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WEIGHT FUNCTIONS WHICH ADMIT TCHEBYCHEFF QUADRATURE

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We describe a class of weight functions, which admit Tchebycheff quadrature.

Let integers $n, m \in \mathbb{N}_0$, $m \leq 2n-1$, and a nonnegative continuous weight function w on $(-1, +1)$ be given. We call a quadrature formula of the type

$$(1) \qquad \int_{-1}^{+1} f(x)w(x)dx = \sum_{i=1}^{n} \lambda_i f(x_i) + R_n(f)$$

a $(2n-1-m, n, w)$ Tchebycheff quadrature formula on $[-1, +1]$, if the following three conditions are fulfilled: $-1 < x_1 < x_2 < \ldots < x_n < 1$, $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1/n$, and $R_n(f) = 0$ for all $f \in \mathbb{P}_{2n-1-m}$, where $\mathbb{P}_{2n-1-m}$ is the set of polynomials of degree at most $2n-1-m$. We say that the weight function w admits T-quadrature, if for each $n \in \mathbb{N}$ there exist nodes $x_1, \ldots, x_n$, such that (1) is a (n, n, w) Tchebycheff quadrature formula.

Ullman [5] has shown that the weight function

$$w_\alpha(x) = \frac{1}{\pi\sqrt{1-x^2}} \frac{1+\alpha x}{1+\alpha^2+2\alpha x}, \quad x \in (-1, +1), \quad \alpha \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$

admits T-quadrature. So far these are the only known weight functions

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with this property (compare [1]). In this paper we describe a class of weight functions, which admit T-quadrature.

We denote the closure and the boundary of a set $E \subset \mathbb{C}$ by $\bar{E}$ and ∂E respectively. The open disk of radius r $\{z \in \mathbb{C} \mid |z| < r\}$ is denoted by U_r. As usual the index r is omitted for $r = 1$. Furthermore T_k, $k \in \mathbb{N}_0$, denotes the Tchebycheff polynomial of first kind of degree k.

For the following theorem see also Theorem 1 of [4].

THEOREM 1. *The following three properties are equivalent:*

(a) a quadrature formula based on nodes $x_1, \ldots, x_n \in \mathbb{R}$, $-1 < x_1 < x_2 < \ldots < x_n < 1$, *is a* $(2n-1-m, n, w)$ *Tchebycheff quadrature formula on* $[-1, +1]$;

(b) $$1 - \frac{z t_{2n}'(z)}{n t_{2n}(z)} = 1 + \sum_{k=1}^{2n-1-m} c_k z^k + O\left(z^{2n-m}\right) \quad \text{for } z \in U, \text{ where}$$

$$t_{2n}(z) = \prod_{j=1}^{n} \left(z^2 - 2\cos\varphi_j z + 1\right), \quad \varphi_j = \arccos x_j, \text{ and}$$

$$c_k = 2\int_{-1}^{+1} T_k(x) w(x)dx \quad \text{for } k = 1, \ldots, 2n-1-m;$$

(c) $$t_{2n}(z) = \exp\left(n\int_0^z \frac{1-F(\xi)}{\xi}\, d\xi\right) + O\left(z^{2n-m}\right) \quad \text{for } z \in U, \text{ where}$$

$$F(\xi) = 1 + \sum_{k=1}^{\infty} c_k \xi^k \text{ and } c_k = 2\int_{-1}^{+1} T_k(x) w(x)dx \text{ for}$$

$k \in \mathbb{N}$.

Proof. *(a)* $\Rightarrow$ *(b)*. Since

$$c_k = 2\int_{-1}^{+1} T_k(x) w(x)dx = \frac{2}{n}\sum_{j=1}^{n} \cos k\varphi_j \quad \text{for } k = 0, \ldots, 2n-1-m,$$

we deduce that

$$\frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{1-e^{i\varphi_j}z} + \frac{1}{1-e^{-i\varphi_j}z} - 1\right) = 1 + \sum_{k=1}^{2n-1-m} c_k z^k + O\left(z^{2n-m}\right) \quad \text{for } z \in U.$$

Using the fact that

$$\frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{1-e^{i\varphi_j}z}+\frac{1}{1-e^{-i\varphi_j}z}-1\right] = 1-\frac{zt'_{2n}(z)}{nt_{2n}(z)} \tag{2}$$

the implication is proved.

(b) ⇒ *(c)*. Taking into consideration the facts that t_{2n} has all zeros on the circumference and that $t_{2n}(0) = 1$, we obtain

$$\frac{1}{n}\ln t_{2n}(z) = \int_0^z \frac{1-F(\xi)}{\xi}\,d\xi + O\left(z^{2n-m}\right)$$

from which the assertion follows.

(c) ⇒ *(a)*. Since

$$1-\frac{zt'_{2n}(z)}{nt_{2n}(z)} = 1+\sum_{k=1}^{2n-1-m} c_k z^k + O\left(z^{2n-m}\right)$$

we get with the aid of (2) that

$$c_k = \frac{2}{n}\sum_{j=1}^{n}\cos k\varphi_j \quad \text{for} \quad k = 0, \ldots, 2n-1-m .$$

Henceforth we denote by

$$q_n^*(z) = z^n\bar{q}\left(z^{-1}\right) = \bar{\gamma}\prod_{i=1}^{n}\left(1-\bar{z}_i z\right)$$

the reciprocal polynomial of

$$q_n(z) = \gamma\prod_{i=1}^{n}\left(z-z_i\right), \quad \gamma, z, z_i \in \mathbf{C} .$$

As a consequence of Theorem 1 we obtain (compare also [2, Theorem 1])

COROLLARY 1. *Let*

$$F(z) = 1+\sum_{k=1}^{\infty} c_k z^k ,$$

where

$$c_k = 2\int_{-1}^{+1} T_k(x)w(x)dx$$

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for $k \in \mathbb{N}$. *Further let* p_n^* *be defined by*

$$p_n^*(z) = \exp\left(n \int_0^z \frac{1-F(\xi)}{\xi}\, d\xi\right) + O\left(z^{n+1}\right)$$

and assume that p_n *has all zeros in* $\overline{U}_{\frac{1}{2}}$. *The quadrature formula based on nodes which are the zeros of the polynomial*

$$q_n(x) = \operatorname{Re}\left\{p_n\left(e^{i\varphi}\right)-\left(p_n(0)/2\right)\right\}, \quad x = \cos\varphi , \quad \varphi \in [0, \pi] ,$$

is a (n, n, w) *Tchebycheff quadrature formula on* $[-1, +1]$.

Proof. Since p_n has all zeros in $\overline{U}_{\frac{1}{2}}$, it follows from [3, p. 80] that all zeros of $p_n - p_n(0)/2$ lie in the circle

$$|z| \le \tfrac{1}{2} + |p_n(0)/2|^{1/n} < 1 .$$

Thus the cosine polynomial $\operatorname{Re}\left\{p_n\left(e^{i\varphi}\right)-\left(p_n(0)/2\right)\right\}$ has n simple zeros in $(0, \pi)$. Setting

$$t_{2n}(z) = p_n^*(z) + z^n\left(p_n(z)-p_n(0)\right)$$

the assertion follows from Theorem 1 *(c)* and the fact that

$$z^{-n}t_{2n}(z)/2 = \operatorname{Re}\left\{p_n(z)-\left(p_n(0)/2\right)\right\} \quad \text{for} \quad z = e^{i\varphi} , \quad \varphi \in [0, \pi] .$$

The following corollary gives us an extension of some results of Ullman [5] and Geronimus [2].

COROLLARY 2. *Let* $l \in \mathbb{N}$ *be fixed and let*

$$p_l(z) = \prod_{i=1}^{l} \left(z-\alpha_i\right) ,$$

$\alpha_i \in \overline{U}_{\frac{1}{2}}$ *real or complex conjugate. Put*

$$w(x) = \frac{1}{\pi\sqrt{1-x^2}} \frac{1}{l} \sum_{i=1}^{l} \frac{1+\alpha_i x}{1+\alpha_i^2+2\alpha_i x} \quad \textit{for} \quad x \in (-1, +1) .$$

The quadrature formula based on nodes which are the zeros of the polynomial $(n \in \mathbb{N})$

$$q_{ln}(x) = \mathrm{Re}\left\{\left[p_l\left(e^{i\varphi}\right)\right]^n - \left[\left(p_l(0)\right)^n/2\right]\right\}, \quad x = \cos\varphi, \quad \varphi \in [0, \pi],$$

is a (ln, ln, w) *Tchebycheff quadrature formula on* $[-1, +1]$.

Proof. Using the notation of Corollary 1 we obtain $\left(z = e^{i\varphi},\right.$ $\left.\varphi \in [0, \pi]\right)$

$$\pi w(\cos\varphi)\sin\varphi = \mathrm{Re}\left\{\frac{1}{l}\sum_{i=1}^{l}\frac{1}{1-\alpha_i z}\right\} = \mathrm{Re}\, F(z) .$$

Hence

$$\exp\left[ln\int_0^z \frac{1-F(\xi)}{\xi}\,d\xi\right] = \prod_{i=1}^{l}\left(1-\alpha_i z\right)^n .$$

In view of Corollary 1 the assertion is proved.

LEMMA 1. *Let* E_1, E_2, E, $\bar{E}_1 \subset \bar{E}_2 \subset E$, $0 \in E_1$, *be bounded simple connected domains in the complex plane, such that*

$$F(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

is analytic on E *and*

$$\delta := \left(\max_{z\in\partial E_1}|z|\right)/\left(\min_{\xi\in\partial E_2}|\xi|\right) < 1 .$$

Put $G(z) = \sum_{k=1}^{\infty}\left(c_k/k\right)z^k$. *If*

$$-\frac{1}{n}\ln\left(-1+(1/\delta)\right) + \ln\delta \le \min_{\xi\in\partial E_2}\mathrm{Re}\, G(\xi) - \max_{z\in\partial E_1}\mathrm{Re}\, G(z) ,$$

then p_n^*, *defined by* $p_n^*(z) = \exp\left(-nG(z)\right) + O\left(z^{n+1}\right)$, *has no zero in* E_1 .

Proof. Put

$$H(z) = \exp\left[\int_0^z \frac{1-F(\xi)}{\xi}\,d\xi\right] = \exp\left(-G(z)\right) \quad \text{for} \quad z \in E$$

and assume that p_n^* has a zero $v \in E_1$. Since

$$\left(H(z)\right)^n = p_n^*(z) + \frac{1}{2\pi i}\int_{\partial E_2} (z/\xi)^n\left(\frac{z}{\xi-z}\right)\left(H(\xi)\right)^n \frac{d\xi}{\xi} \quad \text{for} \quad z \in E_2$$

we obtain, by using the inequality

$$\frac{1-\delta}{\delta} \leq |\xi/z| - 1 \leq |(\xi/z)-1| \quad \text{for} \quad z \in \partial E_1 , \quad \xi \in \partial E_2 ,$$

that

$$1 = \left|\frac{1}{2\pi i}\int_{\partial E_2} (v/\xi)^n\left(\frac{v}{\xi-v}\right)\left(\frac{H(\xi)}{H(v)}\right)^n \frac{d\xi}{\xi}\right|$$

$$< \left(\max_{\xi\in\partial E_2} |H(\xi)|^n\right)/\left(\min_{z\in\partial E_1} |H(z)|^n\right) \cdot \delta^n \cdot \frac{\delta}{(1-\delta)} .$$

Taking into consideration the fact that

$$|H(\xi)| = \exp\left(-\operatorname{Re} G(\xi)\right)$$

we get

$$\max_{\xi\in\partial E_2} |H(\xi)| = 1/\left(\exp\left(\min_{\xi\in\partial E_2} \operatorname{Re} G(\xi)\right)\right) ,$$

respectively

$$\min_{z\in\partial E_1} |H(z)| = 1/\left(\exp\left(\max_{z\in\partial E_1} \operatorname{Re} G(z)\right)\right) .$$

Thus it follows that

$$1 < \left(\max_{\xi\in\partial E_2} |H(\xi)|\right)/\left(\min_{z\in\partial E_1} |H(z)|\right) \cdot \delta \cdot \left(\frac{\delta}{1-\delta}\right)^{1/n}$$

$$= \exp\left(\max_{z\in\partial E_1} \operatorname{Re} G(z) - \min_{\xi\in\partial E_2} \operatorname{Re} G(\xi)\right) \cdot \delta \cdot \left(\frac{\delta}{1-\delta}\right)^{1/n} \leq 1$$

which is a contradiction.

THEOREM 2. *(a) Let* σ *be an odd bounded nondecreasing function on* $[-\pi, \pi]$, *such that* $\int_{-\pi}^{+\pi} d\sigma(t) = 1$. *Suppose that* $\alpha \in [-1/8, 1/8]$. *Then*

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$$w(x) = \frac{1}{\pi\sqrt{1-x^2}} \operatorname{Re} \int_{-\pi}^{+\pi} \frac{d\sigma(t)}{1-\alpha e^{it}z} , \quad x = \tfrac{1}{2}\left(z + \frac{1}{z}\right) , \quad z = e^{i\varphi} , \quad \varphi \in (0, \pi) ,$$

admits T-quadrature on $[-1, +1]$.

(b) *Let* $\Phi(z) = a_0 + a_1 z + a_2 z^2 + \ldots$, $a_i \in \mathbb{R}$ *for* $i \in \mathbb{N}_0$, *be analytic on* U_8 *and suppose that* $|\Phi(z)| \le 1/8$ *for* $z \in U_8$. *Then*

$$w(x) = \frac{1}{\pi\sqrt{1-x^2}} \operatorname{Re} \frac{1}{1+z\Phi(z)} , \quad x = \tfrac{1}{2}\left(z + \frac{1}{z}\right) , \quad z = e^{i\varphi} , \quad \varphi \in (0, \pi)$$

admits T-quadrature on $[-1, +1]$.

Proof. (a) Let

$$F(z) = \int_{-\pi}^{+\pi} \frac{d\sigma(t)}{1-\alpha e^{it}z} = 1 + \sum_{k=1}^{\infty} c_k z^k .$$

Then F is analytic on $U_{1/|\alpha|}$ and

$$\operatorname{Re} F(z) = \int_{-\pi}^{+\pi} \operatorname{Re} \frac{1}{1-\alpha e^{it}z} \, d\sigma(t) \ge \frac{1}{1+|\alpha|} \quad \text{for} \quad |z| \le 1 .$$

Thus $\pi\sqrt{1-x^2}\, w(x) = \operatorname{Re} F\left(e^{i\varphi}\right)$, $x = \cos\varphi$, $\varphi \in (0, \pi)$, is positive and continuous on $(-1, +1)$. Furthermore we have that

$$c_k = \frac{2}{\pi} \int_0^{\pi} \operatorname{Re} F\left(e^{i\varphi}\right) e^{-in\varphi} d\varphi = 2 \int_{-1}^{+1} T_k(x) w(x) dx .$$

In view of Corollary 1 it suffices to show that p_n^* has no zero in U_2 . Putting $E_1 = U_2$ and $E_2 = U_4$ in Lemma 1 and observing that

$$G(z) = - \int_{-\pi}^{+\pi} \ln\left(1-\alpha e^{it}z\right) d\sigma(t)$$

we obtain, by using the inequality

$$1 - |\alpha| r \le \left|1-\alpha e^{it}z\right| \le 1 + |\alpha| r \quad \text{for} \quad z = re^{i\varphi} ,$$

that

$$\min_{\xi\in\partial U_4}\int_{-\pi}^{+\pi}\ln 1/|1-\alpha e^{it}\xi|\,d\sigma(t) - \max_{z\in\partial U_2}\int_{-\pi}^{+\pi}\ln 1/|1-\alpha e^{it}z|\,d\sigma(t)$$

$$\geq \ln 1/(1+4|\alpha|) - \ln 1/(1-2|\alpha|) \geq \ln \tfrac{1}{2} = \ln \delta$$

from which the assertion follows.

(b) Let F be defined as in the proof of *(a)*. Then F is analytic on U_8 and $\operatorname{Re} F(z) > \frac{1}{2}$ for $z \in U_8$. Thus $F(z) = 1/\big(1+z\Phi(z)\big)$, where Φ is analytic on U_8 and satisfies $|\Phi(z)| \leq 1/8$ for $z \in U_8$.

EXAMPLE. From Theorem 2 we obtain immediately that the weight function

$$w(x) = \frac{1}{\pi\sqrt{1-x^2}}\,\frac{1}{l}\sum_{i=1}^{l}\frac{1+\alpha_i x}{1+\alpha_i^2+2\alpha_i x}, \quad x \in (-1, +1),$$

which was considered in Corollary 2, admits T-quadrature on $[-1, +1]$, if $|\alpha_i| \leq 1/8$ for $i = 1, \ldots, l$.

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