K \times K$. It is also obvious that $\operatorname{supp}(h) \subseteq K \times K$. We now prove that for every positive ϵ, K can be partitioned into disjoint Borel sets $E_1, \ldots, E_l$ such that E_i contains a point x_i with

(3) $\quad |h(x, y) - h(x_i, y_j)| < \epsilon$

for all $x \in E_i$ and $y \in E_j$, for $1 \leqq i, j \leqq l$. To establish this we first prove that for each $x \in K$ there exists an open neighbourhood U_x of x such that

(4) $\quad |h(x', y) - h(x, y)| < \dfrac{\epsilon}{2}$

for all $x' \in U_x$ and $y \in K$. For every $x, y \in K$, by the continuity of h at (x, y), there exist open neighbourhoods $U_{(x,y)}$ of x and $V_{(x,y)}$ of y such that

(5) $\quad |h(x', y') - h(x, y)| < \dfrac{\epsilon}{4}$

for all $x' \in U_{(x,y)}$ and $y' \in V_{(x,y)}$. It is evident that for each $x \in S$ the collection $\{V_{(x,y)}:y \in K\}$ is an open cover of K, so by the compactness of K there exist $y_1, \ldots, y_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} V_{(x,y_i)}.$$

Let

$$U_x = \bigcap_{i=1}^{n} U_{(x,y_i)}.$$

Then U_x is an open neighbourhood of x. Suppose that $y \in K$, then

$$y \in V_{(x,y_j)} \quad \text{for some } j(1 \leqq j \leqq n).$$

Now if $x' \in U_x$, then $x' \in U_{(x,y_j)}$, and since $x \in U_{(x,y_j)}$ and $y_j \in V_{(x,y_j)}$, with the aid of (5) we have

$$|h(x, y) - h(x, y_j)| < \frac{\epsilon}{4}, \text{ and}$$

$$|h(x', y) - h(x, y_j)| < \frac{\epsilon}{4}.$$

Therefore

$$|h(x', y) - h(x, y)| < \frac{\epsilon}{2},$$

and so (4) is established. From the compactness of K and the fact that the class $\{U_x:x \in K\}$ forms an open cover for K we infer that there exist $x_1, \ldots, x_m \in K$ such that

$$K \subseteq \bigcup_{k=1}^{m} U_{x_k}.$$

Now we can easily find a subset $\{x_{k_1}, \ldots, x_{k_l}\}$ of the set $\{x_1, \ldots, x_m\}$ and a collection $\{E_1, \ldots, E_l\}$ of Borel subsets U_{x_i}, s such that $x_{k_i} \in E_i$ for $i = 1, \ldots, l$, E_i disjoint from E_j for $i \neq j$, and

$$K = \bigcup_{i=1}^{l} E_i.$$

Indeed, for each $i = 1, \ldots, l$ by (2) we have

$$(6) \qquad |h(x', y) - h(x, y)| < \frac{\epsilon}{2}$$

for all $x' \in E_i$ and $y \in K$. It is clear that $\{E_i \times E_j:1 \leqq i, j \leqq l\}$ forms a partition for $K \times K$. Let $(x, y) \in K \times K$. Then $(x, y) \in E_i \times E_j$ for some i, j with $1 \leqq i, j \leqq l$. By (2) and (6) we have

$$|h(x, y) - h(x_{k_i}, x_{k_j})| \leqq |h(x, y) - h(x_{k_i}, y)| + |\overline{h(y, x_{k_i})} - \overline{h(x_{k_j}, x_{k_i})}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

From this we get the desired inequality (3) by putting x_i equal to x_{k_i}, for $i = 1, \ldots, l$.

Suppose again we are given $\epsilon > 0$. Then we choose a partition $\{E_i\}_{i=1}^{l}$ for K into Borel sets such that each E_i contains a point x_i which satisfies

$$(7) \quad |f(x)\overline{f(y)}\phi(xy^*) - f(x_i)\overline{f(x_j)}\phi(x_ix_j^*)| < \frac{\epsilon}{\|\mu\|^2},$$

for all $x \in E_i$ and $y \in E_j$ $(1 \leqq i, j \leqq l)$. Since $E_i \times E_j$'s form a partition of $K \times K$, with the aid of Fubini's theorem we have

$$\int_S \int_S f(x)\overline{f(y)}\phi(xy^*)d\mu(x)d\mu(y) = \sum_{1 \leqq i,j \leqq l} \int_{E_i \times E_j} f(x)\overline{f(y)}\phi(xy^*)d(\mu \times \mu)(x, y).$$

Since by (7)

$$\sum_{1 \leqq i,j \leqq l} \int_{E_i \times E_j} |f(x)\overline{f(y)}\phi(xy^*) - f(x_i)\overline{f(x_j)}\phi(x_ix_j^*)| d(\mu \times \mu)(x, y) < \epsilon,$$

it follows that

$$\int_S \int_S f(x)\overline{f(y)}\phi(xy^*)d\mu(x)d\mu(y)$$

can be approximated by summations of the form

$$\sum_{i,j} \mu(E_i)f(x_i)\overline{\mu(E_j)f(x_j)}\phi(x_ix_j^*).$$

By using the fact that ϕ is positive definite we conclude that

$$T_\phi(\nu * \nu^*) \geqq 0,$$

and therefore T_ϕ is a positive functional.

Now for every $\mu, \nu \in M(S, w)$ we denote $T_\phi(\mu * \nu^*)$ by $\langle \mu, \nu \rangle$. Then $\langle \mu, \mu \rangle \geqq 0$ for all $\mu \in M(S, w)$, and it is also clear that $\langle \mu, \nu \rangle$ is linear in μ. Since ϕ is w-bounded, by the use of (1) and with the aid of Lemma 2.2, we can easily prove that

$$\langle \mu, \nu \rangle = \overline{\langle \nu, \mu \rangle} \quad \text{for all } \mu, \nu \in M(S, w).$$

These are just the properties of the Hilbert space inner product which are needed for the standard proof of the Cauchy-Schwarz inequality. In our case the inequality is

(8) $|\langle \mu, \nu \rangle|^2 \leqq \langle \mu, \mu \rangle \langle \nu, \nu \rangle, \quad (\mu, \nu \in M(S, w)).$

Since $\bar{1}(= \delta_1) \in M(S, w)$, from (8) it follows that

$$|T_\phi(\mu)|^2 = |\langle \mu, \bar{1} \rangle|^2 \leqq \langle \mu, \mu \rangle \langle \bar{1}, \bar{1} \rangle = \phi(1) T_\phi(\mu * \mu^*),$$

for all $\mu \in M(S, w)$. Statement (ii) is obtained by choosing $k = \phi(1)$, and so the proof is completed.

As a consequence of the above lemma we obtain the following result.

THEOREM 2.4. *Let S be a foundation topological semigroup with identity and with a continuous involution $*$. Suppose that w is a weight function on S such that $w(x) = w(x^*)$ for all $x \in S$. A complex-valued function ϕ on S is w-bounded, continuous, and positive definite if and only if there exists a w-bounded continuous $*$-representation V of S by bounded operators on a Hilbert space H such that*

$$\phi(x) = \langle V_x \xi, \xi \rangle \quad (x \in S),$$

for some $\xi \in H$.

Proof. The "if" part is trivial. To prove the converse we suppose that ϕ is a w-bounded continuous positive definite function on S. Then the formula

$$T_\phi(\mu) = \int_S \phi(x) d\mu(x) \quad (\mu \in \tilde{L}(S, w)),$$

defines a bounded positive linear functional T_ϕ on the Banach $*$-algebra $\tilde{L}(S, w)$ which also satisfies the conditions (i) and (ii) of Lemma 2.3. Therefore, by Theorem 21.24 of [**9**], there exists a cyclic $*$-representation T' of $\tilde{L}(S, w)$ by bounded operators on a Hilbert space H, with a cyclic vector ξ such that

(9) $T_\phi(\mu) = \langle T'_\mu \xi, \xi \rangle \quad (\mu \in \tilde{L}(S, w)).$

So, by Theorem 5.4 of [**12**], there exists a w-bounded and continuous $*$-representation V of S by bounded operators on H such that

(10) $\langle T'_\mu \xi, \xi \rangle = \int_S \langle V_x \xi, \xi \rangle d\mu(x) \quad (\mu \in \tilde{L}(S, w)).$

Now, from (9) and (10) it follows that

(11) $\int_S \phi(x) d\mu(x) = \int_S \langle V_x \xi, \xi \rangle d\mu(x) \quad (\mu \in \tilde{L}(S, w)).$

Since both functions ϕ and $x \to \langle V_x \xi, \xi \rangle$ are continuous, from (11) we have

$$\phi(x) = \langle V_x\xi, \xi \rangle \quad (x \in S).$$

This completes the proof of the theorem.

It seems probable that the above result is true for all topological semigroups; it is certainly true if $w \equiv 1$ (see [**13**, Theorem 3.2]).

A combination of the above theorem (with $w \equiv 1$) and Theorem 3.4 of [**12**] leads us to the following result.

THEOREM 2.5. *Let S be a foundation topological semigroup with identity and with a continuous involution *. Then the following are equivalent*:

(i) *$M(S)$ is *-semisimple*;

(ii) *$\tilde{L}(S)$ is *-semisimple*;

(iii) *the set of all bounded and continuous *-representations on S separates the points of S*;

(iv) *the set of all bounded continuous positive definite functions on S separates the points of S.*

We shall now prove a theorem which is useful in the sequel.

THEOREM 2.6. *Let S be a commutative topological semigroup with identity and with a continuous involution *. Suppose that $\hat{S}$, with the topology of uniform convergence on compact subsets of S, is a locally compact (Hausdorff) space. Let Γ be a closed subset of S^*. If $\lambda \in M(\Gamma)$ is such that*

$$\int_\Gamma \chi(x) d\lambda(\chi) = 0$$

for all $x \in S$, then $\lambda = 0$.

Proof. Suppose that $\lambda \neq 0$. For each $x \in S$ we define $\tilde{x}$ on Γ by $\tilde{x}(\chi) = \chi(x)$ for all $\chi \in \Gamma$. It is clear that A, the algebra generated by the set $\{\tilde{x}:x \in S\}$, defines a self-conjugate subalgebra of $C(\Gamma)$ which separates the points of Γ and contains the constant function $\tilde{1}$. So, A is (L^1-norm) dense in $L^1(\Gamma, |\lambda|)$, by Lemma 3.5 of [**4**]. Let f be a Borel function on Γ with $|f| = 1$, $d\lambda = fd|\lambda|$ and $d|\lambda| = \bar{f}d\lambda$. Put $\epsilon = \frac{1}{2}||\lambda||$.

Then there exists a $g \in A$ such that

$$\int_\Gamma |f - g| d|\lambda| < \epsilon.$$

Hence

$$\left| \int_\Gamma \bar{f} d\lambda - \int_\Gamma \bar{g} d\lambda \right| < \epsilon.$$

From the fact that

$$\int_\Gamma \tilde{x}(\chi) d\lambda(\chi) = 0 \quad \text{and}$$

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$$\int_{\Gamma} \overline{\tilde{x}(\chi)} d\lambda(\chi) = 0 \quad \text{for every } x \in S,$$

it follows that

$$\int_{\Gamma} \bar{g} d\lambda = 0.$$

Thus

$$\left| \int_{\Gamma} \bar{f} d\lambda \right| < \epsilon.$$

Therefore

$$2\epsilon = \|\lambda\| = \int_{\Gamma} \bar{f} d\lambda < \epsilon.$$

This contradiction shows that $\lambda = 0$, which completes the proof of the theorem.

Our next proposition shows that when S is considered without involution and Γ is chosen to be the whole of $\hat{S}$, then the above theorem may fail.

PROPOSITION 2.7. *Let S be a commutative topological semigroup such that $\hat{S}$ with the topology of uniform convergence on compact subsets of S is a locally compact (Hausdorff) space. If there exist a $\chi_0 \in S$ and a point $x_0 \in S$ such that $0 < |\chi_0(x_0)| < 1$, then there exists a nonzero measure λ in $M(\hat{S})$ which satisfies*

$$\int_{\hat{S}} \chi(x) d\lambda(\chi) = 0$$

for all $x \in S$.

Proof. It is evident that for every $\chi \in \hat{S}$ and $z \in \mathbf{C}$ with $\operatorname{Re} z > 0$, $|\chi|^z$ again belongs to $\hat{S}$. Let

$$\Omega = \{x \in \mathbf{C}: \operatorname{Re} z > 0\}$$
$$\cap \{z \in \mathbf{C}: \pi/_{\log|\chi_0(x_0)|} < \operatorname{Im} z < -\pi/_{\log|\chi_0(x_0)|}\}.$$

Then the mapping $z \to |\chi_0(x_0)|^z$ $(z \in \Omega)$ defines an analytic and one-to-one function on Ω. Now, for each $z \in \Omega$, we put $\chi_z = |\chi_0|^z$ and we prove that the function $h:z \to \chi_{z'}$ defines a homeomorphism of Ω onto the subset

$$h(\Omega) = \{\chi_z : z \in \Omega\}$$

of $\hat{S}$. It is clear that h is one-to-one. Suppose that (χ_{z_α}) is a net in $h(\Omega)$ which converges to χ_{z_0} $(z_0 \in \Omega)$ in compact open topology of $\hat{S}$. This implies that

$$|\chi_0(x_0)|^{z_\alpha} \to |\chi_0(x_0)|^{z_0}.$$

Therefore $z_\alpha \rightarrow z_0$. Conversely, suppose that (z_α) is a net in Ω which converges to $z_0 \in \Omega$. Thus (y^{z_α}) converges uniformly to y^{z_0}, for $y \in [0, 1]$. This shows that (χ_{z_α}) converges uniformly to χ_{z_0} on S. Therefore Ω is homeomorphic to $h(\Omega)$. Let C be a fixed circle inside Ω, and let μ denote the one-dimensional Lebesgue measure on C. We define λ in $M(\hat{S})$ by

$$\lambda(B) = \mu(B \cap h(C)),$$

for every Borel subset B of $\hat{S}$. It is obvious that $\lambda \neq 0$. Since for every $x \in S$, $z \rightarrow |\chi_0(x)|^z$ $(z \in \mathbf{C})$ is an entire function, we have

$$\int_{\hat{S}} \chi(x)d\lambda(\chi) = \int_{h(\Omega)} \chi(x)d\lambda(\chi) = \int_C |\chi_0(x)|^z d\mu(z) = 0,$$

for all $x \in S$. The proof is now complete.

We now include a result, which is of interest in its own right, although no direct use of it will be made.

PROPOSITION 2.8. *Let S be a commutative topological semigroup with identity and with a continuous involution * such that S^* separates the points of S, and suppose that $\hat{S}$ with the topology of uniform convergence on compact subsets of S is a locally compact (Hausdorff) space. Then S is an inverse semigroup with $x^{-1} = x^*$ $(x \in S)$, where x^{-1} is the inverse of x in S, if and only if the only $\lambda \in M(\hat{S})$ with*

$$\int_{\hat{S}} \chi(x)d\lambda(\chi) = 0$$

for all $x \in S$, is $\lambda = 0$.

Proof. Suppose that S in an inverse semigroup such that $x^* = x^{-1}$ for every $x \in S$. Since S^* separates the points of S, from Theorem 3.4 of [**1**] it follows that $S^* = \hat{S}$. Therefore if λ is any measure in $M(\hat{S})$ such that

$$\int_{\hat{S}} \chi(x)d\lambda(\chi) = 0 \quad \text{for all } x \in S,$$

then $\lambda = 0$, by Theorem 2.6. Conversely, if for every $\lambda \in M(\hat{S})$ from

$$\int_{\hat{S}} \chi(x)d\lambda(\chi) = 0 \quad (x \in S)$$

it follows that $\lambda = 0$, then by Proposition 2.7, for each $\chi \in \hat{S}$ we have $|\chi| = 1$ or 0. Since S^* separates the points of S, from Lemma 3.2 of [**1**] it follows that S is an inverse semigroup with $x^{-1} = x^*$ for all $x \in S$.

We now return to our main line of this section in which we shall develop the Bochner theorem.

THEOREM 2.9. *Let S be a commutative topological semigroup with a weight w. Suppose that Γ_w, with the topology of uniform convergence on compact subsets of S, is a locally compact (Hausdorff) space. Let Γ'_w be a closed subset of Γ_w. If λ is a nonnegative measure in $M(\Gamma'_w)$, then the mapping ϕ which is given by*

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$$\phi(x) = \int_{\Gamma'_w} \chi(x) d\lambda(\chi) \quad (x \in S),$$

*defines a w-bounded continuous function on S. Moreover, if S has an involution * and $\Gamma'_w \subseteq \Gamma^*_w$, then ϕ is positive definite.*

Proof. It is obvious that ϕ is w-bounded. To prove the continuity of ϕ we may (and so will) assume that $\lambda \neq 0$. Let x_0 be a fixed point of S and V_0 be a compact neighbourhood of x_0 and suppose that

$$k = \sup\{w(x): x \in V_0\}.$$

For every positive ϵ, by the regularity of λ, there exists a compact subset K of Γ'_w such that

$$\lambda(\Gamma'_w \backslash K) < \epsilon/4k.$$

By the use of Ascoli's theorem [**11**, p. 233, Theorem 17], we conclude that K is equicontinuous. Therefore there exists an open neighbourhood W_0 of x_0 such that

$$|\chi(x) - \chi(x_0)| < \epsilon/2\|\lambda\|,$$

for all $\chi \in K$ and $x \in W_0$. Let $U_0 = W_0 \cap V_0$. Then for every $x \in U_0$ we have

$$|\phi(x) - \phi(x_0)| \leqq \int_K |\chi(x) - \chi(x_0)| d\lambda(\chi) + \int_{\Gamma'_w \backslash K} |\chi(x) - \chi(x_0)| d\lambda(\chi) < \epsilon\lambda(K)/2\|\lambda\| + 2k\lambda(\Gamma'_w \backslash K) < \epsilon.$$

This proves the continuity of ϕ. The last assertion is easy to establish.

We now give a useful theorem which generalizes Theorem 4.4 of [**4**].

THEOREM 2.10. *Let S be a commutative foundation topological semigroup with identity and with a weight function w. Then there exists a one-to-one correspondence between Γ_w and $\tilde{L}(S, w)^\wedge$ which is defined as follows*:

(i) *for each $\chi \in \Gamma_w$ the corresponding h in $\tilde{L}(S, w)^\wedge$ is given by*

$$(12) \quad h(\mu) = \int_S \chi(x) d\mu(x) \quad (\mu \in \tilde{L}(S, w)),$$

and

(ii) *for each $h \in \tilde{L}(S, w)^\wedge$ the corresponding χ in Γ_w is given by*

$$(13) \quad \chi(x) = \frac{h(\mu * \bar{x})}{h(\mu)} \quad (x \in S),$$

where μ is any element of $\tilde{L}(S, w)$ for which $h(\mu) \neq 0$.

Moreover, if Γ_w and $\tilde{L}(S, w)^\wedge$ are identified by means of this correspondence, then the Gelfand topology and the topology of uniform convergence on compact subsets of S coincide on Γ_w.

Proof. The result can be obtained from the proof of Theorem 4.4 of [**4**] with a slight modification.

Remark. The above theorem shows that for a foundation topological semigroup with identity and with a weight function w, Γ_w with the compact open topology is a locally compact Hausdorff space.

We now give a generalization of Theorem 33.2 of [**10**], with an alternative proof. This result enables us to prove our version of Bochner's theorem.

THEOREM 2.11. *Let A be a commutative Banach *-algebra, and $\hat{A}$ denote the set of all nonzero multiplicative linear functionals on A. Suppose that*

$$\Delta = \{\tau \in \hat{A} : \tau(x^*) = \overline{\tau(x)}, \text{ for all } x \in A\}.$$

Let p be a positive functional on A satisfying conditions (i) *and* (ii) *of Lemma* 2.3 (*with $p = T_\phi$*). *Then there exists a nonnegative measure $\lambda \in M(\Delta)$ such that*

$$(14) \quad p(x) = \int_\Delta \hat{x}(\tau) d\lambda(\tau)$$

for all $x \in A$. Conversely, every functional of the form (14) *is positive and linear and satisfies* (i) *and* (ii) *of Lemma* 2.3.

Proof. The proof easily follows by adjoining an identity to A and using the results given on page 120 of [**15**], and the integral form of the Krein-Milman theorem [**15**, p. 6].

The next result is our main result of this section.

THEOREM 2.12. (The Bochner theorem for w-bounded positive definite functions on foundation topological semigroups). *Let S be a commutative foundation topological semigroup with identity and with a continuous involution *. Suppose that w is a weight function on S such that $w(x^*) = w(x)$ for all $x \in S$. Then a function $\phi: S \to \mathbf{C}$ is w-bounded and continuous positive definite if and only if there exists a nonnegative measure λ_ϕ in $M(\Gamma_w^*)$ such that*

$$(15) \quad \phi(x) = \int_{\Gamma_w^*} \chi(x) d\lambda_\phi(\chi) \quad (x \in S).$$

Moreover, if either $w = 1$ on S or λ_ϕ has a compact support, then λ_ϕ is unique.

Proof. The "if" part follows from Theorem 2.9. Conversely, suppose that ϕ is a w-bounded continuous positive definite function on S. Therefore, T_ϕ, which is is given by

$$(16) \quad T_\phi(\mu) = \int_S \phi(x) d\mu(x) \quad (\mu \in \tilde{L}(S, w)),$$

defines a bounded positive linear functional on the Banach *-algebra $\tilde{L}(S, w)$, which also satisfies (i) and (ii) of Lemma 2.3. Thus by Theorem 2.11, there exists a positive measure $\lambda_\phi \in M(\Delta)$ such that

$$(17) \quad T_\phi(\mu) = \int_\Delta h(\mu) d\lambda_\phi(h)$$

for all $\mu \in \tilde{L}(S, w)$, where Δ is the set of all bounded nonzero *-multiplicative linear functionals on $\tilde{L}(S, w)$. Now, if we apply the identification formulae (12) and (13) of Theorem 2.10, we see that Δ is homeomorphic to Γ_w^*, moreover, for every $\mu \in \tilde{L}(S, w)$ from (16) and (17) we obtain

$$\begin{aligned} \int_S \phi(x) d\mu(x) &= \int_\Delta h(\mu) d\lambda_\phi(h) \\ &= \int_{\Gamma_w^*} \int_S \chi(x) d\mu(x) d\lambda_\phi(\chi) \\ &= \int_S \int_{\Gamma_w^*} \chi(x) d\lambda_\phi(\chi) d\mu(x) \quad \text{(by Fubini's theorem).} \end{aligned}$$

By Theorem 2.9 the map

$$x \rightarrow \int_{\Gamma_w^*} \chi(x) d\lambda_\phi(\chi)$$

is a w-bounded and continuous function, and since ϕ is also w-bounded and continuous, from the above equalities it follows that

$$\phi(x) = \int_{\Gamma_w^*} \chi(x) d\lambda_\phi(\chi) \quad (x \in S).$$

This justifies the formula (15). If $w \equiv 1$ on S, then the uniqueness of λ_ϕ follows from Theorem 2.6. In the case when λ_ϕ has compact support, the proof of uniqueness of λ_ϕ is essentially the same as part (ii) of the proof of Theorem 2.1 of [**6**].

Remark. Let S be the topological semigroup ([0, 1], min) (with the usual topology and with the natural involution $x^* = x$ for all $x \in S$). Let ϕ be a continuous nonconstant increasing positive function on S. From Proposition 7.1 of [**5**] it follows that ϕ is positive definite. Now, if the Bochner theorem holds for S, then ϕ must be constant, since $S^* = \{1\}$. This contradiction shows that our version of Bochner's theorem for foundation topological semigroups fails for this non-foundation semigroup.

3. The spectral theory of w-bounded continuous positive definite operator-valued functions. In this section we shall give an integral representation of the w-bounded continuous positive definite operator-valued functions, in terms of the operator-valued measures on Γ_w^*, on those commutative foundation topological semigroups with identity which have a continuous involution *, and a weight function w. For the most parts of this section, our notation and definitions will follow the standard lines of [**14**]. However, we shall state some of the less well known definitions of this reference.

Definition 3.1. Let A be a commutative Banach *-algebra, and let H be a Hilbert space. Then a linear transformation $U:A \to B(H)$ is said to be an *operator-valued transformation of A by bounded operators on H*. Moreover, if U has also the following properties:

(i) U_{xx^*} is a positive operator on H, for every $x \in A$, and

(ii) there exists a positive constant k such that

$$\|U_x\| \leqq k\|x\| \quad \text{for all } x \in A,$$

then U is said to be a *positive operator-valued transformation.*

Remark. The original form of this definition is due to P. H. Maserick [**14**]. As that author has suggested, the above definition is that which is needed there, and which we shall need.

Definition 3.2 [**14**]. Let S be a *-semigroup, and let H be a Hilbert space. An operator-valued function $\phi:S \to B(H)$ is said to be *positive-definite* if for all $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbf{C}$, the sum

$$\sum_{i,j=1}^{n} c_i\bar{c}_j\phi(s_is_j^*)$$

is a positive definite operator.

Definition 3.3. Let S be a semigroup with a weight function w, and let H be a Hilbert space. An operator-valued function $\phi:S \to B(H)$ is said to be *w-bounded*, if there exists a positive constant k such that

$$\|\phi(x)\| \leqq kw(x) \quad \text{for all } x \in S.$$

Definition 3.4. Let S be a topological semigroup, and let H be a Hilbert space. Suppose that ϕ is an operator-valued mapping of S into $B(H)$. If for every $\xi, \eta \in H$ the scalar-valued function

$$x \to \langle \phi(x)\xi, \eta\rangle$$

is continuous on S, then ϕ is said to be *weakly continuous.*

The following theorem is the main result of this section.

THEOREM 3.5. *Let S be a commutative foundation topological semigroup with identity and with a continuous involution *. Suppose that w is a weight function on S such that* $w(x) = w(x^*)$ *for all* $x \in S$. *Then an operator-valued mapping* ϕ *of S into the bounded operators on a Hilbert space H is w-bounded, weakly continuous, and positive definite if and only if there exists a positive operator-valued measure E on* Γ_w^* *such that*

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$$\phi(x) = \int_{\Gamma_w^*} \chi(x) dE_\chi \quad (x \in S).$$

Furthermore, if either $w = 1$, or for every Borel subset B of Γ_w^* the support of $E(B)$ is compact, then E is unique.

*Moreover, ϕ is a w-bounded continuous *-representation if and only if E is a spectral measure.*

Proof. Using the techniques of the proof of Theorem 2.9, we can easily prove that for each positive operator-valued measure E on Γ_w^*, the mapping ϕ which is given by

$$\langle \phi(x)\xi, \eta \rangle = \int_{\Gamma_w} \chi(x) d\langle E_\chi(\cdot)\xi, \eta\rangle \quad (x \in S, \xi, \eta \in H)$$

defines a w-bounded continuous positive definite operator-valued function on S. To prove the converse we suppose that ϕ satisfies the conditions given in the statement of the theorem. For each $\xi \in H$ the function

$$\phi_\xi : x \to \langle \phi(x)\xi, \xi\rangle$$

defines a w-bounded continuous positive definite function on S. Therefore, by Lemma 2.3, T_ξ which is given by

$$T_\xi(\nu) = \int_S \phi_\xi(x) d\nu(x) \quad (\nu \in \widetilde{L}(S, w))$$

defines a bounded positive linear functional on the Banach *-algebra $\widetilde{L}(S, w)$ which also satisfies conditions (i) and (ii) of that lemma. By Theorem 21.20 of **[9]**, T_ξ can be extended to a bounded positive linear functional $T_\xi^\dagger$ on $(\widetilde{L}(S, w))_u$, the Banach *-algebra obtained from $\widetilde{L}(S, w)$ by adjoining a unit u. For every $\nu \in (\widetilde{L}(S, w))_u$ and every $\xi \in H$, let the symbol $\langle U_\nu\xi, \xi\rangle$ be defined by

$$\langle U_\nu\xi, \xi\rangle = T_\xi^\dagger(\nu).$$

Then, for each $\nu \in (\widetilde{L}(S, w))_u$, ξ and $\eta \in H$, $\langle U_\nu\xi, \eta\rangle$ can be defined through the polarization identity by

$$4\langle U_\nu\xi, \eta\rangle = \langle U_\nu(\xi + \eta), \xi + \eta\rangle - \langle U_\nu(\xi - \eta), \xi - \eta\rangle + i\langle U_\nu(\xi + i\eta), \xi + i\eta\rangle - i\langle U_\nu(\xi - i\eta), \xi - i\eta\rangle.$$

It is easy to see that U defines a positive operator-valued transformation of the Banach *-algebra $(\widetilde{L}(S, w))_u$. Therefore, by Theorem 2.1 of **[14]**, there exists a (unique) positive operator-valued measure $E^\dagger$ on Δ_u, the space of all nonzero, *-multiplicative linear functionals on $(\widetilde{L}(S, w))_u$, such that

$$\langle U_\mu\xi, \eta\rangle = \int_{\Delta_u} \tau(\mu) d\langle E_\tau^\dagger(\cdot)\xi, \eta\rangle$$

for all $\nu \in (\widetilde{L}(S, w))_u$, and all $\xi, \eta \in H$. From this it follows that

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$$\langle U_\mu \xi, \xi \rangle = \int_\Delta \tau(\mu(d\langle E_\tau(\cdot)\xi, \xi\rangle$$

for all $\mu \in \tilde{L}(S, w)$ and all $\xi \in H$, where Δ denotes the space of all nonzero, *-multiplicative bounded linear functionals on $\tilde{L}(S, w)$, and E_τ is the restriction of $E^\dagger$ to Δ. Therefore for every $\mu \in \tilde{L}(S, w)$ and every $\xi \in H$ we have

$$\int_S \phi_\xi(x)d\mu(x) = \int_\Delta \tau(\mu)d\langle E_\tau(\cdot)\xi, \xi\rangle$$

$$= \int_{\Gamma_w^*} \int_S \chi(x)d\mu(x)d\langle E_\chi(\cdot)\xi, \xi\rangle$$

(by Theorem 2.10)

$$= \int_S \int_{\Gamma_w^*} \chi(x)d\langle E_\chi(\cdot)\xi, \xi\rangle d\mu(x)$$

(by Fubini's theorem).

Since both the functions

$$x \to \int_{\Gamma_w^*} \chi(x)d\langle E_\chi(\cdot)\xi, \xi\rangle$$

and ϕ_ξ are w-bounded and continuous, from these equalities it follows that

$$\phi_\xi(x) = \int_{\Gamma_w^*} \chi(x)d\langle E_\chi(\cdot)\xi, \xi\rangle$$

for all $x \in S$ and $\xi \in H$. Now an application of the polarization identity gives

$$\langle \phi(x)\xi, \eta\rangle = \int_{\Gamma_w^*} \chi(x)d\langle E_\chi(\cdot)\xi, \eta\rangle$$

for all $x \in S$ and all $\xi, \eta \in H$. The proof of the uniqueness of E is similar to the proof of the uniqueness of λ in Theorem 2.12.

Suppose now that E is a spectral measure. Then it is easy to see that ϕ is a w-bounded continuous *-representation. Conversely, if ϕ is a w-bounded continuous *-representation, then by Theorem 2.1 of [**12**] and Theorem 2.1 of [**14**], E is a spectral measure. This completes the proof of the theorem.

Remark. The above theorem generalizes Theorem 3.2 of [**14**] and Theorem 2.6 of [**6**] which deal with discrete semigroups.

4. The semisimplicity of $M(S, w)$ and $\tilde{L}(S, w)$. In this section we shall be concerned with establishing a theorem concerning the semisimplicity of $M(S, w)$ and $\tilde{L}(S, w)$, in terms of Γ_w, for a commutative foundation topological semigroup S with identity and with a weight w. This theorem will provide us with new proofs for some well-known results. We shall first establish the following lemma.

LEMMA 4.1. *Let S be a commutative foundation topological semigroup. If the set of continuous semicharacters* (*not necessarily bounded*) *separates the points of S then $\hat{S}$ separates the points of S.*

Proof. Suppose that $y, z \in S$ with $y \neq z$, and let χ be a continuous semicharacter on S such that $\chi(y) \neq \chi(z)$. If $|\chi(y)| \neq |\chi(z)|$, then we can choose a positive real number α such that

$$\alpha(\log|\chi(y)| - \log|\chi(z)|)$$

is not of the form $2k\pi$ for any integer k. It follows from Proposition 4.8 of [**4**] that the function χ_1 which is given by

$$\chi_1(x) = \begin{cases} \exp(i\alpha \log|\chi(x)|) & \text{if } \chi(x) \neq 0 \\ 0 & \text{if } \chi(x) = 0 \end{cases}$$

belongs to $\hat{S}$ and $\chi_1(y) \neq \chi_1(z)$. In the case when $\operatorname{Arg} \chi(y) \neq \operatorname{Arg} \chi(z)$, then again by Proposition 4.8 of [**4**] the function χ_2 which is given by

$$\chi_2(x) = \begin{cases} \chi(x)/|\chi(x)| & \text{if } \chi(x) \neq 0 \\ 0 & \text{if } \chi(x) = 0 \end{cases}$$

defines an element in $\hat{S}$ with $\chi_2(y) \neq \chi_2(z)$. The proof is now complete.

THEOREM 4.2. *Let S be a commutative foundation topological semigroup with identity and with a weight function w. Consider the following statements*:

(i) *$M(S, w)$ is semisimple*;
(ii) *$\tilde{L}(S, w)$ is semisimple*;
(iii) *Γ_w separates the points of S*;
(iv) *$\hat{S}$ separates the points of S*;
(v) *$\tilde{L}(S)$ is semisimple*;
(vi) *$M(S)$ is semisimple.*

Then, we have that (i) *implies* (ii), (ii) *implies* (iii), *and* (iii) *implies* (iv); *moreover,* (iv), (v), *and* (vi) *are equivalent. Furthermore, in the case when Γ_w contains a semicharacter χ such that $\chi(x) \neq 0$ for all $x \in S$, then all the above statements are equivalent.*

Proof. It is trivial that (i) implies (ii). That (ii) implies (iii) follows from Theorem 2.10. By Lemma 4.1, (iii) implies (iv). By Theorem 3.6 and Theorem 3.7 of [**4**], (iv), (v) and (vi) are equivalent. Therefore to complete the proof it suffices to show that (iv) implies (i) in the case Γ_w contains a semicharacter χ with $\chi(x) \neq 0$ for all $x \in S$. Without loss of generality we may assume that $\chi(x) > 0$ for all $x \in S$. Thus w/χ is a weight function on S. In order to establish the theorem we shall first prove that $M(S, w/\chi)$ is semisimple. Let $0 \neq \nu \in M(S, w/\chi)$. Then by Lemma 3.5 of [**4**], $[\hat{S}]$, the algebra generated by $\hat{S}$ is dense in $L^1(S, |\nu|)$. Applying the same

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argument as in the proof of Lemma 3.3 of [**4**], we infer that there exists a semicharacter $\rho \in S$ such that

$$\int \rho(x) d\nu(x) \neq 0.$$

It is clear that $\rho \in \Gamma_{w/\chi}$. It follows that the map h which is given by

$$h(\mu) = \int_S \rho(x) d\mu(x) \quad (\mu \in M(S, w/\chi))$$

defines a bounded multiplicative linear functional on $M(S, w/\chi)$ with $h(\nu) \neq 0$. Thus $M(S, w/\chi)$ is semisimple. Now, it is easy to see that the mapping

$$\Phi : M(S, w) \rightarrow M(S, w/\chi)$$

which is defined by

$$\Phi(\mu) = \chi\mu \quad (\mu \in M(S, w))$$

gives an isometric isomorphism of $M(S, w)$ onto $M(S, w/\chi)$. This completes the proof of the theorem.

As a consequence of the above theorem we obtain a new proof for the following two well-known results. The second of these seems to be known previously only for continuous w.

COROLLARY 4.3. *Let* $(\mathbf{Z}+, +)$ *be the additive semigroup of nonnegative integers with a weight function* $w > 0$. *Then,* $l^1(\mathbf{Z}+, w)$ *is semisimple if and only if* $(w(n)^{1/n})$ *converges to some positive real number* α.

Proof. Suppose first that $l^1(\mathbf{Z}+, w)$ is semisimple. Since

$$l^1(\mathbf{Z}+, w) = \tilde{L}(\mathbf{Z}+, w),$$

from Theorem 4.2 it follows that Γ_w separates the point of $\mathbf{Z}_+$. Therefore there exists a $\chi \in \Gamma_w$ with $\chi(1) \neq 0$. Since

$$\inf\{w(n)^{1/n} : n \in \mathbf{N}\} = \lim_{n \rightarrow \infty} w(n)^{1/n}$$

(see [**8**, p. 16]), we have

$$\lim w(n)^{1/n} = \alpha \geqq |\chi(1)|.$$

Conversely, suppose that

$$\alpha = \lim_{n \rightarrow \infty} w(n)^{1/n} > 0.$$

Let Θ be a positive real number such that $\exp(\Theta) \leqq \alpha$. Then the mapping

$$n \rightarrow \exp(-n\Theta) \quad (n \in \mathbf{Z}_+)$$

defines a semicharacter χ in Γ_w with $\chi(n) \neq 0$ for all $n \in \mathbf{Z}_+$. Therefore by Theorem 4.2, $l^1(\mathbf{Z}_+, w)$ is semisimple.

COROLLARY 4.4. *Let* $(\mathbf{R}_+, +)$ *be the additive topological semigroup consisting of the nonnegative real numbers with the usual topology. Then the weighted convolution Banach algebra* $L^1(\mathbf{R}_+, w)$ *with the convolution product*:

$$(f * g)(x) = \int_0^x f(x - t)g(t)dt \quad (f, g \in L^1(\mathbf{R}_+, w), x \in \mathbf{R}_+)$$

is semisimple if and only if $\lim_{t\to\infty} -\frac{1}{t} \log w(t)$ *is a real number.*

Proof. The proof is similar to that of Corollary 4.3.

5. The Hausdorff moment theorem for commutative foundation topological semigroups. The Hausdorff moment theorem has been studied extensively in the case of certain classical semigroups and discrete commutative semigroups (see Section 3.3 of [7]). In this section, we shall develop this theorem on commutative foundation topological semigroups.

Definition 5.1. On a commutative topological semigroup S with identity 1, for each $n \in \mathbf{Z}_+$ (the set of nonnegative integers), define the operator Δ_n on the space $C^{\mathbf{R}}(S)$ of continuous real-valued functions inductively by

$$\Delta_0 f(x) = f(x)$$

and

$$\Delta_n f(x; h_1, \ldots, h_n) = \Delta_{n-1} f(x; h_1, \ldots, h_{n-1}) - \Delta_{n-1} f(xh_n; h_1, \ldots, h_{n-1}),$$

$(f \in C^{\mathbf{R}}(S);\ x, h_1, \ldots, h_n \in S)$. A function $f \in C^{\mathbf{R}}(S)$ is said to be *completely monotone* if and only if

$$\Delta_n f \geqq 0 \quad \text{for all } n \in \mathbf{Z}_+.$$

Remark. It is easy to see that every $\chi \in \hat{S}_+$ is completely monotone. Moreover, if f is completely monotone on S, then $0 \leqq f(x) \leqq f(1)$ $(x \in S)$.

Definition 5.2. Let S be a commutative topological semigroup, and let A be a subalgebra of $M(S)$. Suppose that F is a linear functional on A. For each $n \in \mathbf{Z}_+$ we define $\Delta_n F$ inductively by

$$\Delta_0 F(\mu) = F(\mu) \quad (\mu \in A),$$

and

$$\Delta_n F(\mu; \nu_1, \dots, \nu_n) = \Delta_{n-1} F(\mu; \nu_1, \dots, \nu_{n-1}) - \Delta_{n-1} F(\mu * \nu_n; \nu_1, \dots, \nu_{n-1})$$

($\mu, \nu_1, \dots, \nu_n \in A$). The linear functional F is said to be *completely monotone*, if for every $n \in \mathbf{Z}_+$,

$$\Delta_n F(\mu; \nu_1, \dots, \nu_n) \geqq 0$$

whenever $\mu, \nu_1, \dots, \nu_n$ are probability measures in A.

The next theorem will enable us to prove the Hausdorff moment theorem for commutative foundation topological semigroups.

THEOREM 5.3. *Let S be a commutative topological semigroup with identity, and let K denote the set of all completely monotone linear functionals F on $M(S)$ with $\|F\| \leqq 1$. Then K is convex and weak *-compact in $M(S)^*$, the dual space of $M(S)$. If F is an extreme point of K, then*

$$F(\mu * \nu) = F(\mu)F(\nu) \quad \textit{for all } \mu, \nu \in M(S)$$

and $F(\mu) \geqq 0$ for all nonnegative measures μ in $M(S)$.

Proof. It is obvious that K is convex. That K is weak *-compact follows from Alaoglu's theorem. Let F be an extreme point of K. In order to prove that

$$F(\mu * \nu) = F(\mu)F(\nu) \quad \text{for every } \mu, \nu \in M(S),$$

it suffices to establish this equality for all probability measures μ, ν in $M(S)$. For each $\nu \in M(S)$ we define F_ν by

$$F_\nu(\mu) = F(\mu * \nu) \quad (\mu \in M(S)).$$

One can easily prove that for each $n \in \mathbf{Z}_+$,

$$\Delta_n (F - F_\nu)(\mu; \nu_1, \dots, \nu_n) = \Delta_{n+1} F(\mu; \nu_1, \dots, \nu_n, \nu); \quad (\mu, \nu_1, \dots, \nu_n \in M(S)).$$

Therefore $F - F_\nu$ is also completely monotone. Thus we have

(18) $\quad 0 \leqq F(\mu * \nu) \leqq F(\nu)$, for all probability measures $\mu, \nu \in M(S)$,

and

$$0 \leqq (F - F_\nu)(\mu * \mu_1) \leqq (F - F_\nu)(\mu), \quad \text{for all probability measures } \mu, \nu, \mu_1 \in M(S).$$

Putting $\mu = \overline{1}$ in the above inequality and using the fact that $F(\overline{1}) \leqq 1$, we obtain

(19) $\quad 0 \leqq (F - F_\nu)(\mu_1) \leqq F(\overline{1}) - F(\nu) \leqq 1 - F(\nu)$,

for all probability measures μ_1 and ν in $M(S)$. To prove that $F(\mu * \nu) = F(\mu)F(\nu)$ for each pair of probability measures μ and ν we consider three cases. Firstly, suppose that $F(\nu) = 0$. Then

$$F(\mu * \nu) = 0 = F(\mu)F(\nu),$$

by (18). Secondly, suppose that $F(\nu) = 1$. Then

$$(F - F_\nu)(\mu) = 0,$$

by (19). Thus,

$$F(\mu * \nu) = F(\mu) = F(\nu)F(\mu).$$

Thirdly, suppose that $0 < F(\nu) < 1$. Then we write

$$F = (1 - F(\nu))\frac{F - F_\nu}{1 - F(\nu)} + F(\nu)\frac{F_\nu}{F(\nu)}.$$

From (19) it follows that $(F - F_\nu)_{/1 - F(\nu)} \in K$, and (18) implies that $F_{\nu/F(\nu)}$ also belongs to K. Since F is an extreme point, it follows that $F = F_{\nu/F(\nu)}$. Thus

$$F(\mu)F(\nu) = F(\mu * \nu).$$

That $F(\mu) \geqq 0$ for all $0 \leqq \mu \in M(S)$ is trivial. The proof is now complete.

THEOREM 5.4. (The Hausdorff moment theorem for commutative foundations topological semigroups). *Let S be a commutative foundation topological semigroup with identity* 1. *Then a function f on S is continuous and completely monotone if and only if there exists a unique nonnegative measure λ_f in $M(\hat{S}_+)$ such that*

$$f(x) = \int_{\hat{S}+} \chi(x) d\lambda_f(\chi) \quad (x \in S).$$

Proof. The "if" part follows from the fact that each $\chi \in \hat{S}_+$ is completely monotone and that $\lambda_f \geqq 0$. To prove the converse we suppose that f is a completely monotone and continuous function on S. Thus $0 \leqq f(x) \leqq f(1)$ for all $x \in S$. Without loss of generality we may assume that $f(1) = 1$, because the case $f = 0$ is trivial. Let K denote the set of all completely monotone linear functionals F on $M(S)$ with $\|F\| \leqq 1$. By Theorem 5.3, K is convex and weak *-compact. Define τ_f on $M(S)$ by

$$\tau_f(\mu) = \int_S f(x) d\mu(x) \quad (\mu \in M(S)).$$

Then $\tau_f \in M(S)^*$ and $\|\tau\| \leqq 1$. It is also easy to see that

$$\Delta_n \tau_f(\mu; \nu_1, \ldots, \nu_n) = \int_{\underset{(n+1)\text{-times}}{S \times \ldots \times S}} \Delta_n f d(\mu \times \nu_1 \times \ldots \times \nu_n)$$

for all probability measures $\mu, \nu_1, \ldots, \nu_n$ in $M(S)$. Since f is completely monotone, it follows that τ_f is a completely monotone functional on $M(S)$. Thus $\tau_f \in K$. Now, from the integral form of the Krein-Milman theorem it follows that τ_f is the barycenter of a regular probability measure λ_f which is supported by the closure of the extreme points of K. From Theorem 5.3 and Theorem 2.10 we infer that the closure of the extreme points of K is homeomorphic to a closed subset of $\hat{S}_+$. Therefore, if we regard λ_f as a measure on $\hat{S}_+$, then we have

$$\tau f(\mu) = \int_{\hat{S}^+} \hat{\mu}(\chi) d\lambda_f(\chi) \quad (\mu \in \tilde{L}(S)).$$

Hence

$$\int_S f(x) d\mu(x) = \int_{\hat{S}^+} \int_S \chi(x) d\mu(x) d\lambda_f(\chi)$$
$$= \int_S \int_{\hat{S}^+} \chi(x) d\lambda_f(\chi) d\mu(x) \quad (\mu \in \tilde{L}(S)).$$

Since

$$x \to \int_{\hat{S}^+} \chi(x) d\lambda_f(\chi)$$

is continuous by Theorem 2.9, and f also is continuous, it follows that

$$f(x) = \int_{\hat{S}^+} \chi(x) d\lambda_f(\chi) \quad (x \in S).$$

The uniqueness of λ_f follows from Theorem 2.6.

The above theorem will allow us to give an easy proof for the following well-known Hausdorff moment theorem for $\mathbf{R}_+$.

COROLLARY 5.5. (Hausdorff moment theorem for $\mathbf{R}_+$). *Let $\mathbf{R}_+$ be the additive topological semigroup of nonnegative real numbers with the usual topology. Then a function f on $\mathbf{R}_+$ is continuous and completely monotone if and only if there exists a nonnegative measure λ_f in $M([0, 1])$ such that*

$$f(x) = \int_0^1 t^x d\lambda_f(t)$$

for all $x \in \mathbf{R}_+$.

Proof. The result follows from Theorem 5.4 and the fact that the mapping

$$\psi:[0, 1] \to (\mathbf{R}_+)_+$$

which is given by

$$t \to \psi_t \quad (t \in [0, 1]) \text{ with } \psi_t(x) = t^x (x \in \mathbf{R}_+)$$

defines a homeomorphism of $[0, 1]$ onto $(\mathbf{R}_+)_+$.

PROPOSITION. *Let S be the topological semigroup* ([0, 1], min). *Then a real-valued function f on S is completely monotone if and only if it is nonnegative and nondecreasing.*

Proof. The proof is straightforward.

Remark. The above proposition shows that the Hausdorff moment theorem fails in the same way as Bochner's theorem, for the non-foundation topological semigroup $S = ([0, 1]), \min)$.

Questions. (a) Let S be a commutative foundation topological semigroup with identity and with a weight function w. Is is true in general that $M(S)$ is semisimple if and only if Γ_w separates the points of S?

(b) In the Bochner's theorem (Theorem 2.12) is the representing measure λ_ϕ always unique?

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