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ENGEL-LIKE ELEMENTS IN INFINITE SOLUBLE GROUPS

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In [1] the first author considered the following Engel-like condition for a pair of elements x, y of a group G.

$$\textit{There exist } r=r(x,y)\geqq 0 \textit{ and } d=d(x,y)\geqq 1 \textit{ such that } [x,{}_r y]=[x,{}_{r+d} y]. \qquad (*)$$

He studied the situation where (*) is satisfied by all pairs of elements in a soluble group and proved that this is precisely equivalent to the group being locally finite-by-nilpotent, a result analogous to the fact, established by Gruenberg in [3], that a soluble Engel group is locally nilpotent.

Just as in the case of the stronger Engel condition, (*) gives rise for an arbitrary group to two sets of elements: $A(G)$, the set of all y in G such that (*) holds for all x in G, so that $A(G)$ contains the set $L(G)$ of all left Engel elements of G, and $B(G)$, the set of all x in G such that (*) holds for all y in G, so that $B(G)$ contains the set $R(G)$ of all right Engel elements of G. Moreover, as in [4], one can show that $x\in B(G)$ implies $x^{-1}\in A(G)$.

The object of this paper is to study some of the properties of $A(G)$ and $B(G)$ for soluble groups G. In [3] Gruenberg showed that if G is a soluble group both $L(G)$ and $R(G)$ are subgroups. However it is not hard to see (Example 1) that $A(G)$ is not in general a subgroup, not even for metabelian groups G. Not all is lost though; for any group G, if we define $A^*(G)$ to be the set of all z in G such that (*) holds for all powers $y=z^i$ of z and all x in G, then we have, as a direct consequence of the proof of Proposition 2 below, the fact that:

If G is a locally soluble group then $A^*(G)$ is a subgroup, in fact the unique largest normal locally finite-by-nilpotent subgroup of G.

For $B(G)$ the situation is better. As our first main result we prove:

Theorem A. *Let G be a locally soluble group. Then $B(G)$ is a characteristic subgroup of G and $B(G/B(G))=1$.*

In [2] the second author proved that in a finitely generated soluble group G the set of right Engel elements coincides with the hypercentre $Z_\infty(G)$ of G. Using the same basic method we shall prove:

Theorem B. *Let G be a finitely generated soluble group. Then $F(G) \leqq B(G)$ and $B(G)/F(G) = Z_\infty(G/F(G))$.*

Here $F(G)$ denotes the subgroup generated by all finite normal subgroups of G.

The behaviour of Engel-like elements is well-illustrated by the class of abelian-by-cyclic groups. Let A be a torsion-free abelian normal subgroup of a group H so that there is an element x in H with $H = \langle A, x \rangle$. The following example shows that it is possible for x to lie in $A(G)$ without x^{-1} doing so.

Example 1. Let

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \alpha \neq 0 \right\}$$

and let

$$x = \begin{pmatrix} 1+i & 0 \\ 0 & 1 \end{pmatrix}.$$

As $-1 + x = (-1 + x)^5$ we see that $x \in A(H)$. However

$$-1 + x^{-1} = \begin{pmatrix} (-1-i)/2 & 0 \\ 0 & 0 \end{pmatrix}$$

and $|(-1-i)/2| < 1$. So $(-1 + x^{-1})^r = (-1 + x^{-1})^{r+d}$ if and only if $d = 0$. Hence $x^{-1} \notin A(H)$.

On the other hand making the stronger assertion that $x \in A^*(H)$ is enough to ensure that x is actually a left Engel element of H and so $\langle x \rangle \leqq L(H)$. This is immediate from the following basic result about Engel-like elements.

Lemma 1. ([1, Lemma 1]). *Let A be a torsion-free abelian group and let Y be a cyclic subgroup of* Aut(A). *Assume that for each $a \in A$ and each $y \in Y$ there exist positive integers $r < s$ such that $a^{(-1+y)^r} = a^{(-1+y)^s}$. If Y is finite then $Y = 1$; if Y is infinite then for each $a \in A$ there exists a positive integer $r = r(a)$ with $a^{(-1+y)^r} = 1$ for all $y \in Y$.*

Lemma 1 can also be used as follows to deduce that if $x \in B(H)$ then it is also a left Engel element.

Lemma 2. *Let A be a torsion-free abelian normal subgroup of a group H and let $x \in B(H)$. Then for all $a \in A$ there exists a positive integer $r = r(a, x)$ such that $a^{(-1+x)^r} = 1$. Moreover if $x^t \in C_H(a)$ for some $t \geqq 1$ then $[a, x] = 1$.*

Proof. For $a \in A$ and $i \in \mathbb{Z}$ we have $[x, {}_k x^i a] = [x, a, {}_{k-1} x^i]$. Hence there exist positive integers $r = r(i, a) < s = s(i, a)$ such that

$$a^{(-1+x)(-1+x^i)^{r-1}} = a^{(-1+x)(-1+x^i)^{s-1}},$$

and so

$$a^{(-1+x^i)^r} = a^{(-1+x^i)^s}.$$

So Lemma 1 implies $a^{(-1+x)^r} = 1$ for some $r = r(1, a)$, and $[a, x] = 1$ if $x^t \in C_H(a)$ for some $t \geqq 1$.

Proofs. Our first objective will be to prove the first part of Theorem A which we split off as:

Proposition 1. *Let G be a locally (soluble-by-finite) group. Then the set $B(G)$ is a subgroup.*

This fact will be deduced from:

Proposition 2. *Let H be a locally (soluble-by-finite) group. Then the subgroup generated by finitely many elements of $B(H)$ is finite-by-nilpotent.*

To facilitate notation, let $\mathscr{F}$ and $L\mathscr{F}$ respectively denote the classes of finite-by-nilpotent and locally (finite-by-nilpotent) groups.

We shall need the following factorisation theorem.

Lemma 3. *Let $\Gamma = \langle x_1, \dots, x_n \rangle$ be an $\mathscr{F}$-group. Then $\Gamma = \langle y_1 \rangle \cdots \langle y_m \rangle$ where $y_i \in \{x_1, \dots, x_n\}$ for all i.*

Proof. As the torsion subgroup of Γ is finite, it lies in a product of finitely many subgroups $\langle x_i \rangle$, so assume that Γ is torsion-free nilpotent of class c, say. Let $A = \gamma_c(\Gamma)$. By induction on c we have $\Gamma/A = \langle Ay_1 \rangle \cdots \langle Ay_m \rangle$ where each y_i is some x_j. Setting $P = \langle y_1 \rangle \cdots \langle y_m \rangle$ we thus have $\Gamma = AP$. Now there exist elements $a_i = [g_i, h_i]$ where $g_i \in \gamma_{c-1}(\Gamma)$, $h_i \in \Gamma$ and $A = \langle a_1 \rangle \cdots \langle a_s \rangle$. For each $r \in \mathbb{Z}$ we have $a_i^r = [g_i^r, h_i]$. As g_i^r and h_i are contained in AP, we have $a_i^r \in PPPP$ and so A is contained in a product of cyclic subgroups generated by some x_i.

Lemma 3 is a variation on Proposition 1 of Kropholler [5]. Here though, by restricting to $\mathscr{F}$-groups, it is possible to be more precise about the generators of the cyclic subgroups in the factorisation. This is necessary because in Lemma 4 we are given information only about a particular generating set, and not about all elements, of Γ.

Lemma 4. *Let $\Gamma = \langle x_1, \dots, x_n \rangle$ be a finitely generated $\mathscr{F}$-group and let A be a finitely* *generated $\mathbb{Z}\Gamma$-module. Suppose that $a\mathbb{Z}\langle x_i\rangle$ is a finitely generated $\mathbb{Z}$-module for every $a\in A$, $1\leqq i\leqq n$. Then A is a finitely generated $\mathbb{Z}$-module.*

Proof. By Lemma 3, we have $\Gamma=\langle y_1\rangle\cdots\langle y_m\rangle$ where each y_j is some x_i. Let $a\in A$. Then $a\mathbb{Z}\Gamma=a\mathbb{Z}\langle y_1\rangle\cdots\langle y_m\rangle$ and so a simple induction shows that $a\mathbb{Z}\Gamma$ is a finitely generated $\mathbb{Z}$-module. The result follows as A is finitely generated as $\mathbb{Z}\Gamma$-module.

The next result provides a sufficient criterion for a finitely generated soluble group to act nilpotently.

Lemma 5. *Let A be a free abelian group of finite rank acted upon by a finitely generated soluble-by-finite group $\Gamma=\langle x_1,\dots,x_n\rangle$. Suppose for each $1\leqq i\leqq n$ and each $a\in S$ that $[a,{}_rx_i]=1$ for some $r=r(a,i)\geqq 1$. Then $[A,{}_t\Gamma]=1$ for some $t\geqq 1$.*

Proof. It is enough to show that if Γ acts faithfully and rationally irreducibly on A then $\Gamma=1$. By Mal'cev's Theorem (see [7]), Γ is abelian-by-finite, so there exists an abelian normal subgroup Γ_0 of Γ of finite index m. This implies that $\Gamma_1=\langle x_1^m,\dots,x_n^m\rangle^\Gamma$ is abelian and can be generated by finitely many elements each of which has the form $y=(x_i^m)^\gamma$ for some i and $\gamma\in\Gamma$. Moreover for any $a\in A$ there is a positive integer r such that $[a,{}_ry]=1$.

By Clifford's Theorem, $A\otimes\mathbb{Q}$ is a direct sum of irreducible $\mathbb{Q}\Gamma_1$-modules. Since Γ_1 is abelian and generated by elements acting unipotently, it acts trivially on such a simple $\mathbb{Q}\Gamma_1$-module, so $\Gamma_1=1$. Hence Γ is finite and Lemma 2 finally implies $\Gamma=1$.

We can now prove Proposition 2. Note that the following argument also proves the statement about $A^*(G)$ in the introduction, provided that the definition of $A^*(G)$ is appealed to, instead of Lemma 2.

Proof of Proposition 2. Let H be a counterexample and let $x_1,\dots,x_n\in B(H)$ be such that $\langle x_1,\dots,x_n\rangle$ is not an $\mathscr{F}$-group. Without loss we may assume $H=\langle x_1,\dots,x_n\rangle$. As finitely generated $\mathscr{F}$-groups are finitely presented, we may assume that every proper quotient of H is an $\mathscr{F}$-group. Let A be a nontrivial abelian normal subgroup of H. As H/A is finitely presented, A is a finitely generated $\mathbb{Z}(H/A)$-module. So Lemma 2 and Lemma 4 imply that A is a finitely generated abelian group. By the choice of H the group A is torsion-free and so Lemma 5 implies $A\leqq Z_t(H)$ for some $t\geqq 1$. If T_1/A denotes the torsion subgroup of H/A then T_1/A is finite and the above implies that T_1 is an $\mathscr{F}$-group. In particular the torsion subgroup T_2 of T_1 is finite, so $T_2=1$ again by the choice of H. This implies that H is nilpotent, a final contradiction.

In order to deduce Proposition 1 from Proposition 2, we need another series of lemmas.

Lemma 6. *Let u and v be elements of a group G and let $0\leqq r\leqq\infty$. Then $\langle[u,{}_kv]\mid 0\leqq k\leqq r\rangle=\langle u^{v^k}\mid 0\leqq k\leqq r\rangle$.*

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Proof. By induction on k one proves that $[u,_k v]$ is a product of elements of the form u^{v^j} and their inverses where $j \leqq k$ and there is exactly one such term with $j=k$. The result follows.

Lemma 7. *Let G be a group, let $a,b \in B(G)$ and let $x \in G$. Define*

$$N = \langle [a,_k x], [a,_k x^{-1}], [b,_k x], [b,_k x^{-1}] \mid k \geqq 0 \rangle$$

and set $U = \langle N, x \rangle$. Then N and U are finitely generated and N is a normal subgroup of U.

Proof. As $a,b \in B(G)$, only finitely many of the generators of N are distinct. Moreover, Lemma 6 implies $N = \langle a^{x^k}, b^{x^k} \mid k \in \mathbb{Z} \rangle$ and hence $N = N^x$.

Lemma 8. *U/N' is an $\mathscr{F}$-group.*

Proof. Let $\bar{U} = U/N'$. We show that every pair of elements in $\bar{U}$ satisfies (*). As U/N is abelian, it suffices to show that for any $c \in \bar{N}$ and every power x^t of x there exist positive integers $r<s$ such that $[c,_r \bar{x}^t] = [c,_s \bar{x}^t]$. As $\bar{N}$ is abelian, it will be enough to prove this for all elements c contained in some system of generators of $\bar{N}$. For example, let $c = [\bar{a},_k \bar{x}^e]$ where $e = \pm 1$. As $a \in B(G)$, there exist $r<s$ such that $[a,_r x^t] = [a,_s x^t]$. Hence we have

$$[c,_r \bar{x}^t] = [\bar{a},_k \bar{x}^e,_r \bar{x}^t] = [\bar{a},_r \bar{x}^t,_k \bar{x}^e] = [\bar{a},_s \bar{x}^t,_k \bar{x}^e] = [c,_s \bar{x}^t]$$

as $\bar{U}$ is metabelian. The claim now follows from Lemma 7 and the main result of [1].

Proof of Proposition 1. Let $a,b \in B(G)$ and let $x \in G$. Adopting the notation of Lemma 7, we know that U/N' is an $\mathscr{F}$-group. Moreover, Lemma 7 and Proposition 2 imply $N \in \mathscr{F}$. Finally, a result of Hall type due to Lennox [6] implies $U \in \mathscr{F}$ and the claim follows.

Factoring out the hypercentre of a group always yields a group with trivial hypercentre. We now prove that there is a corresponding result for the function B.

Proposition 3. *Let H be a locally (soluble-by-finite) group. If $H/B(H) \in L\mathscr{F}$ then $H \in L\mathscr{F}$.*

Proof. Let K be a counterexample and let L be a finitely generated subgroup of K that is not an $\mathscr{F}$-group. As $L \cap B(K) \leqq B(L)$, we infer from the main result of [1] that $L/B(L) \in \mathscr{F}$. So we may assume K is finitely generated. If $\bar{K} = K/N$ is a quotient of K then $\overline{B(K)} \leqq B(\bar{K})$ implies $\bar{K}/B(\bar{K}) \in \mathscr{F}$. Thus, as in the proof of Proposition 2, we may assume that every proper quotient of K is an $\mathscr{F}$-group. Moreover $B(K) \neq 1$.

Now Proposition 1 implies that $B(K)$ is a subgroup, and so we can choose a nontrivial abelian normal subgroup N of K contained in $B(K)$.

We may choose N to be either torsion or torsion-free, so first assume that N is a torsion group, let $a \in N$, $a \neq 1$ and let $x \in K$. As $a \in B(K)$, the subgroup $\langle [a,_k x] \mid k \geqq 0 \rangle$ of N is finitely generated and so it is finite. Now Lemma 6 implies that a has only finitely many conjugates of the form a^{x^k} and so some nontrivial power of x centralises a. Hence [**1**, Lemma 3] implies that a has only finitely many conjugates in K and so K has a nontrivial finite normal subgroup F. As K/F is finite-by-nilpotent, the claim follows.

Now allow N to be torsion-free and let T/N be the torsion subgroup of K/N. So T/N is finite and Lemma 1 implies $N \leqq Z(T)$. Hence T' is a finite normal subgroup of K and, as above, we see that T is torsion-free abelian. Moreover K/T is torsion-free nilpotent.

We now show that K satisfies an Engel condition. Let $x, y \in K$. Then $b = [x,_k y] \in T$ for some $k \geqq 1$. As T/N is finite, we have $b^n \in N$ for some $n \geqq 1$. Now Lemma 1 implies $[b^n,_l y] = 1$ as $b^n \in B(K)$. As T is abelian, this yields $[b,_l y]^n = 1$ and we get $[x,_{k+l} y]^n = 1$. But T is torsion-free and so $[x,_{k+l} y] = 1$. This shows that K is nilpotent, a final contradiction.

Proposition 4. *Let G be a locally (soluble-by-finite) group. Then $B(G/B(G)) = 1$.*

Proof. Let $R/B(G) = B(G/B(G))$ and let $a \in R$. For $x \in G$ we consider $U = \langle a, x \rangle$ and $\bar{U} = UB(G)/B(G)$. As $\bar{a} \in B(\bar{U})$, we infer that $\bar{U}/B(\bar{U}) = \langle \bar{x}B(\bar{U}) \rangle$ is cyclic. Hence Proposition 3 shows $\bar{U} \in L\mathscr{F}$ and another application of Proposition 3 proves $U \in L\mathscr{F}$. Hence there exist $r < s$ such that $[a,_r x] = [a,_s x]$ and so $a \in B(G)$ as required.

This proves Theorem A. For Theorem B we need some more preparation.

Lemma 9. *Let G be a finitely generated group and let $P(G)$ be the join of all normal polycyclic subgroups of G. Then $G/P(G)$ contains no nontrivial polycyclic normal subgroups.*

Proof. Let $P = P(G)$ and let $x \in G \setminus P$ such that $\langle x^G \rangle P/P$ is polycyclic. Let $G = \langle g_1, \ldots, g_n \rangle$ and $\langle x^G \rangle P = \langle x_1, \ldots, x_m \rangle P$ with $x_1 = x$. Consider the subgroup $U = \langle x_1, \ldots, x_m, [x_i, g_j^{\pm 1}] \mid 1 \leqq i \leqq m, 1 \leqq j \leqq n \rangle$. So U is finitely generated and $U/U \cap P$ is polycyclic. Thus $U \cap P$ is finitely generated as a U-group.

Let $V = (U \cap P)^G$, so V is finitely generated as a G-group and because $V \leqq P$, it is polycyclic. We claim that $W = V\langle x_1, \ldots, x_m \rangle$ is a normal subgroup of G. In fact, it is enough to establish that $W^{g_j^{\pm 1}} \leqq W$ for each j. Since V is normal in G, we only need to prove $x_i^{g_j^{\pm 1}} \in W$. We have $x_i^{g_j^{\pm 1}} = x_i [x_i, g_j^{\pm 1}]$. Now $[x_i, g_j^{\pm 1}] \in U \leqq \langle x_1, \ldots, x_m \rangle P$ by definition of U. Hence we get $[x_i, g_j^{\pm 1}] = h_1 h_2$ for some $h_1 \in \langle x_1, \ldots, x_m \rangle$ and $h_2 \in P$. Now $h_2 = h_1^{-1}[x_i, g_j^{\pm 1}] \in U$ implies $h_2 \in U \cap P \leqq V$ and so $x_i^{g_j^{\pm 1}} \in x_i h_1 V \subseteq \langle x_1, \ldots, x_m \rangle V = W$.

We now prove that W is polycyclic. Indeed, we have

$$W \cap P = V\langle x_1, \ldots, x_m \rangle \cap P = V(\langle x_1, \ldots, x_m \rangle \cap P) \leqq V(U \cap P) = V$$

by Dedekind's law, and so $W \cap P$ is polycyclic. Moreover, by definition of W we have $W/W \cap P \cong \langle x_1, \ldots, x_m \rangle P/P$, so $W/W \cap P$ is polycyclic and hence W is polycyclic.

As $x \in W$ and W is normal in G, we have $\langle x^G \rangle \leqq W$. So $\langle x^G \rangle$ is a polycyclic normal subgroup of G and thus $x \in P$.

Proof of Theorem B. Let $C/F(G) = Z_\infty(G/F(G))$ and suppose $C \lneqq B(G)$. Let D/C be a nontrivial abelian normal subgroup of G/C with $D \leqq B(G)$. We are going to construct a certain $\mathbb{Z}G$-module M of the form $M = \langle x^G \rangle C/C$ for some $x \in D \backslash C$. Indeed, if D/C contains an element xC of prime order, then we choose this x. Otherwise we choose any element $x \in D \backslash C$. For $a \in M$, $y \in G$ we have $[a, {}_r y] = [a, {}_s y]$ for some $r < s$ and Lemma 1 implies that a is annihilated by $f(y)$ where f is some nontrivial integral polynomial. Hence M is a finitely generated constrained $\mathbb{Z}G$-module in the sense of **[2]** and hence M is a finitely generated abelian group. In particular, M is a polycyclic normal subgroup of G/C. Now a threefold application of Lemma 9 shows that $F(G)$, C and finally $\langle x^G \rangle C$ lie in P. In particular $\langle x^G \rangle$ is finitely generated and $x \in B(G)$ implies that $\langle x^G \rangle$ is finite-by-nilpotent. Let T be its torsion subgroup. By Lemma 5, applied to the upper central factors of $\langle x^G \rangle / T$ (which are torsion-free), we have $[\langle x^G \rangle, {}_t G] \leqq T$ for some $t \geqq 1$. But $T \leqq F(G)$ and so $\langle x^G \rangle F(G)/F(G) \leqq C/F(G)$. Thus $x \in C$, a contradiction.

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