Riemann Surfaces, by Lars V. Ahlfors and Leo Sario. Princeton Mathematics Series 26, Princeton University Press, 1960. xi + 382 pages.

Hermann Weyl' s "Idee der Riemannschen Fläche" was followed only recently by other presentations of the subject, after research in the theory of several complex variables had revealed the unique position in many respects of analytic manifolds of one complex dimension and thus led to its definite form. Since 1953 the following comprehensive treatments of the subject have appeared: Nevanlinna, "Uniformisierung"; Schiffer-Spencer, "Functionals of Finite Riemann Surfaces"; the third revised edition of Weyl' s classical treatise; Springer, "Introduction to Riemann Surfaces"; Pfluger, "Theorie der Riemannschen Flächen"; and a chapter in Behnke-Sommer, "Theorie der analytischen Funktionen einer komplexen Veränderlichen". It is inevitable that some parallels will be drawn between the present latest addition and its predecessors.

Springer, Behnke, and the present volume have the appearance of textbooks (though only Springer contains exercises), while the other works are addressed rather to specialists only. The necessary topology, group theory, and Hilbert space theory are all developed from the beginning. An outstanding feature is the careful separation of topological, conformal, and metric aspects of the theory, which will be appreciated most by those readers who have tried to disentangle the first three chapters in Pfluger in this respect. The first chapter on the topology of surfaces occupies more than 100 pages, and in my opinion it would be will worth basing an introductory course in topology on this chapter alone. Simplicial homology precedes singular homology, and happily these concepts are placed in their geometrical context rather than being based on line integrals (notwithstanding Weyl' s and Pfluger' s "ursprüngliche Auffassung"). Besides containing standard material on covering surfaces and classification of polyhedrons this chapter is remarkable for a detailed discussion of the concept of ideal boundary along the lines of Freudenthal' s "Endentheorie". Finally the existence of a triangulation on every surface satisfying the second axiom of countability is shown by purely topological methods. Cohomology remains formally undefined, although it is later implicitly used.

Methods used to prove the existence of harmonic and analytic functions and differentials are threefold: Perron' s subharmonic functions, Weyl' s orthogonal projection in Hilbert space, and Sario' s principal functions. The alternating method (Nevanlinna' s main tool) is rejected as "too tedious".

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In the chapter on classification theory use is made of Parreau's methods concerning positive harmonic functions, as well as of the method of extremal length and Pfluger's analytical modules. The tricky and spicy examples devised by Japanese and other mathematicians in order to differentiate between the various 0-classes receive careful treatment — a very welcome feature since they are not easily accessible elsewhere.

The chapter on differentials contains recent generalizations of the classical theory to open surfaces, the most remarkable of which is Ahlfors' extension of Abel's theorem. Behnke's and Stein's results are mentioned, but without proof. The book terminates with the most extensive bibliography of all the previously mentioned works (more than forty pages in small print).

There are a few missing topics which I would have liked to be included: Heinz Huber's definition and properties of surfaces with discrete modular spectrum, his generalization of the great Picard theorem, some notions on quasi-conformal mappings and Teichmüller space, and some geometric aspects of the mapping theory between Riemann surfaces, such as analytic homotopy classes. On the last subject, however, we have been promised a new publication in the Ergebnisse series by M. Heins.

I noticed very few errors or misprints; it is unfortunate, however, that an outstanding article by Alfred Huber (On subharmonic functions and differential geometry in the large) is attributed to Heinz Huber.

Although in general motivation is carefully provided, the book occasionally suffers from a weakness shared by many advanced textbooks: a new concept will not be properly understood unless examples not only for the occurrence of the phenomenon in question are presented but also examples for its non-occurrence. Surely a student reading for the first time the definition of "locally countable space" is eager to have an example of space which does not have this property (or at least wishes to be told where to find one).

It is regrettable that in many universities the theory of Riemann surfaces is taught merely as a short appendix to the course in elementary complex analysis, and then often by the scissors-and-glue method only. A course or seminar based on the present book would provide the advanced student with a beautiful sense of unity between such various fields as topology, algebra, differential geometry, Hilbert space theory, and numerous other aspects of analysis.

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