BULL. AUSTRAL. MATH. SOC. 10H08, 12B05, 32A30
VOL. 23 (1981), 49-63.

ON MEASURES OF POLYNOMIALS IN SEVERAL VARIABLES

C.J. Smyth

The measure of a polynomial is defined as the exponential of a certain intractable-looking integral. However, it is shown how the measures of certain polynomials can be evaluated explicitly: when all their irreducible factors are linear, and belong to one of two special classes. Asymptotic values for the measures of two sequences of polynomials in large numbers of variables are also found. The proof of this result uses a quantitative form of the central limit theorem.

1. Introduction

For a polynomial $P\left(x_1, \ldots, x_n\right) \in R\left[x_1, \ldots, x_n\right]$, Mahler [6] in 1962 defined the measure $M(P)$ of P as $M(0) = 0$ and for $P \not\equiv 0$,

$$(1) \quad M(P) = \exp\left\{\frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} \log\left|P\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)\right| d\theta_1 \ldots d\theta_n\right\} .$$

Interest in polynomial measures has recently been revived by Boyd [3], who conjectured that the set

$$L^{\#} = \{M(P) : P \text{ has integer coefficients}\}$$

is closed in R. The algebraic nature of elements of $L^{\#}$ is also of interest, and in this connection it is necessary to have explicit formulae (that is, formulae not involving integrals) for measures of polynomials.

Received 6 August 1980. This work was done while the author was visiting the University of British Columbia, Vancouver. I would like to express my appreciation to Professor David Boyd for many helpful discussions on the subject. He also suggested the problem in Section 4, and supplied reference [1].

One such formula is

$$M\left(a_0 \prod_i (x-\alpha_i)\right) = |a_0| \prod_i \max(1, |\alpha_i|) , \tag{2}$$

which shows that the measure of any element of $Z[x]$ is algebraic. For polynomials in more than one variable, there appears to be no such general formula. However, in Section 2 we generalise (Theorem 1) equation (2) to a class of polynomials in several variables, which shows that polynomials in this class also have algebraic measure.

In Section 3 we show how to find an explicit formula for the measure of any $P \in Z[x, y]$, all of whose irreducible factors are linear of the form $x + y + 2 \cos q\pi$, where q is rational. In particular, we give (Theorem 2) the formula for the measure of $T_n(x+y) + l$, where T_n is the nth Chebyshev polynomial, and $l = 0, \pm 1, \pm 2$.

Another polynomial whose measure has been explicitly calculated is $1 + x_1 + x_2 + x_3$, with measure $\exp\left(7\zeta(3)/\left(2\pi^2\right)\right)$ (see [3, Appendix 1]). While there appears to be no reason why this measure or the measures of Theorem 2 should be algebraic, no one has yet found a polynomial with integer coefficients whose measure can be proved to be transcendental.

For an irreducible polynomial $P\left(x_1, \ldots, x_n\right) = \sum_{\mathbf{i} \in Z^n} a(\mathbf{i})\mathbf{x}^{\mathbf{i}}$, define the dimension $\dim P$ of P as the dimension in R^n of the convex hull of the non-zero $a(\mathbf{i})$. Now for all P with integer coefficients, $M(P) \geq 1$, and all P with $M(P) = 1$ have been found ([2], [4], [7]). The irreducible ones of these all have $\dim P = 1$, and Boyd [3] asked whether the function

$$\lambda(n) = \inf\{M(P) : P \text{ irreducible and } \dim P = n\}$$

tends to infinity with n . In Section 4 we show (Theorem 3) that $M\left(P_n\right) \to \infty$ $(n \to \infty)$ for two sequences of polynomials P_n of dimension n , which might have been expected to give small measures for all n .

2. Polynomials having algebraic measures

We have

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

THEOREM 1. *Suppose* $P(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ *factorizes as*

$$(3) \qquad P(x_1, \ldots, x_n) = \prod_{j=1}^{N} (\alpha_{0j} + \alpha_{1j}x_1 + \ldots + \alpha_{nj}x_n)$$

where $\alpha_{ij} \in \mathbb{C}$ *, and that for each* j *there is a* $k(j)$ *such that*

$$(4) \qquad |\alpha_{k(j)j}| \geq \sum_{\substack{i=1 \\ i \neq k(j)}}^{n} |\alpha_{ij}| .$$

Then

$$(5) \qquad M(P) = \prod_{j=1}^{n} |\alpha_{k(j)j}| .$$

In particular, if also P *has integer coefficients, then* $M(P)$ *is algebraic.*

Note that for $n = 1$ the theorem reduces to (2).

The proof is an immediate application of Jensen's formula

$$(6) \qquad \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta}-\alpha|d\theta = \log_+|\alpha| ,$$

where $\log_+ x = \max(0, \log x)$. Hence for any polynomial $Q(x_1, \ldots, x_{n-1})$ we have

$$(7) \quad \log M(Q(x_1, \ldots, x_{n-1})+x_n)$$
$$= \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \log_+|Q(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}})| d\theta_1 \ldots d\theta_{n-1}$$
$$= \log M_+(Q)$$

say.

Suppose without loss of generality that $|\alpha_n| \geq |\alpha_0| + \ldots + |\alpha_{n-1}|$.

Then $\log_+\left|\left(\alpha_0 + \alpha_1 e^{i\theta_1} + \ldots + \alpha_{n-1}e^{i\theta_{n-1}}\right)/|\alpha_n|\right| = 0$, so that

$$M(\alpha_0 + \alpha_1x_1 + \ldots + \alpha_nx_n)$$
$$= M(\alpha_n)M((\alpha_0 + \alpha_1x_1 + \ldots + \alpha_{n-1}x_{n-1})/\alpha_n) = M(\alpha_n) ,$$

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

using (7). Applying this result to each factor of P, we obtain (5).

If P has integer coefficients, then it is easily seen that all the α_{ij} may be taken to be algebraic, so that $M(P)$ is algebraic.

EXAMPLE 1. $M(x+y+k) = |k|$ for $|k| \geq 2$. Note that $M(x+y) = 1$, and that $M(x+y\pm 1)$ is given in Example 5.

EXAMPLE 2. $M\left((x+y)^N \pm k\right) = |k|$ for $|k| \geq 2^N$.

EXAMPLE 3. $M\left(x^2-y^2+xy+3x-y+1\right) = \rho^2$, where $\rho = \frac{1}{2}(1+\sqrt{5})$, since

$$x^2 - y^2 + xy + 3x - y + 1 = \left(x+\rho y+\rho^2\right)\left(x-\rho^{-1}y+\rho^{-2}\right) ,$$

and

$$\rho^2 = 1 + \rho , \quad 1 = \rho^{-1} + \rho^{-2} .$$

EXAMPLE 4. Let α_0 be an algebraic integer with conjugates $\alpha_0, \alpha_1, \ldots, \alpha_n$. Choose algebraic integers $\xi_{i0} \in Q(\alpha_0)$ with conjugates $\xi_{ij} \in Q(\alpha_j)$, and $|\xi_{ii}| \geq n$, $|\xi_{ij}| \leq 1$ $(i \neq j)$. Then

$$M\left(\prod_{i=0}^{n} \left(\xi_{i0} + \sum_{j=1}^{n} \xi_{ij}x_j\right)\right) = \prod_{i=1}^{n} |\xi_{ii}| .$$

3. Explicit formulae

The methods of this section enable us to evaluate the measures of integral polynomials, all of whose factors are of the form $x + y + 2\cos q\pi$, q rational. As an indication of the method, we show that, for the Chebyshev polynomials T_n, U_{n-1} defined by $T_n(2\cos\theta) = 2\cos n\theta$, $U_{n-1}(2\cos\theta) = \sin n\theta/\sin\theta$:

THEOREM 2. *(a) We have*

$$M\left(U_{n-1}(x+y)\right) = \exp\left(2s(n)\right) , \tag{8}$$

where

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

$$(9)\quad s(n) = \sum_{k=1}^{[\frac12(n-1)]} \left(1-\frac{2k}{n}\right)\log\left|2\cos\frac{k\pi}{n}\right| + \frac{1}{\pi n^2}\sum_{\substack{2<d|n\\ n/d\ odd}} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi\ odd}} \lambda_\chi L(2, \bar\chi)$$

with

$$(10)\qquad \lambda_\chi = \begin{cases} \left(\frac12-\chi(2)\right) \displaystyle\sum_{j=1}^{\frac12(d-1)} \chi(j)\tan\frac{\pi j}{d} & (d\ \ odd), \\ 2\displaystyle\sum_{j=1}^{\frac12 d-1} \chi(j)\cos\frac{\pi j}{d} & (d\ \ even). \end{cases}$$

Here χ *is a character* mod d , *and* $L(2, \chi) = \sum_{j=1}^{\infty} \chi(j) j^{-2}$.

(b) *For* $l = 0, \pm1, \pm2$,

$$(11)\qquad M\left(T_n(x+y)+l\right) = 2^* \exp\left(f_l(n)\right)$$

where

$$(12)\qquad 2^* = \begin{cases} 4 , & l = -2 , \quad n \quad even, \\ 2 , & l = 2 \quad , \quad n \quad odd, \\ 1 , & otherwise, \end{cases}$$

and

$$(13)\quad f_l(n) = \begin{cases} 2s(n) , & n\ \ odd, \quad l = \pm2 , \\ s(3n) - s(n) , & n\ \ odd, \quad l = \pm1 , \\ s(2n) , & n\ \ odd, \quad l = 0 , \\ 4\left(s(n)-s(n/2)\right) , & n\ \ even, \quad l = -2 , \\ 2\left(\left(s(3n)-s(3n/2)\right)-\left(s(n)-s(n/2)\right)\right) , & n\ \ even, \quad l = -1 , \\ 2\left(s(2n)-s(n)\right) , & n\ \ even, \quad l = 0 , \\ 2\left(s(3n/2)-s(n/2)\right) , & n\ \ even, \quad l = 1 , \\ 4s(n/2) , & n\ \ even, \quad l = 2 . \end{cases}$$

We can compute some values of s as follows:

$$s(1) = s(2) = 0 , \quad s(3) = \frac{3\sqrt3}{4\pi} L\left(2, \chi_3\right) , \quad s(4) = \tfrac12 \log 2 + \frac{1}{\pi} L\left(2, \chi_4\right) ,$$

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

$$s(5) = \tfrac{2}{5}\log\rho + \frac{5^{3/4}}{4\pi}\,\mathcal{Re}\Big[\big(\rho^{3/2}+i\rho^{-3/2}\big)L(2,\chi_5)\Big]$$

where $\rho = \tfrac{1}{2}(1+\sqrt{5})$ and $\chi_5(2) = i$;

$$s(6) = \tfrac{1}{3}\log 3 + \frac{5\sqrt{3}}{4\pi}L(2,\chi_3) ,$$

$$s(8) = \tfrac{1}{2}\log 2 + \tfrac{1}{4}\log(1+\sqrt{2}) + \frac{1}{\pi}\big(L(2,\chi_4)+\sqrt{2}L(2,\chi_8)\big) .$$

Here χ_d $(d = 3, 4, 5, 8)$ is the uniquely specified odd character mod d .

EXAMPLE 5.

$$M(x+y\pm1) = M\big(T_1(x+y)\pm1\big) = \exp\left(\frac{3\sqrt{3}}{4\pi}L(2,\chi_3)\right) .$$

EXAMPLE 6.

$$M\big((x+y)^2\pm3\big) = M\big(T_2(x+y)-1\big) = 3^{2/3}\exp\left(\frac{\sqrt{3}}{\pi}L(2,\chi_3)\right) .$$

EXAMPLE 7.

$$M\big((x+y)^2\pm2\big) = M\big(T_2(x+y)\big) = \sqrt{2}\exp\left(\frac{2}{\pi}L(2,\chi_4)\right) .$$

EXAMPLE 8.

$$M\big((x+y)^2+(x+y)-1\big) = \rho^{2/5}\exp\left(\frac{5^{3/4}}{4\pi}\,\mathcal{Re}\Big[\big(\rho^{3/2}+i\rho^{-3/2}\big)L(2,\chi_5)\Big]\right) .$$

$\big(\rho = \tfrac{1}{2}(1+\sqrt{5})\ ,\ \chi_5(2) = i\big)$, since $T_5(x) - 2 = (x-2)\big(x^2+x-1\big)^2$.

EXAMPLE 9.

$$M\big((x+y)^2+(x+y)+1\big) = M\big((x+y-\omega)(x+y-\omega^2)\big) = M(x+y+1)^2 = \exp\left(\frac{3\sqrt{3}}{2\pi}L(2,\chi_3)\right) .$$

The proof of Theorem 2 follows directly from Lemmas 1, 2, 3.

LEMMA 1. *If* $0 \le \alpha \le \pi/2$ *then*

$$M(x+y+2\cos\alpha) = \exp\left(\left(1-\frac{2\alpha}{\pi}\right)\log(2\cos\alpha) + \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\sin 2j\alpha\right) .$$

Proof. Now we can change variables from x, y to x^{-1}, $x^{-1}y^{-1}$

without affecting the measure, so that

$$\begin{aligned}\log M(x+y+2\cos\alpha) &= \log M\left(x^{-1}y+x^{-1}y^{-1}+2\cos\alpha\right)\\ &= \log(2\cos\alpha) + \log M\left(x + \frac{y+y^{-1}}{2(\cos\alpha)}\right)\\ &= \log(2\cos\alpha) + \frac{1}{\pi}\int_0^{\pi} \log_+\left|\frac{2(\cos\frac{1}{2}\theta)}{2(\cos\alpha)}\right| d\theta \quad \text{using (7)},\\ &= \left(1-\frac{2\alpha}{\pi}\right)\log(2\cos\alpha) + \frac{1}{\pi}\int_0^{2\alpha} \log(2\cos\tfrac{1}{2}\theta)\,d\theta\ .\end{aligned}$$

Then, on expanding $\log(2\cos\frac{1}{2}\theta)$ as a series in $e^{i\theta}$ and integrating, the result follows.

Let $S(n)$ denote the set of absolute values of the roots of $U_{n-1}(x) = 0$, excluding $2\cos\frac{\pi}{2} = 0$:

$$(14) \qquad S(n) = \left\{2\cos\frac{k\pi}{n},\ k = 1, \ldots, [\tfrac{1}{2}(n-1)]\right\}\ .$$

LEMMA 2. *We have*

$$(15) \qquad \sum_{2(\cos\alpha)\in S(n)} \log M(x+y+2\cos\alpha) = s(n)$$

where $s(n)$ *is given by* (9).

Proof. From Lemma 1 it is necessary only to prove that

$$(16) \quad S := \sum_{2(\cos\alpha)\in S(n)} \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^2} \sin 2j\alpha$$

$$= \frac{1}{\pi n^2} \sum_{\substack{2<d|n\\ n/d \text{ odd}}} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi \text{ odd}}} \lambda_\chi L(2, \bar{\chi})\ .$$

Firstly, note that for $n^* = [\frac{1}{2}(n-1)]$,

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

(17) $$g\left(\frac{j}{n}, n\right) = \sum_{k=1}^{n^*} \sin 2k\pi \cdot \frac{j}{n} = \frac{\sin\pi j(n^*+1)/n \cdot \sin\pi jn^*/n}{\sin\pi j/n}$$

$$= \begin{cases} 0 & , n \text{ even}, \ j \text{ even}, \\ \cot\frac{\pi j}{n} & , n \text{ even}, \ j \text{ odd}, \\ -\frac{1}{2}\tan\frac{\pi j}{2n} & , n \text{ odd}, \ j \text{ even}, \\ \frac{1}{2}\cot\frac{\pi j}{2n} & , n \text{ odd}, \ j \text{ odd}, \end{cases}$$

on simplification. Hence for n, j not both even, $g(j/n, n)$ is a function of j/n only, say $g(j/n)$. Then

$$S = \frac{1}{\pi} \sum_{\substack{j=1 \\ (2,j,n)=1}}^{\infty} \frac{(-1)^{j-1}}{j^2} g\left(\frac{j}{n}\right) = \frac{1}{\pi n^2} \sum_{\substack{d|n \\ n/d \text{ odd}}} d^2 \sum_{\substack{j=1 \\ (j,d)=1}}^{\infty} \frac{(-1)^{j-1}}{j^2} g\left(\frac{j}{d}\right) .$$

For d even, $(-1)^{j-1}g(j/d) = \cot j\pi/d$ has period d, while for d odd $(-1)^{j-1}g(j/d)$ has period $2d$ (as a function of j). However, since

$$j^{-2}\tan\frac{j\pi}{2d} = -j^{-2}\tan\frac{\pi j}{2d} + 2\left(2\cdot\frac{j}{2}\right)^{-2}\tan\frac{2\pi\cdot(j/2)}{2d}$$

for j even, we have for d odd

$$\sum_{\substack{j=1 \\ (j,d)=1}}^{\infty} \frac{(-1)^{j-1}}{j^2} g\left(\frac{j}{d}\right) = \frac{1}{2} \sum_{\substack{j=1 \\ (j,d)=1}}^{\infty} j^{-2}\left(c_d(j)+\frac{1}{2}\tan\frac{j\pi}{d}\right)$$

where

$$c_d(j) = \begin{cases} (-1)^{j-1}\tan\frac{j\pi}{2d} & (j \text{ even}) \\ (-1)^{j-1}\cot\frac{j\pi}{2d} & (j \text{ odd}) \end{cases}$$

has period d. So, on putting

(18) $$t_d(j) = \begin{cases} c_d(j) + \frac{1}{2}\tan\frac{j\pi}{d} & , d \text{ odd}, \\ \cot\frac{j\pi}{d} & , d \text{ even}, \end{cases}$$

t_d has period d, and is odd:

(19) $$t_d(d-j) = -t_d(j) .$$

Also

(20) $$S = \frac{1}{\pi n^2} \sum_{\substack{d|n \\ n/d \text{ odd}}} d^2 \sum_{\substack{j=1 \\ (j,d)=1}}^{\infty} j^{-2} t_d(j) .$$

It remains only to express the inner sum as a sum of L-functions. Write t_d as a sum over the odd characters mod d :

(21) $$t_d(j) = \sum_{\chi \text{ odd}} a_\chi \bar{\chi}(j) .$$

Then in the usual way we have, by (21), changing the order of summation, and orthogonality, that

(22) $$a_\chi = \frac{1}{\phi(d)} \sum_{j=1}^{d-1} \chi(j) t_d(j) .$$

Finally, since

$$\sum_{j=1}^{d} \chi(j) \cot \frac{\pi j}{d} = 2 \sum_{j=1}^{\frac{1}{2}d-1} \chi(j) \cot \frac{\pi j}{d} \quad (d \text{ even})$$

and

$$\tfrac{1}{2} \sum_{j=1}^{d} \chi(j) \left(c_d(j) + \tfrac{1}{2} \tan \frac{j\pi}{d} \right) = \left(\tfrac{1}{2} - \chi(2) \right) \sum_{j=1}^{\frac{1}{2}(d-1)} \chi(j) \tan \frac{\pi j}{d} \quad (d \text{ odd}),$$

we obtain, from (20), (21) and (22), that S is given by (16), as required.

This proves part *(a)* of Theorem 2. For part *(b)*, we first look at the factors of the form $x \pm 2$ of $T_n(x) + l$. We find that $T_n(x) - 2$ has $(x+2)(x-2)$ as a factor for n even, and for n odd $T_n(x) \pm 2$ has $x \pm 2$ as a factor, and that in no other case is $x \pm 2$ a factor of $T_n(x) + l$. Since $M(x+y\pm 2) = 2$ by Example 1, this accounts for the factor 2^* in (11). For the other roots of $T_n(x) + l$, we have

LEMMA 3. *Let* $F_l(n)$ *be the set of absolute values, excluding* 0

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

and 2 *, of the zeros of* $T_n(x)+l$ *. Then* $F_l(n)$ *is given by* (13) *on replacing* f *by* F *,* s *by* S *,* $+$ *by* $\cup$ *,* $-$ *by* $\setminus$ *, and interpreting the factors* 2, 4 *as the number of times each element of* $F_l(n)$ *is repeated.*

Proof. Since $T_n(2\cos\theta) = 2\cos n\theta$, we must find the θ in $(0, \pi/2)$ such that $\cos n\theta = -\frac{1}{2}l$ or $\cos n(\pi-\theta) = -\frac{1}{2}l$. The tedious details are left to the reader.

This completes the proof of Theorem 2.

4. Asymptotic formulae

In this section we prove the following theorem.

THEOREM 3. *As* n *tends to infinity*

$$M(x_0+\ldots+x_n) = c\sqrt{n}+O(1) , \tag{23}$$

where

$$c = \exp\left(\tfrac{1}{2}\int_1^\infty u^{-1}e^{-u}du\right) = 1.115934 ,$$

and

$$M\left(x_0+\prod_{i=1}^{n}(1+x_i)\right) = \exp\left(\sqrt{\frac{\pi n}{24}}+O(\log n)\right) . \tag{24}$$

Equation (21) shows that the function $\lambda(n)$, defined in Section 1, is $O(\sqrt{n})$. Since $M(x_0) = M\left(\prod_{i=1}^{n}(1+x_i)\right) = 1$, equation (22) is an example of how ill-conditioned the function M is with respect to small changes in polynomial coefficients. A one-variable example of the same phenomenon was mentioned to me by Boyd: $M\left((z-1)^{2n}\right) = M(z^n) = 1$, but $M\left((z-1)^{2n}+z^n\right) = \beta^{2n+O(1)}$, where $\beta = M(x+y+1) = 1.38135\ldots$. These examples show that for polynomials P and Q one cannot hope to bound $M(P+Q)$ as a function of $M(P)$ and $M(Q)$ only. However

$$M(P+Q) \le 2M_+(P)M_+(Q) , \tag{25}$$

where M_+ is defined by (7). This result is an immediate consequence of

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

the inequality

(26) $$\log|a+b| \le \log 2 + \log_+|a| + \log_+|b| .$$

For Boyd's example cited above, this inequality is only out by at most a constant factor, since $M_+\left((z-1)^{2n}\right) = \beta^{2n}$. Also note that the constant 2 in (25) cannot be removed, as the example $P = Q = 1$ shows.

For the proof of Theorem 3, we need the following notation: let $\|x\|$ denote the length of $x \in R^k$. For a function $f : R^k \to R$, define $B(f) = \sup\limits_{x \in R^k} \frac{|f(x)|}{1+\|x\|}$. For a random variable X with values in R^k , with distribution function $G(x)$, denote

$$\zeta_s = E\left(\|X\|^s\right) = \int_{R^k} \|x\|^s dG(x) .$$

Also, let Φ be the standard k-dimensional normal distribution function, $\phi(x) = (2\pi)^{-\frac{1}{2}k} \exp\left(-\frac{1}{2}\|x\|^2\right)$, and

$$\omega_f(\varepsilon) = \int_{\|y_i - x\| < \varepsilon} |f(y_1)-f(y_2)|\, d\Phi(x) .$$

We also need the following quantitative Central Limit Theorem, which is a slightly specialised version of one in Bhattacharya and Rao.

LEMMA 4 [*1*, Corollary 15.5]. *Let* $X_1, \ldots, X_n$ *be* n *independent identically distributed random variables with values in* R^k *, each component of each* X_i *having mean* 0 *and variance* 1 *, and any two distinct components being independent (that is, the autocorrelation matrix is the identity). Suppose that* G *is the distribution function of the* X_i *, and*

(27) $$\int_{\|x\| > (2/3)n^{\frac{1}{2}}} \|x\|^3 dG(x) \le c_1 n^{\frac{1}{2}} .$$

Then for every Borel-measurable f *with* $B(f) < \infty$ *we have*

(28) $$\left|\int_{R^k} fd\left(Q_n(x)-\Phi(x)\right)\right| \le c_2 B(f)\zeta_3\left[1+\left(c_3\zeta_3\right)^{k+2}\right]n^{-\frac{1}{2}} + \omega_f\left(c_4\zeta_3 \frac{(1+\log n)}{n^{\frac{1}{4}}}\right) .$$

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

Here Q_n *is the distribution function of* $\frac{1}{n^{\frac{1}{2}}}(X_1 + \ldots + X_n)$ *, and* C_1, C_2, C_3, C_4 *are constants depending only on* k *.*

Proof. From (7), because

$$\log M(x_0 + \ldots + x_n) = \frac{1}{(2\pi)^n}\int_0^{2\pi} \ldots \int_0^{2\pi} \log_+|e^{i\theta_1} + \ldots + e^{i\theta_n}|d\theta_1 \ldots d\theta_n$$

$$= \frac{1}{(2\pi)^n}\int_0^{2\pi} \ldots \int_0^{2\pi} f(C(n, \theta), S(n, \theta))d\theta_1 \ldots d\theta_n$$

$$+ \frac{\frac{1}{2}\log n}{(2\pi)^n}\int_0^{2\pi} \ldots \int_0^{2\pi} \chi_n(C(n, \theta), S(n, \theta))d\theta_1 \ldots d\theta_n ,$$

where

$$C(n, \theta) = n^{-\frac{1}{2}}(\sqrt{2}\cos\theta_1 + \ldots + \sqrt{2}\cos\theta_n) ,$$

$$S(n, \theta) = n^{-\frac{1}{2}}(\sqrt{2}\sin\theta_1 + \ldots + \sqrt{2}\sin\theta_n) ,$$

$$f(x, y) = \tfrac{1}{2}\log_+\left(\tfrac{1}{2}(x^2+y^2)\right) ,$$

and

$$\chi_n = \begin{cases} 1 & \text{if } x^2 + y^2 \geq 1/n , \\ 0 & \text{otherwise,} \end{cases}$$

(29) $$\log M(x_0 + \ldots + x_n) = \int_{R^2} f(\mathrm{x})dQ_n(\mathrm{x}) + \int_{R^2} \chi_n(\mathrm{x})dQ_n(\mathrm{x}) ,$$

where $\mathrm{x} = (x, y)$ and $X_i = (\sqrt{2}\cos\theta_i, \sqrt{2}\sin\theta_i)$ $(i = 1, \ldots, n)$. Now apply Lemma 4 with $k = 2$. It is easy to verify that the conditions of the lemma are satisfied, and that $B(f) < 1$, and $w_f(\varepsilon) = O(\varepsilon)$. Hence

$$\left|\int fd(Q_n - \Phi)\right| = O(\log n/n^{\frac{1}{2}}) .$$

Also

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

$$\int f d\Phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}\log_+\left(\tfrac{1}{2}(x^2+y^2)\right) \cdot \frac{1}{2\pi}\, e^{-\frac{1}{2}(x^2+y^2)}dxdy$$

$$= \int_{\sqrt{2}}^{\infty} \log(r/\sqrt{2})re^{-\frac{1}{2}r^2}dr = \int_{\sqrt{2}}^{\infty} r^{-1}e^{-\frac{1}{2}r^2}dr = \tfrac{1}{2}\int_1^{\infty} u^{-1}e^{-u}du = \log c ,$$

so that

$$\int fdQ_n = \log c + O\left(\log n/n^{\frac{1}{2}}\right) . \tag{30}$$

To estimate $\int \chi_n dQ_n$, write

$$\left|\int \chi_n dQ_n - 1\right| \le \left|\int \chi_n d(Q_n-\Phi)\right| + \left|\int (\chi_n-1)d\Phi\right| \tag{31}$$

and estimate the two integrals separately. Then apply Lemma 4 to χ_n ,

$$\left|\int \chi_n d(Q_n-\Phi)\right| = O\left(n^{-\frac{1}{2}}\right) + \omega_{\chi_n}(\varepsilon) ,$$

where $\varepsilon = C_4\zeta_3(1 + \log n)n^{-\frac{1}{2}}$. To estimate $\omega_{\chi_n}(\varepsilon)$, note that

$$\sup_{\substack{\|y_i-x\|<\varepsilon\\ i=1,2}} \left|\chi_n(y_1)-\chi_n(y_2)\right| = 0 \quad \text{if} \quad \|x\| > n^{-\frac{1}{2}} + \varepsilon$$

so that

$$\omega_{\chi_n}(\varepsilon) < \int_{\|x\|<n^{-\frac{1}{2}}+\varepsilon} 1.d\Phi = \int_0^{n^{-\frac{1}{2}}+\varepsilon} re^{-\frac{1}{2}r^2}dr = O\left((\log n)^2/n\right) . \tag{32}$$

Hence

$$\left|\int (\chi_n-1)d\Phi\right| = \int_{\chi_n=0} \Phi < \int_{\|x\|<n^{-\frac{1}{2}}+\varepsilon} d\Phi = O\left((\log n)^2/n\right) \tag{33}$$

and so from (31) and (33),

$$\int \chi_n dQ_n = 1 + O\left(n^{-\frac{1}{2}}\right) . \tag{34}$$

Combining (30) and (34), we get from (29) that

$$\log M\left(x_0 + \ldots + x_n\right) = \tfrac{1}{2}\log n + \log c + O\left(\log n/n^{\frac{1}{2}}\right) ,$$

https://doi.org/10.1017/S0004972700006894 Published online by Cambridge University Press

which lead to (23), on exponentiation.

The analysis for (24) is somewhat easier. We have from (7) that

$$\log M\left(x_0 + \prod_{j=1}^{n} (1+x_j)\right)$$

$$= \frac{1}{(2\pi)^n}\int_0^{2\pi} \cdots \int_0^{2\pi} \log\left|\prod_{j=1}^{n} \left(1+e^{i\theta_j}\right)\right| d\theta_1 \ldots d\theta_n$$

$$= \frac{1}{(2\pi)^n}\int_0^{2\pi} \cdots \int_0^{2\pi} \max\left(0, \sum_{j=1}^{n} \log 2 \cos \tfrac{1}{2}\theta_j\right) d\theta_1 \ldots d\theta_n$$

$$= \frac{\sigma\sqrt{n}}{(2\pi)^n}\int_0^{2\pi} \cdots \int_0^{2\pi} \max\left(0, n^{-\frac{1}{2}} \sum_{j=1}^{n} Y_j\right) d\theta_1 \ldots d\theta_n ,$$

where

$$\sigma^2 = \frac{1}{\pi}\int_0^{\pi} \log^2(2 \cos \tfrac{1}{2}\theta)d\theta = \frac{\pi^2}{12}$$

(see [5, equation (6.81), p. 152]), and $Y_j = \sigma^{-1}|\log 2 \cos \frac{1}{2}\theta_j|$ has mean 0 and variance 1 . Hence

$$\log M\left(x_0 + \prod_{j=1}^{n} (1+x_j)\right) = \sigma\sqrt{n}\int_R g(x)dP_n(x)$$

where $g(x) = \max(0, x)$, and $P_n(x)$ is the distribution function of $n^{-\frac{1}{2}} \sum_{j=1}^{n} Y_j$. Since (27) is satisfied for the Y_j , we apply Lemma 4 with $k = 1$, getting

$$\left|\int_R gd\left(P_n-\Phi\right)\right| = O\left(n^{-\frac{1}{2}}\right) + w_g\left(C_4\zeta_3(1 + \log n)n^{-\frac{1}{2}}\right) = O\left(\log n/n^{\frac{1}{2}}\right) .$$

Since

$$\int_R gd\Phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \max(0, x)e^{-\frac{1}{2}x^2}dx = \frac{1}{\sqrt{2\pi}} ,$$

we get

$$\log M\left(x_0 + \prod_{j=1}^{n} (1+x_j)\right) = \sqrt{\frac{\pi n}{24}} + O(\log n) .$$

References

[1] R.N. Bhattacharya and R. Ranga Rao, *Normal approximations and asymptotic expansions* (John Wiley & Sons, New York, London, Sydney, 1976).

[2] David W. Boyd, "Kronecker's theorem and Lehmer's problem for polynomials in several variables", *J. Number Theory* (to appear).

[3] David W. Boyd, "Speculations concerning the range of Mahler's measure", *Canad. Math. Bull.* (to appear).

[4] Wayne M. Lawton, "A generalization of a theorem of Kronecker", *J. Sci. Fac. Chiang Mai Univ.* **4** (1977), 15-23.

[5] L. Lewin, *Dilogarithms and associated functions* (Macdonald, London, 1958).

[6] K. Mahler, "On some inequalities for polynomials in several variables", *J. London Math. Soc.* **37** (1962), 341-344.

[7] C.J. Smyth, "A Kronecker-type theorem for complex polynomials in several variables", *Canad. Math. Bull.* (to appear).

Department of Mathematics,
James Cook University,
Townsville,
Queensland 4811,
Australia.