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JARNÍK TYPE THEOREMS ON MANIFOLDS

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Abstract

Let ψ be a decreasing function. We prove zero-infinity Hausdorff measure criteria for the set of dual ψ-approximable points and for the set of inhomogeneous multiplicative ψ-approximable points on nondegenerate planar curves. Our results extend theorems of Huang ['Hausdorff theory of dual approximation on planar curves', *J. reine angew. Math.* **740** (2018), 63–76] and Beresnevich and Velani ['A note on three problems in metric Diophantine approximation', in: *Recent Trends in Ergodic Theory and Dynamical Systems*, Contemporary Mathematics, 631 (American Mathematical Society, Providence, RI, 2015), 211–229] from s-Hausdorff measure, where $s \in \mathbb{R}$, to the more general g-Hausdorff measure, where g is a suitable class of dimension functions.

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1. Introduction

Khintchine's theorem (1924) is a fundamental result in the metric theory of Diophantine approximation. It asserts that the Lebesgue measure of the set

$$W(\psi) := \{x \in [0, 1) : |qx - p| < \psi(q) \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N}\}$$

of ψ-approximable numbers is either zero or full according as the sum $\sum_{q=1}^{\infty} \psi(q)$ converges or diverges. Here and throughout, $\psi : \mathbb{N} \to [0, \infty)$ is a decreasing function such that $\psi(q) \to 0$ as $q \to \infty$, referred to as an *approximating function*. Throughout, we identify $\mathbb{I} = [0, 1)$.

There are various higher dimensional generalisations of $W(\psi)$ leading to simultaneous, dual and multiplicative approximation. In this note, we are concerned with the dual (one linear form) and multiplicative problems on manifolds. We define the sets for the inhomogeneous setting which is considered to be more general than the homogeneous setting and where results are often more difficult to prove than

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in the homogeneous setting. Fix $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_n) \in \mathbb{I}^n$. Denote $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n$ and $\|\mathbf{q}\| := \max\{|q_1|, \ldots, |q_n|\}$. Define the sets

$$\mathcal{S}_n^{\boldsymbol{\theta}}(\psi) = \{\mathbf{x} \in \mathbb{I}^n : \max_{1\le i\le n} |qx_i - p_i - \theta_i| < \psi(q) \text{ for i.m. } (\mathbf{p}, q) \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}\},$$

$$\mathcal{D}_n^{\boldsymbol{\theta}}(\psi) = \{\mathbf{x} \in \mathbb{I}^n : |\mathbf{q} \cdot \mathbf{x} - p - \theta| < \psi(\|\mathbf{q}\|) \text{ for i.m. } (p, \mathbf{q}) \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}\},$$

$$\Lambda_n^{\boldsymbol{\theta}}(\psi) = \Big\{\mathbf{x} \in \mathbb{I}^n : \prod_{i=1}^{n} |qx_i - p_i - \theta_i| < \psi(q) \text{ for i.m. } (\mathbf{p}, q) \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}\Big\}.$$

Here and throughout, i.m. stands for *infinitely many*. The sets, $\mathcal{S}_n^{\boldsymbol{\theta}}(\psi)$, $\mathcal{D}_n^{\boldsymbol{\theta}}(\psi)$ and $\Lambda_n^{\boldsymbol{\theta}}(\psi)$ are called the set of simultaneous, dual and multiplicative ψ-approximable points. A fundamental aim in Diophantine approximation is to quantify the 'size' of these sets in terms of Lebesgue measure, Hausdorff measure and Hausdorff dimension, which we generally refer to as the metrical theory. The metrical theory for the above sets, termed the independent variable setting, is relatively well developed as compared to the dependent variable theory (that is, when $\mathbf{x}$ is restricted to a manifold). Indeed, to estimate the size of the intersections of the above sets with a k-dimensional, nondegenerate submanifold $\mathcal{M} \subseteq \mathbb{R}^n$ is an intricate and challenging problem. However, some remarkable advances have been made. We state the latest results for all three sets in terms of Hausdorff measure. The Lebesgue measure or Hausdorff measure of the sets typically split into two parts: the convergence case and the divergence case. The convergence/divergence of the corresponding series depend upon the properties of the approximating function ψ and/or dimension function g. Before we discuss the state of the art and our results, we briefly summarise the notation used. For the definitions of Hausdorff measure and dimension, see Section 1.5.

1.1. Notation. Throughout, by a dimension function g, we mean an increasing continuous function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$. By $\mathcal{H}^g$-measure, we mean the g-dimensional Hausdorff measure which is proportional to the standard Lebesgue measure when $g(r) = r^n$. In the case where the dimension function is of the form $g(r) := r^s$ for some $s < k$, $\mathcal{H}^g$ is simply denoted as $\mathcal{H}^s$. For real quantities A, B and a parameter t, we write $A \lesssim_t B$ if $A \le c(t)B$ for a constant $c(t) > 0$ that depends on t only (while A and B may depend on other parameters). We write $A \asymp_t B$ if $A \lesssim_t B \lesssim_t A$. If the constant $c > 0$ depends only on parameters that are constant throughout a proof, we simply write $A \lesssim B$ and $B \asymp A$.

1.2. Dual approximation. The convergence Lebesgue measure result for $\mathcal{D}_n^{\mathbf{0}} \cap \mathcal{M}$ was established first in [3] and then, for any nondegenerate manifold, in a generalised form in [11]. The divergence Lebesgue measure was established for any nondegenerate manifold in [10]. The divergence case for the $\mathcal{H}^g$-measure of $\mathcal{D}_n^{\mathbf{0}} \cap \mathcal{M}$ was established in [6] as a consequence of the ubiquity framework. Regarding the convergence case for $\mathcal{H}^g$-measure of $\mathcal{D}_n^{\mathbf{0}} \cap \mathcal{M}$, progress has been made for various manifolds but not generalised as in the divergence case. The $\mathcal{H}^g$-measure for convergence for the

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parabola was proved in [20] under some mild assumptions on the dimension function g, and then Huang [18] proved that $\mathcal{H}^s(\mathcal{D}_2^{\mathbf{0}}(\psi) \cap C) = 0$ for all nondegenerate planar curves C. We refer the reader to [1] for the inhomogeneous variant of Huang's result.

In a recent paper [23], we (with David Simmons) proved the $\mathcal{H}^g$-measure convergence result for hypersurfaces of dimension at least 3 for both homogeneous and inhomogeneous settings with a nonmonotonic multivariable approximating function. The results of [23] have been extended in [21] to certain classes of nondegenerate sub-manifolds of co-dimension greater than one. For co-dimension two or three, examples of manifolds where the dependent variables can be chosen as quadratic forms are provided. In that paper, the method requires the manifold to have even dimension at least a minimum of four and half the dimension of the ambient space. Hence, the results of [21, 23] are not applicable to one-dimensional manifolds (curves). However, results of similar nature for the special class of Veronese curves have been obtained in [22]. For general planar curves, the best result is due to Huang.

THEOREM 1.1 (Huang, [18]). *Let ψ be a decreasing approximating function and $s \in (0, 1]$. Let C be any $C^{(2)}$ planar curve which is nondegenerate everywhere except possibly on a set of zero Hausdorff s-measure. Then,*

$$\mathcal{H}^s(\mathcal{D}_2^{\mathbf{0}} \cap C) = 0 \quad \text{if} \sum_{q=1}^{\infty} \psi^s(q) q^{2-s} < \infty.$$

A natural problem is to extend Theorem 1.1 to $\mathcal{H}^g$-measure on nondegenerate planar curves. By elaborating ideas presented by Huang [18], it is possible to do so with some restrictions on g.

For convenience, we will assume that the planar curve C is the graph of a smooth map $h : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}$ is a connected bounded open set. Assume that the second derivative h'' of the map h satisfies

$$\mathcal{H}^g(S_C \stackrel{\text{def}}{=} \{x \in U : h''(x) = 0\}) = 0. \tag{I}$$

Condition (I) was implicitly stated in Huang's paper but, for clarity, we state it explicitly and for the $\mathcal{H}^g$-measure. For a detailed discussion about the corresponding condition in higher dimensions, see [23, Section 3]. We prove the following result.

THEOREM 1.2. *Let ψ be a decreasing approximating function. Let h be a $C^{(2)}$ function satisfying condition (I) and let C be the graph of h. Let g be a dimension function such that $r \mapsto r^{-1}g(r)$ is decreasing and $r^{-1}g(r) \to \infty$ as $r \to 0$. Assume that g has the property that, for some $\epsilon > 0$ and for any $r \in (0, 1)$,*

$$g(r) \leq r^{2/3+\epsilon}. \tag{1.1}$$

Then,

$$\mathcal{H}^g(\mathcal{D}_2^{\mathbf{0}}(\psi) \cap C) = 0 \quad \text{if} \sum_{q=1}^{\infty} q^2 g\Big(\frac{\psi(q)}{q}\Big) < \infty.$$

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Observe that (1.1) becomes stronger as ϵ increases since $r < 1$ and it does not make sense with exponent larger than 1 (that is, for $\epsilon > 1/3$) since the dimension of the curve is only 1. By combining Theorem 1.2 with the corresponding divergence theorem proved in [6], we obtain the following complete dichotomy statement.

THEOREM 1.3. *Let ψ be a decreasing approximating function. Let h be a $C^{(2)}$ function satisfying condition (I) and let C be the graph of h. Let g be a dimension function such that $r^{-1}g(r)$ is decreasing and $r^{-1}g(r) \to \infty$ as $r \to 0$. Assume that g has the property (1.1). Then,*

$$\mathcal{H}^g(\mathcal{D}_2^{\mathbf{0}}(\psi) \cap C) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} g\Big(\dfrac{\psi(q)}{q}\Big)q^2 < \infty, \\ \infty & \text{if } \sum_{q=1}^{\infty} g\Big(\dfrac{\psi(q)}{q}\Big)q^2 = \infty. \end{cases}$$

An immediate corollary of Theorem 1.3, which was proved explicitly in [2], is the following Hausdorff dimension result: for $\tau > 2$, if we write $\mathcal{D}_2^{\mathbf{0}}(\tau)$ for $\mathcal{D}_2^{\mathbf{0}}(q \to q^{-\tau})$, we have

$$\dim_{\mathcal{H}}(\mathcal{D}_2^{\mathbf{0}}(\tau) \cap C) = \frac{3}{\tau + 1}.$$

Moreover, $\mathcal{H}^s(\mathcal{D}_2^{\mathbf{0}}(\tau) \cap C) = \infty$ for $s = 3/(\tau + 1)$. To give a more subtle example, let

$$\log_+^i q = \underbrace{\log_+ \cdots \log_+}_{i \text{ times}} q, \quad \log_+(q) := \max\{1, \log q\} \tag{1.2}$$

and for some $\varepsilon > 0$, $\tau > 2$ and $\alpha_2, \ldots, \alpha_t \in \mathbb{R}$, let

$$\psi_\varepsilon(q) = q^{-\tau} \cdot (\log_+ q)^{-\varepsilon} \prod_{i=2}^{t} (\log_+^i q)^{\alpha_i}.$$

Then we have the following exact logarithmic order statement for approximation on the parabola.

COROLLARY 1.4. *Let $\varepsilon > 0$. Then for any fixed α_j and with ψ_ε as defined above, there is a dimension function g_ε such that*

$$\mathcal{H}^{g_\varepsilon}(\mathcal{D}_2^{\mathbf{0}}(\psi_0) \cap C) = \infty \quad \textit{while } \mathcal{H}^{g_\varepsilon}(\mathcal{D}_2^{\mathbf{0}}(\psi_\varepsilon) \cap C) = 0.$$

Consequently, the set $(\mathcal{D}_2^{\mathbf{0}}(\psi_0) \setminus \mathcal{D}_2^{\mathbf{0}}(\psi_\varepsilon)) \cap C$ is not empty and indeed uncountable.

We refer the reader to [5, page 257] for explicit choices of g_ε.

1.3. Multiplicative approximation. For the set $\Lambda_n^{\theta}(\psi)$, hardly anything is known beyond a nondegenerate planar curve C and this time, the most recent result is by Beresnevich and Velani [9]. Even in the independent variable cases, most of the progress has been achieved quite recently (see for example [14, 15, 24]).

THEOREM 1.5 (Beresnevich and Velani, [9]). *Let ψ be any approximating function and $s \in (0,1)$. Let C be a $C^{(3)}$ curve in $\mathbb{R}^2$ with nonzero curvature everywhere apart from a set of s-dimensional Hausdorff measure zero. Then,*

$$\mathcal{H}^s(\Lambda_2^{\theta}(\psi) \cap C) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi^s(q) < \infty, \\ \infty & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi^s(q) = \infty. \end{cases}$$

Note that the case $s = 1$ is not covered by this theorem and represents a challenging open question. We refer the reader to [16] for further details.

We extend this theorem to the g-dimensional Hausdorff measure by combining arguments of this paper with arguments used by Beresnevich and Velani [9]. For this improvement, we first introduce a condition on the dimension function g.

CONDITION G. For some $\alpha \in (0,1)$, the dimension function g is such that $g(t) \cdot t^{-\alpha}$ is nonincreasing. Equivalently, for this value of $\alpha \in (0,1)$,

$$g(xy) \leq x^{\alpha} g(y) \quad \text{for } x \geq 1,\ y > 0. \tag{1.3}$$

For power functions $g(t) = t^s$ with $s \in (0,1)$, the condition holds with $\alpha = s$. More generally, for example, we may consider any function of the form

$$g(t) = t^{\alpha}(\log_+ t^{-1})^{-\beta}(\log_+ \log_+ t^{-1})^{-\gamma}, \quad \beta \geq 0, \gamma \geq 0,$$

with the notation $\log_+$ as in (1.2). By a small twist of condition G, we can in fact take any $\gamma \in \mathbb{R}$ in the following results unless $\beta = 0$.

THEOREM 1.6. *Let ψ be any approximating function. Let C be a $C^{(3)}$ curve in $\mathbb{R}^2$ with nonzero curvature everywhere apart from a set of g-dimensional Hausdorff measure zero. Let g be a dimension function satisfying condition G. Then,*

$$\mathcal{H}^g(\Lambda_2^{\theta}(\psi) \cap C) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q g\Big(\frac{\psi(q)}{q}\Big) < \infty, \\ \infty & \text{if } \sum_{q=1}^{\infty} q g\Big(\frac{\psi(q)}{q}\Big) = \infty. \end{cases}$$

1.4. Some remarks on simultaneous approximation. We first quote a result which is a combined statement of the divergence cases proven by Beresnevich [4] for any nondegenerate manifolds and the convergence case for hypersurfaces by Huang [19].

THEOREM 1.7 (Beresnevich–Huang). *Let $n \geq 3$ and $s > \frac{1}{2}(n-1)$. Let $\mathcal{M}$ denote a compact hypersurface in $\mathbb{R}^n$ with nonvanishing Gaussian curvature except possibly*

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on a set of zero Hausdorff s-measure. Let ψ be an approximating function. Then,

$$\mathcal{H}^s(\mathcal{S}_n^{\mathbf{0}}(\psi)\cap\mathcal{M}) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^n\Big(\frac{\psi(q)}{q}\Big)^{s+1} < \infty, \\ \infty & \text{if } \sum_{q}^{\infty} q^n\Big(\frac{\psi(q)}{q}\Big)^{s+1} = \infty. \end{cases}$$

The divergence part is valid for any nondegenerate analytical submanifold. Proving the convergence part, however, for any nondegenerate manifold represents a challenging problem and progress thus far has been limited to hypersurfaces and only for the s-dimensional Hausdorff measure with some limitations imposed on the real number s. Regarding the more general weighted settings, the $\mathcal{H}^g$-measure on planar curves was established in [25] and the lower bound of the Hausdorff dimension for any nondegenerate manifold in [7].

It is possible to extend the convergence case of Theorem 1.7 to any g-Hausdorff measure without much effort by replacing the $\mathcal{H}^s$-measure with $\mathcal{H}^g$-measure for any dimension function g.

THEOREM 1.8 (Beresnevich–Huang). *Let g be a dimension function with $r^{-1}g(r)\to\infty$ as $r\to 0$. Then, with notation and assumptions of Theorem 1.7, $\mathcal{H}^g(\mathcal{S}_n^{\mathbf{0}}(\psi)\cap\mathcal{M}) = 0$ as soon as*

$$\sum_{q=1}^{\infty} g\Big(\frac{\psi(q)}{q}\Big)q^{n-1}\psi(q) < \infty.$$

However, extending the divergence case of Theorem 1.7 to an arbitrary dimension function is not that straightforward.

1.5. Hausdorff measure and dimension. For completeness, we give a very brief introduction to Hausdorff measures and dimension. For further details, see [12, 17].

Let $\Omega\subset\mathbb{R}^n$. If $0<\rho\le\infty$, any finite or countable collection $\{B_i\}$ of subsets of $\mathbb{R}^n$ such that $\Omega\subset\bigcup_i B_i$ and $\operatorname{diam}(B_i)\le\rho$ is called a *ρ-cover* of Ω. Let

$$\mathcal{H}^g_\rho(\Omega) = \inf\sum_i g(\operatorname{diam}(B_i)),$$

where the infimum is taken over all possible ρ-covers $\{B_i\}$ of Ω. The *g-dimensional Hausdorff measure of* Ω is defined to be

$$\mathcal{H}^g(\Omega) = \lim_{\rho\to 0}\mathcal{H}^g_\rho(\Omega).$$

The map $\mathcal{H}^g:\mathbb{P}(\mathbb{R}^n)\to[0,\infty]$ defines an outer measure on all sets in $\mathbb{R}^n$, which becomes a proper measure when restricted to the subset of $\mathcal{H}^g$-measurable sets, that is, sets A that satisfy $\mathcal{H}^g(B) = \mathcal{H}^g(A\cap B)+\mathcal{H}^g(B\setminus A)$ for any $B\in\mathbb{R}^n$. In the case that $g(r)=r^s$ $(s\ge 0)$, the measure $\mathcal{H}^g$ is denoted $\mathcal{H}^s$ and is called *s-dimensional Hausdorff measure*. For any set $\Omega\subset\mathbb{R}^n$, one can easily verify that there exists a unique critical value of s at which the function $s\mapsto\mathcal{H}^s(\Omega)$ 'jumps' from infinity to zero. The value taken by s at this discontinuity is referred to as the *Hausdorff dimension* of Ω and denoted by $\dim_{\mathcal{H}} \Omega$; that is,

$$\dim_{\mathcal{H}} \Omega := \inf\{s \geq 0 : \mathcal{H}^s(\Omega) = 0\}.$$

The countable collection $\{B_i\}$ is called a *fine cover* of Ω if for every $\rho > 0$, it contains a subcollection that is a ρ-cover of Ω.

We state the Hausdorff measure analogue of the famous Borel–Cantelli lemma (see [12, Lemma 3.10]) which will allow us to estimate the Hausdorff measure of certain sets via calculating the Hausdorff g-sum of a fine cover.

LEMMA 1.9 (Hausdorff–Cantelli lemma). *Let $\{B_i\} \subset \mathbb{R}^n$ be a fine cover of a set Ω and let g be a dimension function such that $\sum_i g(\mathrm{diam}(B_i))$ converges. Then, $\mathcal{H}^g(\Omega) = 0$.*

2. Proof of Theorem 1.2

To keep the length of the proof in control, where it is exactly the same as in [18], we only sketch it with reference to Huang's paper. Since the aim is to extend Huang's proof which is valid for s-dimensional Hausdorff measure to g-dimensional Hausdorff measure, we start by replacing the s-coverings with g-coverings until we encounter nontriviality. As established in [18, Proposition 1], we need to show the convergence of the series

$$\sum_{(q_1,q_2,p)\in\mathbb{Z}^3} g(|\mu(q_1, q_2, p)|),$$

under our assumption. Here,

$$\mu(q_1, q_2, p) = \{x \in I : |q_1 x + q_2 h(x) + p| < \psi(q)\},$$

where $(t, h(t))$ for $t \in I$ is the local parametrisation of the curve C as a function, $M = 1 + \max_{x\in I} |h'(x)|$ and $q = \max\{|q_1|, |q_2|\}$. We notice that Huang's proof essentially works for general measures $\mathcal{H}^g$ until we arrive at (3.15) in his paper, which corresponds to the case $(q_1, q_2) \in \Theta_2$ below. Define

$$\Theta_1 = \{(q_1, q_2) \in \mathbb{Z}^2 : |q_1| > 2M|q_2|\}, \quad \Theta_2 = \mathbb{Z}^2 \setminus (\Theta_1 \cup (0, 0)),$$

and distinguish two cases:

$$(q_1, q_2) \in \Theta_1; \quad (q_1, q_2) \in \Theta_2.$$

Let $(q_1, q_2) \in \Theta_1$. Then using [18, Lemma 2],

$$|\mu(q_1, q_2, p)| \leq \frac{\psi(|q_1|)}{|q_1|}.$$

Since for given q_1 there are only $\ll q_1^2$ choices for the pair (p, q_2), the sum with (q_1, q_2) restricted to Θ_1 can be estimated as

$$\sum_{p\in\mathbb{Z},(q_1,q_2)\in\Theta_1} g(|\mu(q_1, q_2, p)|) \ll \sum_{q_1\in\mathbb{Z},q_1\neq 0} g(\psi(q_1)/|q_1|)q_1^2,$$

which converges by assumption.

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We now treat the more delicate sum with $(q_1, q_2) \in \Theta_2$. As in [18], we distinguish $p \neq p_0$ and $p = p_0$, where p_0 is defined by $F(x_0) - p_0 \in (-1/2, 1/2]$, where $F(x) = q_1 x + q_2 h(x)$. Notice that $q_1/q_2 \in [-2M, 2M]$ since (q_1, q_2) in Θ_2.

When $p \neq p_0$, proceeding as in [18] for a general dimension function g, we see that for fixed $(q_1, q_2) \in \Theta_2$ and $q := \max\{|q_1|, |q_2|\}$,

$$\sum_{p \neq p_0} g(|\mu(q_1, q_2, p)|) \ll \sum_{p \neq p_0} g(\psi(q) \cdot |q(p - p_0)|^{-1/2}).$$

For fixed q, we have $\ll q$ choices of $(q_1, q_2) \in \Theta_2$ and $|p| \leq Cq = C \max\{|q_1|, |q_2|\}$ for an absolute constant $C > 0$, so we have to show the convergence of

$$S := \sum_{q=1}^{\infty} \sum_{p \neq p_0, |p| \leq Cq} q \cdot g(\psi(q)|q(p - p_0)|^{-1/2}).$$

We readily check that

$$S \ll \sum_{q=1}^{\infty} \sum_{1 \leq p \leq Cq} q \cdot g\Big(\frac{\psi(q)}{q} \cdot \sqrt{q/p}\Big).$$

Observe that

$$g(xy) \leq \max\{1, x\} g(y) \quad \text{for } x > 0, y > 0. \tag{2.1}$$

Indeed, if $x > 1$, then by the assumption of the decay of $r \mapsto r^{-1} g(r)$, we have $g(xy)/(xy) \leq g(y)/y$ or equivalently $g(xy) \leq x g(y)$ for any $y > 0$, and if otherwise $0 < x < 1$, then it follows trivially since g is increasing. Application of (2.1) with $y = \psi(q)/q$ and $x = \sqrt{q/p}$ yields

$$S \ll \sum_{q=1}^{\infty} q \cdot g\Big(\frac{\psi(q)}{q}\Big) \sum_{1 \leq p \leq Cq} \max\Big\{\Big(\frac{q}{p}\Big)^{1/2}, 1\Big\}.$$

We show that the inner sum is of order $\ll q$. To see this, we split the sum over $1 \leq p \leq Cq$ into two sums, over $1 \leq p \leq q$ and $q < p \leq Cq$. The sums can be estimated as

$$q^{1/2} \cdot \sum_{p=1}^{q} p^{-1/2} \ll q^{1/2} \cdot \int_1^q p^{-1/2} \ll q^{1/2} \cdot q^{1/2} = q$$

and

$$\sum_{p=q+1}^{Cq} 1 \ll q.$$

Thus,

$$\sum_{q=1}^{\infty} \sum_{p \neq p_0, |p| \leq Cq} q \cdot g(\psi(q)|q(p - p_0)|^{-1/2}) \ll \sum_{q=1}^{\infty} q \cdot q g\Big(\frac{\psi(q)}{q}\Big) = \sum_{q=1}^{\infty} q^2 g\Big(\frac{\psi(q)}{q}\Big),$$

where the rightmost series converges by assumption.

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Now we treat the critical case $p = p_0$. Proceeding as in [18] but for a general dimension function g, we have to show the convergence of

$$\sum_{(q_1,q_2)\in\Theta_2} g(|\mu(q_1, q_2, p_0)|), \tag{2.2}$$

with

$$\Theta_2 = \{(q_1, q_2) \in \mathbb{Z}^2 : |q_1| \le 2M|q_2|\} \setminus \{(0, 0)\},$$

as defined above. Following the method of [18] of estimating (2.2) in this wider generality shows that to prove (2.2), it suffices to show that

$$\sum_{k\ge1} (\psi(2^k)^{1-\epsilon} 2^{2k} + k2^k) g\bigg(\sqrt{\frac{\psi(2^k)}{2^k}}\bigg) < \infty. \tag{2.3}$$

By assumption, $\sum_{q\ge1} q^2 g(\psi(q)/q)$ converges and thus

$$\sum_{k\ge1} g\Big(\frac{\psi(2^k)}{2^k}\Big) 2^{3k} < \infty.$$

Hence, for all sufficiently large k,

$$g(\psi(2^k)/2^k) < 2^{-3k}.$$

This implies $\psi(2^k) < 2^{-2k}$, for sufficiently large k, because of the assumption that $r^{-1}g(r) \to \infty$ as $r \to 0$ and the assumption of (1.1). Thus, we can estimate the argument of g above by

$$\frac{\psi(2^k)}{2^k} < 2^{-3k}.$$

Assume (1.1), which is equivalent to $g(\sqrt{x}) \le x^{\gamma}$ for some $\gamma > \frac{1}{3}$, which we consider fixed in what follows. We see that the series in (2.3) can be bounded by

$$\sum_{k\ge1} (\psi(2^k)^{1-\epsilon} 2^{2k} + k2^k) g\bigg(\sqrt{\frac{\psi(2^k)}{2^k}}\bigg) \le \sum_{k\ge1} \psi(2^k)^{1-\epsilon} 2^{2k-3k\gamma} + \sum_{k\ge1} k2^k 2^{-3k\gamma}.$$

The rightmost series is finite since $\gamma > \frac{1}{3}$. Therefore, it suffices to show the convergence of the series $\sum_{k\ge1} \psi(2^k)^{1-\epsilon} 2^{2k-3k\gamma}$, which can be rewritten as

$$\sum_{k\ge1} \Big(\frac{\psi(2^k)}{2^k}\Big)^{1-\epsilon} 2^{3k-3k\gamma-k\epsilon}. \tag{2.4}$$

Since $x \mapsto g(x)/x$ decreases for $x \in (0, 1)$, we have $g(x)/x \ge g(1)/1$, or equivalently $g(x) \ge g(1)x$. Thus, for any large k, the expression in (2.4) can be bounded by

$$\Big(\frac{\psi(2^k)}{2^k}\Big)^{1-\epsilon} \le \frac{1}{g(1)^{1-\epsilon}} g\Big(\frac{\psi(2^k)}{2^k}\Big)^{1-\epsilon} \le \frac{2^{-3k(1-\epsilon)}}{g(1)^{1-\epsilon}}.$$

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For the sum over k of the products in (2.4) to converge, we require $3 - 3\gamma - \epsilon < 3(1 - \epsilon)$ or equivalently $1 - \gamma < 1 - \frac{2}{3}\epsilon$, which by $\gamma > \frac{1}{3}$ is true as soon as $\epsilon < \frac{1}{2}$. The claim is proved.

3. Proof of Theorem 1.6

The proof of this theorem splits naturally into two parts: the divergence case and the convergence case.

3.1. The divergence case. The divergence case can be proven for any C^2 nondegenerate submanifold of $\mathbb{R}^n$ by using similar arguments to those in [24]. We first quote a result of Bugeaud.

THEOREM 3.1 (Bugeaud, [13]). *Let $n \geq 1$ be an integer. Let ψ be an approximating function. Let g be a dimension function such that $r^{-n}g(r) \to \infty$ as $r \to 0$. Assume that $r \mapsto r^{1+n}g(2\psi(r)/r)$ and $r \mapsto r\psi^n(r)$ are nonincreasing. Then,*

$$\mathcal{H}^g(\mathcal{S}_n^{\boldsymbol{\theta}}(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} r^n g\Big(\dfrac{2\psi(r)}{r}\Big) < \infty, \\ \infty & \text{if } \sum_{r=1}^{\infty} r^n g\Big(\dfrac{2\psi(r)}{r}\Big) = \infty. \end{cases}$$

In fact, the monotonicity of ψ is only needed in the divergence part and when $n = 1$. By Theorem 3.1, and since g is increasing,

$$\mathcal{H}^g(\Lambda_1^{\theta_1}(\psi)) = \mathcal{H}^g(\mathcal{S}_1^{\theta_1}(\psi)) = \infty \quad \text{if} \sum_{r=1}^{\infty} rg\Big(\frac{\psi(r)}{r}\Big) = \infty.$$

As in [9], assume that the planar curve $C := \{(x, \mathcal{F}(x)) : x \in I\}$ is the graph of a $C^{(3)}$ function $\mathcal{F} : I \to \mathbb{R}$ defined on a finite closed interval I and that $\mathcal{F}''$ is continuous and nonvanishing on I. We may for simplicity assume $I = \mathbb{I} = [0, 1]$.

Since any factor in the definition of $\Lambda_2^{\boldsymbol{\theta}}(\psi)$ can be trivially estimated from above by 1 by a proper choice of numerators p_i within the definition of $\Lambda_2^{\boldsymbol{\theta}}(\psi)$, we have the inclusion

$$\Lambda := \{(x, \mathcal{F}(x)) : x \in \Lambda_1^{\theta_1}(\psi) \cap \mathbb{I}\} \subset C \cap \Lambda_2^{\boldsymbol{\theta}}(\psi).$$

Since $\mathcal{F} \in C^{(1)}$, we see that $\mathcal{F}$ is locally bi-Lipschitz and therefore the map $x \mapsto (x, \mathcal{F}(x))$ preserves $\mathcal{H}^g$-measure. Thus, for any dimension function g,

$$\mathcal{H}^g(C \cap \Lambda_2^{\boldsymbol{\theta}}(\psi)) \geq \mathcal{H}^g(\Lambda) = \mathcal{H}^g(\Lambda_1^{\theta_1}(\psi) \cap \mathbb{I}).$$

Therefore,

$$\mathcal{H}^g(C \cap \Lambda_2^{\boldsymbol{\theta}}) = \infty \quad \text{if} \sum_{r=1}^{\infty} rg\Big(\frac{\psi(r)}{r}\Big) = \infty.$$

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3.2. The convergence case. In this section, we prove the convergence case of Theorem 1.6, basically following [9] with some twists towards the end. Assume that the series $\sum_q qg(\psi(q)/q)$ converges. Then, by the monotonicity of ψ and Cauchy condensation,

$$\sum_{q=1}^{\infty} qg(\psi(q)/q) \asymp \sum_{t=1}^{\infty} 2^{2t} g\Big(\frac{\psi(2^t)}{2^t}\Big) < \infty.$$

In particular, for large t,

$$2^{2t} g\Big(\frac{\psi(2^t)}{2^t}\Big) < 1.$$

Recall that the set $\Lambda_2^{\boldsymbol{\theta}}(\psi)$ can be written as

$$\Big\{(x_1, x_2) \in \mathbb{I}^2 : \Big|x_1 - \frac{p_1 + \theta_1}{q}\Big|\Big|x_2 - \frac{p_2 + \theta_2}{q}\Big| < \frac{\psi(q)}{q^2} \text{ for i.m. } (p_1, p_2, q) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}\Big\}.$$

As in [9], we note that for any $\ell \in \mathbb{N}$, the set $\Lambda_2^{\boldsymbol{\theta}}(\psi) \cap C$ can be covered by the union over $t \geq \ell$ of the sets

$$\bigcup_{2^t \leq q < 2^{t+1}} \ \bigcup_{m \in \mathbb{Z}, 2^{|m|}\sqrt{\psi(2^t)} \leq 1} \ \bigcup_{p_1, p_2 = -1}^{q} S_{\boldsymbol{\theta}}(q, m, t, p_1, p_2) \cap C,$$

where $S_{\boldsymbol{\theta}}(q, m, t, p_1, p_2)$ is defined as the set of $(x_1, x_2) \in \mathbb{I}^2$ that satisfy

$$\Big|x_1 - \frac{p_1 + \theta_1}{q}\Big| \leq \frac{2^m\sqrt{2\psi(2^t)}}{2^t}, \quad \Big|x_2 - \frac{p_2 + \theta_2}{q}\Big| \leq \frac{2^{-m}\sqrt{2\psi(2^t)}}{2^t}.$$

(The sets depend on t as well; this is a small formal inaccuracy in [9].)

By continuity and nonvanishing of $\mathcal{F}''$, we see that $\mathcal{F}'$ has at most one local extremum. Hence, each set $C \cap S_{\boldsymbol{\theta}}(q, m, t, p_1, p_2)$ lies in the union of at most two boxes, that is, the projection to the first coordinate lies in two intervals. From a metrical point of view, we may without loss of generality assume that $\mathcal{F}'$ is increasing in $\mathbb{I}$ so that there is in fact only one box. (Again, this argument closes a minor gap from [9]). Then, since $|\mathcal{F}''|$ is bounded from below by a positive constant on the compact interval $\mathbb{I}$, it can easily be seen that

$$\operatorname{diam}(C \cap S_{\boldsymbol{\theta}}(q, m, t, p_1, p_2)) \ll 2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}.$$

Now we quote a result from [8] that counts the number of rational points close to a nondegenerate curve. Since I is compact, this implies that $\mathcal{F}''$ is bounded between two positive constants. Given $\boldsymbol{\theta} \in \mathbb{R}^2, \delta > 0$ and $Q \geq 1$, consider the set

$$A_{\boldsymbol{\theta}}(Q, \delta) := \Big\{(p_1, q) \in \mathbb{Z} \times \mathbb{N} : \begin{array}{l} Q < q \leq 2Q, (p_1 + \theta_1)/q \in I, \\ \|q\mathcal{F}((p_1 + \theta_1)/q) - \theta_2\| < \delta \end{array}\Big\}.$$

Let $N_{\boldsymbol{\theta}}(Q, \delta) = \# A_{\boldsymbol{\theta}}(Q, \delta)$.

THEOREM 3.2 [8]. *Let $\mathcal{F} \in C^{(3)}(I)$, bounded on every $x \in I$. If $\epsilon > 0$, then for any $Q \geq 1$ and $0 \leq \delta \leq 1/2$, $N_\theta(Q,\delta) \ll \delta Q^2 + Q^{1+\epsilon}$, where the implied constant is independent of δ and Q.*

By using this theorem with $\delta = 2^{|m|}\sqrt{\psi(2^t)}$, $Q = 2^t$, for every $\epsilon > 0$, we have

$$N_\theta(Q,\delta) \ll 2^{|m|}\sqrt{\psi(2^t)}2^{2t} + 2^{(1+\epsilon)t}.$$

So from the above cover,

$$\mathcal{H}^g(\Lambda_2^\theta(\psi) \cap C) \ll \sum_{t=1}^{\infty} \sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} g\Big(2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}\Big) \cdot (2^{|m|}\sqrt{\psi(2^t)}2^{2t} + 2^{(1+\epsilon)t}).$$

Until now, the method and calculations have not changed much from [9] with the exception of considering the g-measure instead of the s-measure. So the question is under what 'nice' conditions on g can we make the sum convergent, thus both sums

$$S_1 := \sum_{t=1}^{\infty} \sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} g\Big(2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}\Big) \cdot 2^{|m|}\sqrt{\psi(2^t)}2^{2t}$$

and

$$S_2 := \sum_{t=1}^{\infty} \sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} g\Big(2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}\Big) \cdot 2^{(1+\epsilon)t}$$

convergent. By (1.3), let

$$y = \frac{\psi(2^t)}{2^t}, \quad x = 2^{-|m|}\psi(2^t)^{-1/2} \geq 1.$$

By the same geometric sum argument as in [8],

$$\sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} 2^{(1-\alpha)|m|} \ll (\sqrt{\psi(2^t)})^{\alpha-1}.$$

Using this estimate, along with (1.3) with x, y as above, the sum S_1 can be bounded by

$$\begin{aligned} S_1 &= \sum_{t=1}^{\infty} \sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} g\Big(2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}\Big)2^{|m|}\sqrt{\psi(2^t)}2^{2t} \\ &\leq \sum_{t=1}^{\infty} \sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} g\Big(\frac{\psi(2^t)}{2^t}\Big)2^{-\alpha|m|}(\sqrt{\psi(2t)})^{-\alpha} \cdot 2^{|m|}\sqrt{\psi(2^t)}2^{2t} \end{aligned}$$

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$$= \sum_{t=1}^{\infty} g\Big(\frac{\psi(2^t)}{2^t}\Big) \cdot 2^{2t}(\sqrt{\psi(2t)})^{(1-\alpha)} \sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} 2^{(1-\alpha)|m|}$$

$$\ll \sum_{t=1}^{\infty} g\Big(\frac{\psi(2^t)}{2^t}\Big) 2^{2t}.$$

The last sum converges by hypothesis and Cauchy condensation.

Now we bound the sum S_2. Since the claim of the theorem becomes weaker if we increase ψ, we may assume

$$\psi(2^t) \geq 2^{-(2-2\varepsilon)t/\alpha}$$

or equivalently

$$\psi(2^t)^{-\alpha/2} \leq 2^{t(1-\varepsilon)}.$$

Otherwise, we can just take the pointwise maximum of ψ and the right-hand side function above whose sum over t converges as soon as $\varepsilon < 1$. Observe further that for given t,

$$\#\{m : 2^{|m|}\sqrt{\psi(2^t)} \leq 1\} \ll \log(1/\psi(2^t)) \ll \psi(2^t)^{-\varepsilon_0}$$

for any $\varepsilon_0 > 0$. Hence, given $\epsilon > 0$, we choose any $\varepsilon > \epsilon$ (for example, $\varepsilon = 2\epsilon$) and then we can choose $\epsilon_0 > 0$ small enough so that still

$$\psi(2^t)^{-\alpha/2} \cdot \#\{m : 2^{|m|}\sqrt{\psi(2^t)} \leq 1\} \leq 2^{t(1-\epsilon)}.$$

We can therefore estimate

$$\begin{aligned} S_2 &= \sum_{t=1}^{\infty} \sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} g\Big(2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}\Big) \cdot 2^{(1+\epsilon)t} \\ &\leq \sum_{t=1}^{\infty} \sum_{m\in\mathbb{Z},2^{|m|}\sqrt{\psi(2^t)}\leq 1} g\Big(\frac{\psi(2^t)}{2^t}\Big) 2^{t(1+\epsilon)} \psi(2^t)^{-\alpha/2} 2^{-\alpha|m|} \\ &\ll \sum_{t=1}^{\infty} g\Big(\frac{\psi(2^t)}{2^t}\Big) 2^{t(1+\epsilon)} \psi(2^t)^{-\alpha/2} \cdot \#\{m : 2^{|m|}\sqrt{\psi(2^t)} \leq 1\} \\ &\ll \sum_{t=1}^{\infty} g\Big(\frac{\psi(2^t)}{2^t}\Big) 2^{t(1+\epsilon)} 2^{t(1-\epsilon)} \\ &= \sum_{t=1}^{\infty} g\Big(\frac{\psi(2^t)}{2^t}\Big) 2^{2t}. \end{aligned}$$

Again, the latter sum converges by Cauchy condensation. Hence, both sums converge and, thus, by the Hausdorff–Cantelli lemma,

$$\mathcal{H}^g(\Lambda_2^\theta(\psi) \cap C) = 0 \quad \text{if and only if } \sum_q qg(\psi(q)/q) < \infty.$$

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