Canad. J. Math. Vol. **62** (3), 2010 pp. 543–562
doi:10.4153/CJM-2010-036-3

More Variations on the Sierpiński Sieve

Kevin G. Hare

Abstract. This paper answers a question of Broomhead, Montaldi and Sidorov about the existence of gaskets of a particular type related to the Sierpiński sieve. These gaskets are given by iterated function systems that do not satisfy the open set condition. We use the methods of Ngai and Wang to compute the dimension of these gaskets.

1 Introduction and Basic Definitions

This paper is motivated by the paper of Broomhead, Montaldi and Sidorov [1]. In their paper, the authors looked at a variation of the Sierpiński sieve that allowed some overlap between triangular subregions. They showed that the gasket dimensions could be computed when the overlap was sufficiently nice. In their conclusions, they raised the question of what happens if, instead of triangles, the maps are based on regular n-gons with $n \geq 5$.

This paper answers this question, showing that it is possible to have gaskets of this type for $n = 3, 6, 8$, and 12, and that these are the only values of n for which gaskets of this type can be constructed. We also show how the methods of [7] can be used to compute the dimension of these gaskets and we compute their dimension.

Before summarizing our results, we introduce some notation and definitions.

Definition 1.1 An *iterated function system* (IFS) is a family of contractions $\{f_1, f_2, \dots, f_n\}$. The *attractor* of the IFS is the unique non-empty compact set K such that $K = \bigcup f_i(K)$.

Since their introduction, IFSs and their attractors have been widely studied (see [2, 3] and the references listed therein). A useful condition in the study of IFSs and their attractors is the open set condition.

Definition 1.2 We say that an IFS satisfies the *open set condition* (OSC), if there exists a non-empty open set $\mathcal{O}$ such that

- $f_i(\mathcal{O}) \subset \mathcal{O}$ for all i.
- $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \varnothing$ for all $i \neq j$.

Intuitively one can think of this as saying that $f_i(K)$ and $f_j(K)$ do not have any significant overlap. In practice the construction of this open set $\mathcal{O}$ can be quite complicated. If an IFS satisfies the OSC, then it is relatively straightforward to compute

Received by the editors November 21, 2007.
Published electronically March 18, 2010.
The author's research was supported in part by NSERC of Canada.
AMS subject classification: **28A80**, 28A78, 11R06.

the dimension of its attractor (see for example [2, 3, 5]). The IFSs studied in this paper do not satisfy the OSC.

There are a number of techniques that can be used to compute the dimension of an IFS if the overlapping components $f_i(K)$ and $f_j(K)$ look nice. By "nice", we mean that their construction satisfies the finite type condition that will be defined formally in Section 3. See for instance [6, 7].

It is worth noting that that the box counting dimension and Hausdorff dimension are equal when the f_i are similitudes (as they are in our case). (See [3, Theorem 9.3] and the comments directly following its proof.)

The general construction of these gaskets is based on the contraction maps $f_i(x) = \lambda x + b_i$, where $x \in \mathbb{R}^2$ and the b_i are the vertices of a regular n-gon. If we wish to specify the value of n used in the construction, then we will say n-gasket, instead of gasket. Not all values of λ give interesting gaskets. If λ is too large, then the resulting attractor will be a filled-in n-gon of dimension 2. If λ is too small, then the IFS will satisfy the OSC, and its dimension can be trivially computed.

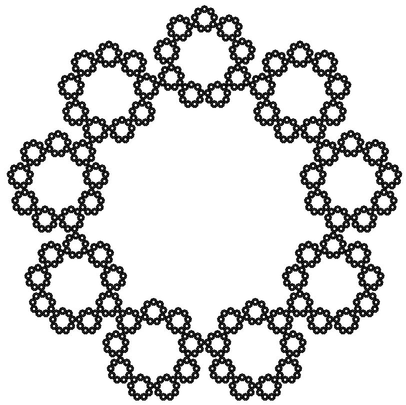
In Section 2 we give a description of the contraction ratios used to find n-gaskets with nice overlap. We also show that this method only works for triangles, hexagons, octagons, and dodecagons. In Section 3, a brief discussion of the algorithm of Ngai and Wang [7] is given. Attention is given to how it can be used to compute the dimension of the gaskets studied in this paper. The interested reader is encouraged to read the original article for a more general discussion. In Section 4 we compute the dimension of these n-gaskets, using the contraction ratios given in Section 2. Some discussion of further directions for this research is given in Section 5.

2 Contraction Ratios

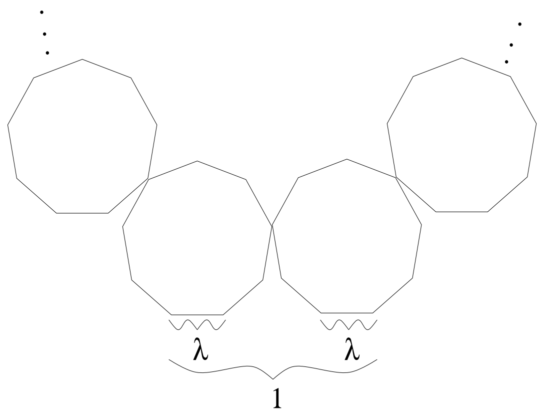
In this section, we show how to determine the contraction ratios used in the construction of n-gaskets with nice overlap. Before we begin, we determine the contraction ratio for the non-overlapping case. The result for the non-overlapping case is not new, but the technique used is useful for demonstration purposes. A variation of this technique is used later for finding the contraction ratios with nice overlaps. Consider a non-overlapping n-gasket with side length 1. We see that this n-gasket is composed of n components, each with side length λ for some λ. These components are scale copies of the original fractal. Our goal is to find this λ, such that these n components touch, but do not overlap. The outside corners of these components are equal to the outside corners of the original n-gasket. In addition, the corner of a component closest to an adjacent component, will touch that adjacent component. We will denote this ratio as $\lambda = \lambda_{n,\infty}$. (This choice of notation will be explained later.) See for example Figure 1.

Consider Figure 2. Let $\theta = \frac{2\pi}{n}$, and let $N = \lfloor \frac{n}{4} \rfloor$. By noticing that the side length of the original gasket is 1, and the side length of the component is λ, we get that

$$1 = 2(\lambda + \lambda\cos(\theta) + \cdots + \lambda\cos(N\theta)) = \lambda\left(1 + \frac{\sin((2N+1)\frac{\theta}{2})}{\sin(\frac{\theta}{2})}\right).$$

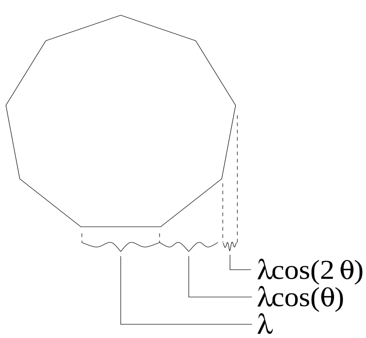


(a) 9-gasket - no overlap

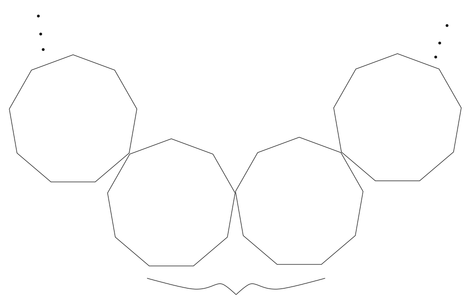


(b) 9-gasket - construction

Figure 1: 9-gon - no overlap



(a) 9-gasket, width



(b) 9-gasket - construction

Figure 2: 9-gon - calculating λ

n	t_n
3	2
4	2
5	$\frac{3+\sqrt{5}}{2} \approx 2.618033989$
6	3
7	3.246979604
8	3.414213562
9	3.879385241
10	4.236067979
11	4.513337092
12	4.732050806
13	5.148114905
14	5.493959207
15	5.783386115
16	6.027339490

Table 1: t_n values

For ease of notation, let

$$t_n = 1 + \frac{\sin((2N+1)\frac{\theta}{2})}{\sin(\frac{\theta}{2})} = 1 + \frac{\sin((2\lfloor \frac{n}{4} \rfloor + 1)\frac{\pi}{n})}{\sin(\frac{\pi}{n})}.$$

Using this notation, the width of an n-gon of side length 1 is $t_n - 1$. Furthermore, by noticing that $1 = \lambda t_n$, we get that $\lambda_{n,\infty} = \frac{1}{t_n}$.

In Table 1 we give the values of t_n for small n. These t_n play an important role in computing $\lambda_{n,m}$, the contraction ratio of an n-gasket with nice overlap. The algebraic value of t_5 is used in the proof of Theorem 2.3, and hence both the algebraic value and a floating point approximation is given here.

We use the term *n-gasket of order m* to mean an n-gasket of side length 1, such that adjacent components of side length λ overlap completely at a subcomponent of side length λ^{m+1}. The contraction ratio for such an n-gasket of order m is denoted by $\lambda_{n,m}$. See Figure 3 for a construction of an 8-gasket of order 1, with contraction ratio $\lambda = \lambda_{8,1}$.

Similar to our analysis in Figure 2 we find a defining equation for $\lambda = \lambda_{n,m}$. Notice that the width of an n-gasket with side length λ^{m+1} is $\lambda^{m+1}(t_n - 1)$. This gives that along the side of length 1 of the original n-gasket of order m we have two components of side length λ minus an overlapping subcomponent of side length λ^{m+1}. This gives

$$\lambda^{m+1}(t_n - 1) - \lambda t_n + 1 = 0, \tag{2.1}$$

where $\lambda \in (0, 1)$. It is worth observing that this gives the same equation as [1] in the case when $n = 3$. It is also worth observing that as $m \to \infty$, so $\lambda_{n,m} \to \lambda_{n,\infty}$, which explains our choice of notation earlier. (Geometrically, this makes sense, as the size of the overlapping section is going to zero.)

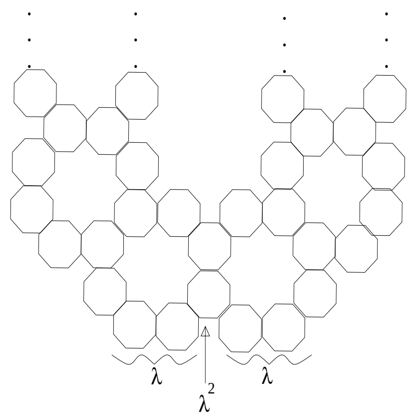
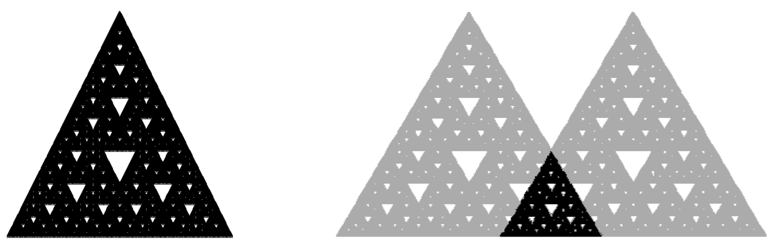


Figure 3: 8-gasket of order 1



(a) gasket (b) gasket - adjacent components and overlap

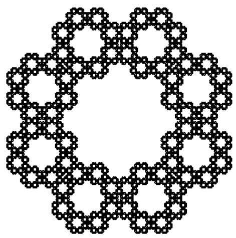
Figure 4: 3-gasket of order 2

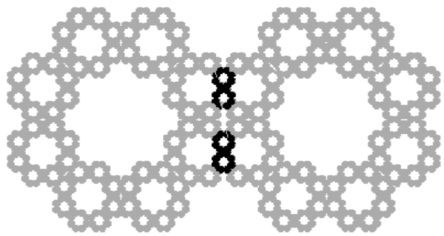
The next question that we need to consider is: what sort of additional overlap do we get? Consider an n-gasket of order m. We know that the two adjacent components of this gasket of side length λ will overlap in a subcomponent of side length λ^{m+1}. The main question is: do we get any other overlapping subcomponents of these gaskets, and if so, how "nice" is this overlap? Here, by nice overlap we mean overlap that is the complete overlap of subcomponents of size λ^k for some $k > m$. This will become clear when we look at the details of Theorems 2.1, 2.2, and 2.3.

In the case of $n = 3$, or $n = 8$, with $m \geq 2$, there is no additional overlap. In the case of $n = 6, 12$ or $n = 8$ with $m = 1$, there is additional overlap, but this additional overlap is the complete overlap of subcomponent of side length λ^k for some larger values of k. For all other n, there is additional overlap, but the overlap is not complete overlap of smaller subcomponents. This is summarized by the next three theorems.

Theorem 2.1 *If $n = 3$ with $m \geq 2$ or $n = 8$ with $m \geq 2$, then there is no additional overlap.*

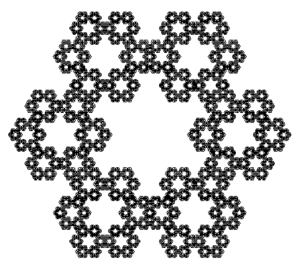
https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press



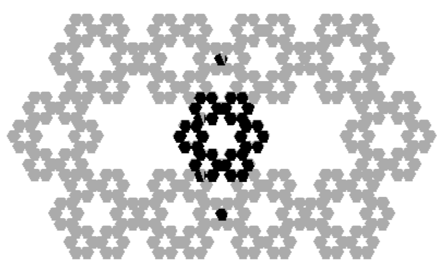


(a) gasket (b) gasket - adjacent components and overlap

Figure 5: 8-gasket of order 2



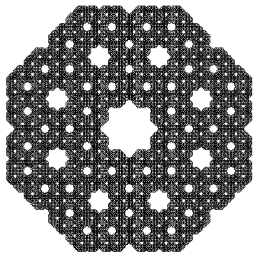
(a) gasket



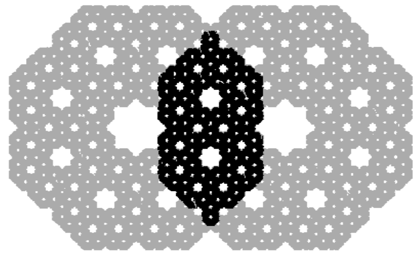
(b) gasket - adjacent components and overlap

Figure 6: 6-gasket of order 3

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press

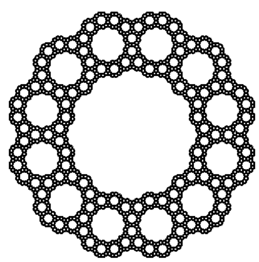


(a) gasket

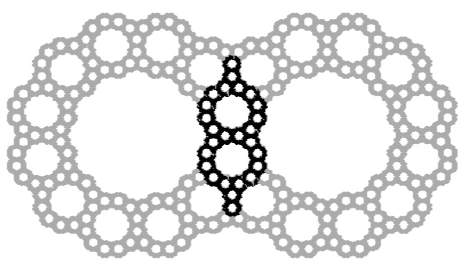


(b) gasket - adjacent components and overlap

Figure 7: 8-gasket of order 1



(a) gasket



(b) gasket - adjacent components and overlap

Figure 8: 12-gasket of order 1

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press

Proof Consider two adjacent components of side length λ of a 3-gasket of order m. The only region of overlap for these two components is a triangle of side length λ^{m+1}, hence there is no additional overlap. See Figure 4. In this case we see that the overlap of the convex hulls of these subcomponents is a triangle of size λ^{m+1}, regardless of how large m is. Thus, even though Figure 4 is for $m = 2$, we see that this argument carries over for all m.

Consider the two adjacent components of side length λ of an 8-gaskets of order m, for $m \geq 2$ We see that the only overlap is 2^m components of side length λ^{m+1}, hence there is no additional overlap. See Figure 5. This case is more complicated than the triangular case. Consider again the overlapping subcomponents of size λ^{m+1}. We see that there would be additional overlap if the neighbours of these subcomponents overlapped. Consider the center of the overlapping subcomponent of size λ^{m+1}, (call this a), and the centers of its two neighbours (say b and c). A quick calculation shows that the distance from a to b, and from a to c is equal to $(1 + \sqrt{2})(\lambda^{m+1} - \lambda^{2m+1})$. As the angle made by bac is a right angle, we then get that the distance between b and c is $(2 + \sqrt{2})(\lambda^{m+1} - \lambda^{2m+1})$. As the subcomponent with centers b and c has width $\lambda^{m+1}(1 + \sqrt{2})$ and

$$\lambda^{m+1}(1 + \sqrt{2}) < (2 + \sqrt{2})(\lambda^{m+1} - \lambda^{2m+1}) \tag{2.2}$$

for $m \geq 2$, we get that there is no additional overlap. For $m = 1$, we have that $\lambda = \frac{1}{1+\sqrt{2}}$ and a quick calculation shows that equation (2.2) does not hold, which is why it is a special case. (See Figure 7 and the discussion in Section 4.2.) ■

Theorem 2.2 *If $n = 6$ with $m \geq 2$, $n = 8$ with $m = 1$, or $n = 12$ with $m \geq 1$, then there is additional overlap. This additional overlap is the complete overlap of subcomponents of side length λ^k for some $k > m$.*

Proof Consider the two adjacent components of side length λ of a 6-gasket of order m. We see that the overlap between these two components is one gasket of side length λ^{m+1}, two gaskets of side length λ^{2m+1}, and in general 2^{k-1} gaskets of side length λ^{km+1}. See Figure 6. Consider the overlapping gasket of size λ^{m+1}. We see that as m increases, the size of the overlap will decrease, but the center of this overlap will always be located in the same location, (midway between the centers of the parent components.) Similarly, if we consider the two overlapping subcomponents of side length λ^{2m+1}, the location of the center will always be the midpoint between the centres of the neighbours of the overlapping subcomponents of side length λ^{m+1}. A similar argument can be used for all of the subcomponents of size λ^{km+1}.

Consider the two adjacent components of side length λ of an 8-gasket of order 1. We see that the overlap between these two regions is two gaskets of side length λ^2, four gaskets of side length λ^3, and in general four gaskets of side length λ^k. See Figure 7. An argument similar to that used for the hexagons can be used to show that this is true for all n.

Consider the two adjacent components of side length λ of a 12-gasket of order m. If $m = 1$, then the overlap is two gaskets of side length λ^2, and four gaskets of side length λ^k for all $k \geq 3$. If $m \geq 2$, then the overlap is 2^m gaskets of side length λ^{m+1}, and in general 2^{m+k-1} gaskets of side length λ^{km+1}. See Figure 8. An argument similar to that used for the hexagons can be used to show that this is true for all n. ■

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press

Theorem 2.3 *For all other n and m there is additional overlap. This additional overlap is not complete overlap of smaller subcomponents.*

Proof Let n be odd, $n \geq 5$. Consider the two adjacent components of an n-gasket of order m, where the components have side length λ, and we have an overlapping subcomponent with side length λ^{m+1}. See Figure 9. Note in Figure 9 the slight overlap between the pentagons is intentional, as I am assuming that there is an overlap of some order. Consider the neighbours of the overlapping subcomponents of side length λ^{m+1}. We claim that these neighbours have additional overlap, and that this additional overlap is not the complete overlap of smaller copies of this gasket. As $n \geq 5$ and n is odd, we see that two of these neighbours will overlap. We notice that the width of this overlap is $(t_n - 2)\lambda^{m+1} + \lambda^{2m+1} = \lambda^{m+1}(t_n - 2 + \lambda^{m+1})$. (We notice that the horizontal distance between A and B, as well as between C and D is $\frac{1}{2}(t_n - 2)\lambda^{m+1}$, and the distance between B and C is λ^{2m+1}.) We see that the width of a gasket with side length λ^k, (with $k \geq m+1$), is $\lambda^k(t_n - 1)$. Letting $k' = k - m + 1$, to show that there is only incomplete overlap, it suffices to show that there are no integer solutions to

$$t_n - 2 + \lambda^m = \lambda^{k'}(t_n - 1) \tag{2.3}$$

for $m, k' \geq 1$.

Notice that

$$\frac{1}{t_n} = \lambda_{n,\infty} \leq \cdots \leq \lambda_{n,3} \leq \lambda_{n,2} \leq \lambda_{n,1} = \frac{1}{t_n - 1}.$$

This shows that the right-hand side of (2.3) is bounded above by $\frac{t_n-1}{t_n-1} = 1$. For $n \geq 7$, we see that the left-hand side of (2.3) is bounded below by $t_7 - 2 \approx 1.24 > 1$. Hence there are no solutions for $n \geq 7$.

So assume that $n = 5$. Recall $t_5 = \frac{\sqrt{5}+3}{2}$. If $k \geq 2$, then the left-hand side is bounded above by $\frac{1}{t_5-1} = t_5 - 2$. The right-hand side of (2.3) is strictly greater than $t_5 - 2$, hence there are no solutions if $n = 5$ and $k \geq 2$.

So assume that $n = 5$ and $k = 1$. Multiplying (2.3) by λ and noticing that $\lambda^{m+1} = \frac{\lambda t_n - 1}{t_n - 1}$, by (2.1) we get

$$\lambda(t_5 - 2) + \frac{\lambda t_5 - 1}{t_5 - 1} = \lambda^2(t_5 - 1).$$

Solving this equation gives $\lambda = \lambda_{5,\infty} = \frac{1}{t_5}$. Hence there are no solutions for n odd, and $n \geq 5$.

A similar argument can be made for when $n \equiv 2 \pmod 4$ and $n \geq 10$ and when $n \equiv 0 \pmod 4$ and $n \geq 16$. The defining equation for $n \equiv 2 \pmod 4$ is

$$t_n - 3 + 2\lambda^m(t_n + 1) = \lambda^k(t_n - 1)$$

and for $n \equiv 0 \pmod 4$ is

$$t_n - 3 - 2\cos\theta + 2\lambda^m(t_n + 1) = \lambda^k(t_n - 1).$$

(Here $\theta = \frac{2\pi}{n}$.) As before, there are no solutions if $n \geq 10$ or $n \geq 16$, respectively, as one side is bounded above by 1, and the other is strictly greater than 1.

■

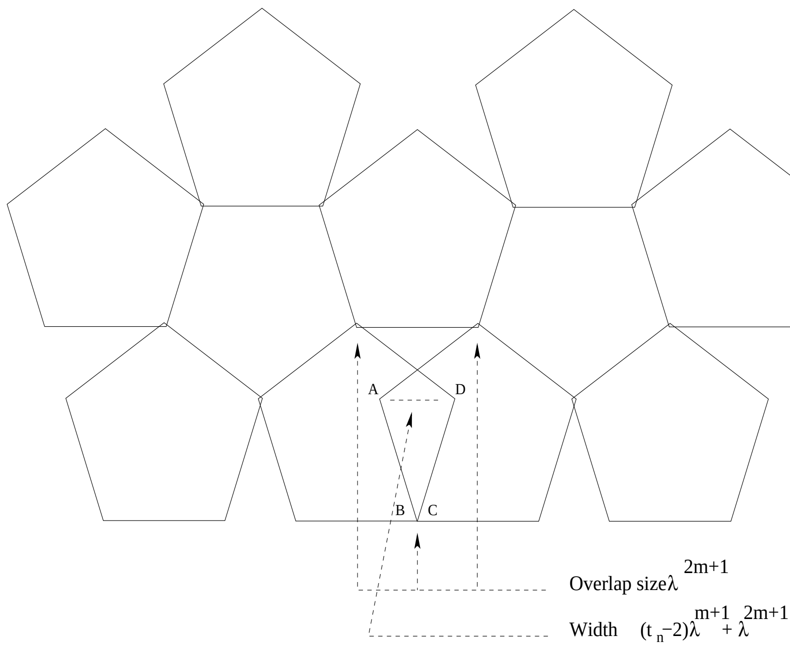


Figure 9: n-gasket for n-odd

3 Algorithm

In this section, we give an overview of the algorithm of Ngai and Wang. We simplify their discussion somewhat, as in our case the contraction ratio for the f_i are all equal. We refer the reader to [7] for a detailed proof as to the correctness of this algorithm.

First, we define formally the maps used to construct an n-gasket of order m. Let $f_j(x) = \lambda x + b_j$, where $x, b_j \in \mathbb{R}^2$. Here the b_j are the vertices of a regular n-gon and $\lambda = \lambda_{n,m}$ is given in equation (2.1). The b_j are scaled such that the attractor has side length 1.

We define $\mathbf{J} \in \{1, 2, \dots, n\}^*$ to be a finite word with symbols $1, 2, \dots, n$. Letting $\mathbf{J} = j_1 j_2 \cdots j_k$ we define $f_{\mathbf{J}} = f_{j_1} \circ f_{j_2} \circ \cdots \circ f_{j_k}$.

Define $R = \frac{1}{1-\lambda}$. Let B_R be the open ball of radius R around 0. We say that $\mathbf{J}_1$ and $\mathbf{J}_2$ are *neighbours* if $|\mathbf{J}_1| = |\mathbf{J}_2|$ and $f_{\mathbf{J}_1}(B_R) \cap f_{\mathbf{J}_2}(B_R) \neq \varnothing$. In our case we can think of each $f_{\mathbf{J}}$ being associated with a particular component of side length $\lambda^{|\mathbf{J}|}$, and we say that two components are neighbours if they are the same size, and they overlap.

We define the *neighbourhood set* of $\mathbf{J}$ as $\Omega(\mathbf{J}) = \{\mathbf{J}' : \mathbf{J}' \text{ is a neighbour to } \mathbf{J}\}$. Based on $\mathbf{J}$, define the map $\tau(x) := \tau_{\mathbf{J}}(x) = \lambda^{-|\mathbf{J}|}(x - f_{\mathbf{J}}(0))$. This map re-scales and re-centers a neighbourhood set $\Omega(\mathbf{J})$, allowing us to compare neighbourhood sets at different depths. We say two neighbourhood sets have the same *neighbourhood type* if there exists a rotational matrix U such that $\tau_{\mathbf{J}_1}(\Omega(\mathbf{J}_1)) = U\tau_{\mathbf{J}_2}(\Omega(\mathbf{J}_2))$.

We say that an IFS is of *finite type* if there is only a finite number of neighbourhood types. If an IFS is of finite type, then it is possible to compute the dimension of the attractor by considering the eigenvalues of an adjacency matrix whose rows and columns are indexed by the neighbourhood types.

Consider a $\mathbf{J} = j_1 j_2 \cdots j_k$ associated with a particular neighbourhood type. We then consider the sequences $\mathbf{J}r = j_1 j_2 \cdots j_k r$, where r takes on the values between 1 and n. We say an edge from $\mathbf{J}$ to $\mathbf{J}r$ is *acceptable* if there are no other $\mathbf{J}'$ with $|\mathbf{J}'| = k+1$ such that $f_{\mathbf{J}'} = f_{\mathbf{J}r}$, where $\mathbf{J}'$ would have precedence. This concept of *precedence* is described in more detail in Section 4. In our implementation, we use a rotational precedence. So, if κ is the largest eigenvalue of the incidence matrix of this graph, then the dimension is

$$\dim(K) = \frac{\log(\kappa)}{-\log(\lambda)}.$$

In the next section, we show that these n-gasket of order m, for $n = 6, 8$, or 12 have finite type, and construct their incidence matrix and compute their dimension.

4 Dimension

In Section 2, we discussed the contraction ratio used for an n-gasket of order m. In Section 3 we gave an overview of the algorithm of Ngai and Wang to compute the dimension of an n-gasket of order m. In this section, we show how to compute the dimension of such gaskets. We do this separately for each n, as the techniques are optimized for each case. Special attention is given to the case $n = 6$, after which we give the highlights only for $n = 8$ and $n = 12$, as the techniques are similar. We do not do the case of $n = 3$, as the results can be found in [1].

4.1 Hexagons

In this subsection we consider the case of a hexagonal gasket of order m for $m \geq 2$. If $m = 1$, then $\lambda_{6,1} = 1/2$, and the resulting gasket is a filled in hexagon with dimension 2.

In a hexagonal gasket of order m, we have that the adjacent components with side length λ overlap at a subcomponent of side length λ^{m+1}. To compute the dimension, we must count the number of subcomponents with side length λ^k, as k goes to infinity. So, in this case, for $k = 0, 1, 2, \ldots, m$ the number of components with side length λ^k is $1, 6, 36, \ldots, 6^m$. In the case when $k = m + 1$, the number of subcomponents is $6^{m+1} - 6$, as there are six subcomponents with side length λ^{m+1} that overlap.

The trick to computing the dimension of this gasket, (or in fact any of these n-gaskets of order m) is to carefully count how many components of side length λ^k there are as $k \to \infty$. Careful consideration needs to be taken about when gaskets overlap completely, as we need to avoid double counting.

For each component of side length λ^k, say A, consider the six subcomponents of side length λ^{k+1}, say $B_1, B_2, \ldots, B_6$, of which A is composed. In this case we say that A is the parent of the B_is, and the B_is are offspring of the A. We similarly define the notion of descendants and ancestors.

Now consider some subcomponent C of side length λ^k such that it is the offspring of two different parents, say A and B. This will happen when C is an overlapping off-

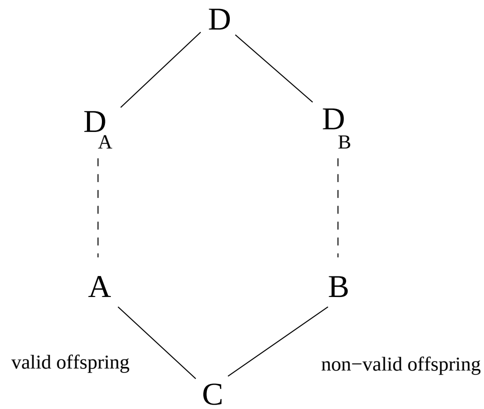


Figure 10: Precedence diagram

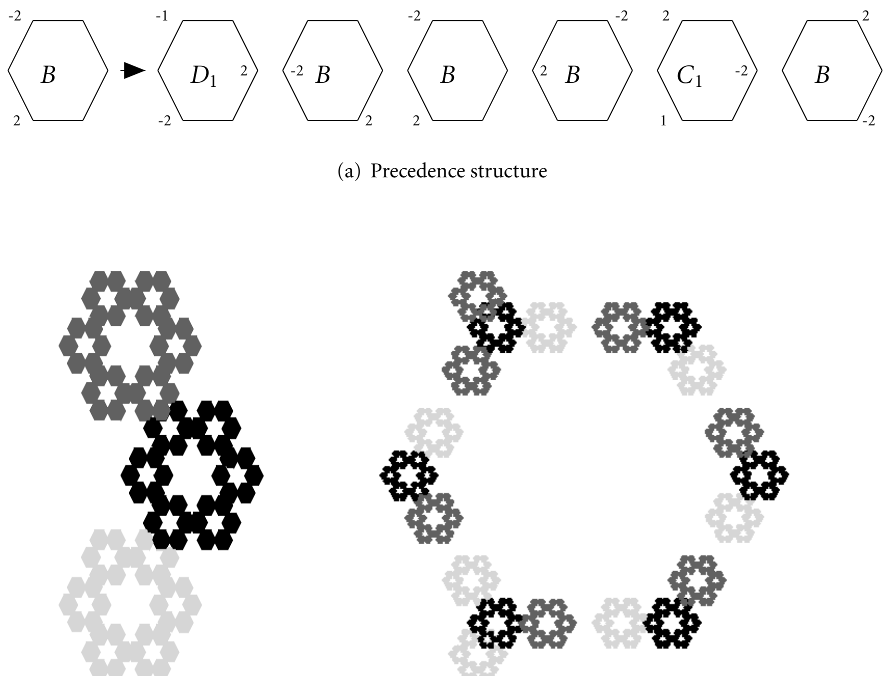
spring subcomponent of two adjacent components. These two parents have a common ancestor D. The nice property, which we can exploit in this case, (but is not true for general IFS), is that we can impose a priority on these parents that is rotational. Let D_A be the offspring of D that contains A and D_B the offspring of D that contains B. If D_A is counter-clockwise to D_B, then we say that A has priority over B. As such, when counting offspring of A and B, we count C as an offspring of A, but not as an offspring of B. This avoids double counting. This is described in the ancestor diagram in Figure 10.

At this point we need to introduce some notation for hexagonal gaskets of order m, (where $m \geq 3$.) We will do this by way of example.

Example 4.1 Consider the following example found by looking at a component of the hexagonal gasket of order 2 in Figure 11. Here black is the component that we are interested in. The light grey and dark grey components are in the neighbourhood of the black component, and included to represent the neighbourhood type of B. We find the six neighbourhood types that are offspring of the neighbourhood type of B.

In Figure 11(a) we have a hexagon (labeled B), with corners labelled -2, 2 or left empty. By labelling the corners with ± 2, we mean that there is a hexagon overlapping this corner, and, moreover, the size of this overlap is λ^2 times that of the parent hexagon. In the case of -2, the overlapping hexagon does *not* have precedence, whereas +2, (or 2), indicates that the overlapping hexagon has precedence. Corners that are left blank indicate that there is no overlapping hexagon with that corner. We can, based on this, determine the six hexagons that the parent hexagon is composed of. This is represented graphically in Figure 11(b) and 11(c), where black is the hexagon we are interested in, light grey represents hexagons with which black has precedence over, and dark grey represents hexagons that have precedence over black.

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press



(a) Precedence structure

(b) Parent

(c) Offspring

Figure 11: Precedence structure and graphical interpretation

If the hexagon gasket has order m, then we see that each corner of a neighbourhood type representation can take the values of empty, or 1 through m, with either positive or negative sign. This means that there are a total of $(2m+1)^6$ possible neighbourhood types. (There are in fact much fewer in practice that we need to worry about.) Hence, these gaskets have finite neighbourhood type. Figure 13 gives our full description of the maps from the neighbourhood types to their offspring. The hexagon A is the neighbourhood type for the starting n-gasket of side length 1.

We notice that because our priority is rotational, we can rotate the existing offspring and treat a number of these gaskets as equivalent.

From this network diagram, we can compute an adjacency matrix, as shown in Figure 12.

We see that $\gamma(k) := [1, 1, \dots, 1]A^k[1, 0, 0, \dots, 0]^T$ gives the number of hexagons of side length λ^k. At this point it suffices to find the maximal eigenvalue of A to tell us what the growth of $\gamma(k)$ looks like.

Let $a, b, \dots,$ be the frequency associated with $A, B, \dots$. Let κ be the eigenvalue

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press

	A	B	C_{m-1}	D_{m-1}	C_{m-2}	D_{m-2}	$\cdots$	C_1	D_1	E	F	G	H	I
A	0	6												
B	0	4	1	1										
C_{m-1}	0	3	1	1	1									
D_{m-1}	0	3	1	1	0	1								
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$							
C_2	0	3	1	1	0	0	$\cdots$	1						
D_2	0	3	1	1	0	0	$\cdots$	0	1					
C_1	0	1	1	1	0	0	$\cdots$	0	0	1	1	1		
D_1	0	1	1	1	0	0	$\cdots$	0	0	0	0	0	1	1
E	0	3	2	1										
F	0	2	3	1										
G	0	3	2	1										
H	0	3	1	2										
I	0	3	0	3										

Figure 12

of A. This tells us that

$$\begin{aligned}
\kappa a &= 0\\
\kappa b &= 4b + 3c_{m-1} + 3d_{m-1} + \cdots + 3c_2 + 3d_2 + c_1 + d_1\\
&\quad + 3e + 2f + 3g + 3h + 3i\\
\kappa c_{m-1} &= b + c_{m-1} + d_{m-1} + \cdots + c_1 + d_1 + 2e + 3f + 2g + h\\
\kappa d_{m-1} &= b + c_{m-1} + d_{m-1} + \cdots + c_1 + d_1 + e + f + g + 2h + 3i\\
\kappa c_{m-2} &= c_{m-1} \cdots \kappa c_1 = c_2\\
\kappa d_{m-2} &= d_{m-1} \cdots \kappa d_1 = d_2\\
\kappa e = c_1 \quad \kappa f &= c_1 \quad \kappa g = c_1 \quad \kappa i = d_1.
\end{aligned}$$

By using the observations that

$$c_k = \kappa^k g, \quad d_k = \kappa^k h, \quad e = g, \quad f = g, \quad i = h,$$

we can rewrite this system to get the relevant equations

$$\begin{aligned}
\kappa b &= 4b + 3(\kappa^{m-1} + \cdots + \kappa^2)(g+h) + \kappa g + \kappa h\\
&\quad + 3g + 2g + 3g + 3h + 3h\\
&= 4b + 3\frac{\kappa^m - \kappa^2}{\kappa - 1}(g+h) + \kappa g + \kappa h + 8g + 6h
\end{aligned} \tag{4.1}$$

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press

$$\begin{aligned}\kappa^m g &= b + (\kappa^{m-1} + \cdots + \kappa)(g+h) + 2g + 3g + 2g + h \\ &= b + \frac{\kappa^m - \kappa}{\kappa - 1}(g+h) + 7g + h\end{aligned} \tag{4.2}$$

$$\begin{aligned}\kappa^m h &= b + (\kappa^{m-1} + \cdots + \kappa)(g+h) + g + g + g + g + 2h + 3h \\ &= b + \frac{\kappa^m - \kappa}{\kappa - 1}(g+h) + 3g + 5h.\end{aligned} \tag{4.3}$$

Subtracting equation (4.2) from (4.3) gives us $0 = (\kappa^m - 4)(h - g)$. Either $\kappa = \sqrt[m]{4}$ or $h = g$. Here we want κ to be the largest eigenvalue, so assume for the moment that $h = g$, to find the other eigenvalues. This then gives us

$$\kappa b = 4b + 6\frac{\kappa^m - \kappa^2}{\kappa - 1}g + 2\kappa g + 14g, \tag{4.4}$$

$$b = \kappa^m g - 2\frac{\kappa^m - \kappa}{\kappa - 1} - 8g. \tag{4.5}$$

Combining (4.4) and (4.5) and noticing that $g \neq 0$ gives the equation for κ.

$$\begin{aligned}&\kappa\Big(\kappa^m - 2\frac{\kappa^m - \kappa}{\kappa - 1} - 8\Big) - \Big(4\Big(\kappa^m - 2\frac{\kappa^m - \kappa}{\kappa - 1} - 8\Big) + 6\frac{\kappa^m - \kappa^2}{\kappa - 1} + 2\kappa + 14\Big) \\ &= \kappa^{m+1} + \frac{-2\kappa^{m+1} + 2\kappa^2}{\kappa - 1} - 8\kappa - 4\kappa^m + \frac{8\kappa^m - 8\kappa}{\kappa - 1} + 32 + \frac{-6\kappa^m + 6\kappa^2}{\kappa - 1} \\ &\qquad - 2\kappa - 14 \\ &= \kappa^{m+1} + \frac{-2\kappa^{m+1} + 2\kappa^m + 8\kappa^2 - 8\kappa}{\kappa - 1} - 4\kappa^m - 10\kappa + 18 \\ &= \kappa^{m+1} - 6\kappa^m - 2\kappa + 18.\end{aligned}$$

By noticing that $\kappa \geq 5$ for all $m \geq 3$, gives that $\kappa \neq \sqrt[m]{4}$. This gives the final result.

Theorem 4.1 *The* 6*-gasket of order* m *has dimension* $\log(\kappa)/-\log(\lambda)$*, where* κ *is the largest root of* $\kappa^{m+1} - 6\kappa^m - 2\kappa + 18$ *and* λ *is the root in* $(0,1)$ *of* $2\lambda^{m+1} - 3\lambda - 1$.

It is worth observing that $\kappa \to 6$ and $\lambda_{6,m} \to \lambda_{6,\infty}$ as $m \to \infty$, and furthermore the dimension of the non-overlapping case is $\log(6)/-\log(\lambda_{6,\infty})$.

4.2 Octagons

The case of octagons is done in a similar way. The main difference is that they are denoted differently, as the overlaps are on the edges, not the corners, and will always effect two octagons. The case of $m = 1$ is a special case, as its overlap structure is unusual (see Theorem 2.2). Its dimension for $m = 1$ was computed directly by a computer implementation of [7], (see [4] for the implementation). This implementation would start with neighbourhood type given by the initial starting octagon. It

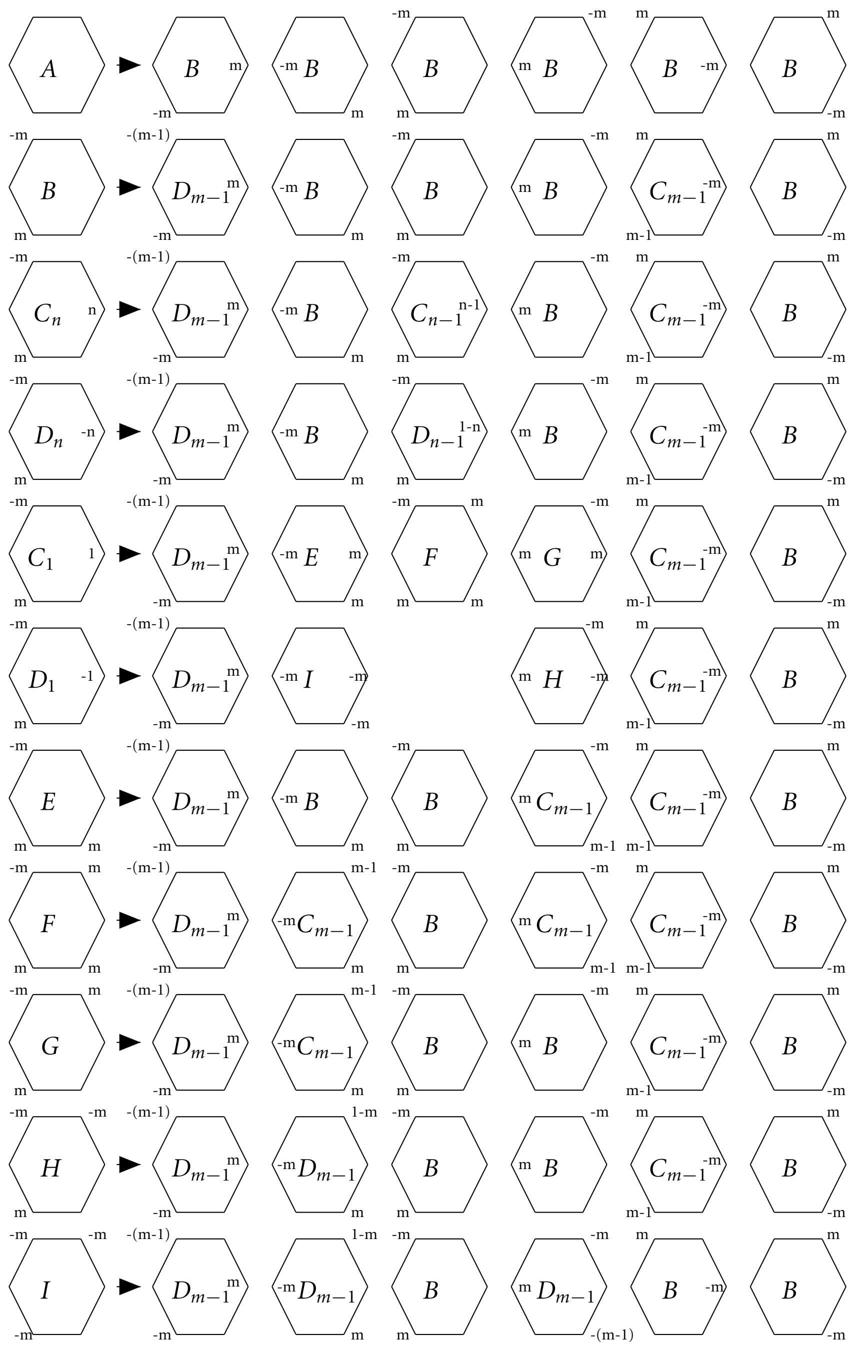
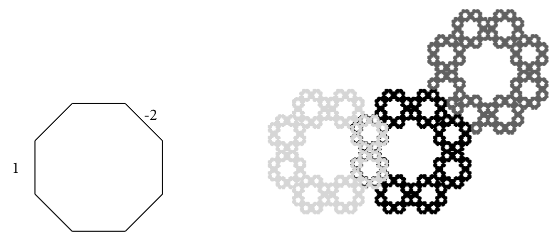


Figure 13

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press

would then recurse into the structure, finding all of the neighbourhood types of all of the descendants of each neighbourhood type. As this fractal has finite type, this process eventually terminates. When this process terminates, we can then compute the adjacency matrix, and finally compute the dimension.



(a) Precedence structure (b) Graphical interpretation

Figure 14: Precedence structure and graphical interpretation

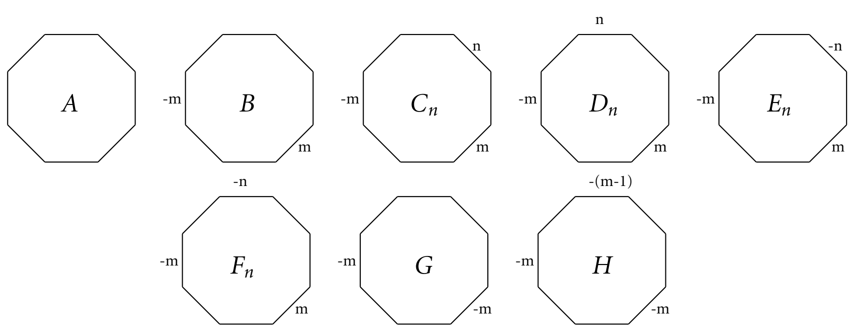


Figure 15: Pictorial Representations

Using the same colour notation as Figure 11, we give an example of a particular 8-gasket of order 2 (see Figure 14). This means that the left side will overlap an adjacent octagon in 1 step, and this octagon will have priority. This also means that the upper right side will overlap an adjacent octagon in 2 steps, and this octagon will not have priority. Lastly, all other sides do not overlap other octagons.

All of the offspring octagons can be determined by the parents with this information. This is done in much that same way as the hexagon case.

To simplify notation, we will not give a pictorial representation as we did for the hexagons. Instead we will list only what is meant by each 8-gasket, and the transitions. See Figure 15. The maps are:

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press

$$
\begin{aligned}
&A \to 8 \times B,\\
&B \to 4 \times B, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&C_n \to 2 \times B, C_{n-1}, D_{n-1}, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1} \quad (n \neq 1),\\
&D_n \to 2 \times B, C_{n-1}, D_{n-1}, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1} \quad (n \neq 1),\\
&C_1 \to 2 \times B, C_m, D_m, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&D_1 \to 2 \times B, C_m, D_m, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&E_n \to 2 \times B, E_{n-1}, F_{n-1}, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1} \quad (n \neq 1),\\
&F_n \to 2 \times B, E_{n-1}, F_{n-1}, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1} \quad (n \neq 1),\\
&E_1 \to B, G, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&F_1 \to 2 \times B, H, C_{m-1}, D_{m-1}, E_{m-1},\\
&G \to 4 \times B, 2 \times E_{m-1}, 2 \times F_{m-1},\\
&H \to 2 \times B, 2 \times E_{m-1}, 2 \times F_{m-1}, E_{m-2}, F_{m-2} \quad (m \neq 2),\\
&H \to 2 \times B, 2 \times E_{m-1}, F_{m-1}, H \quad (m = 2).
\end{aligned}
$$

Using a similar technique to before, we get κ satisfies (for $m \geq 2$)

$$\kappa^{m+1} - 8\kappa^m + 2^{m-1}\kappa + 2^{m+2}.$$

Theorem 4.2 *The 8-gasket of order m has dimension $\frac{\log(\kappa)}{-\log(\lambda)}$, where κ is the largest root of*

- $m = 1$, $\kappa^3 - 6\kappa^2 + \kappa + 12$,
- $m \geq 2$, $\kappa^{m+1} - 8\kappa^m + 2^{m-1}\kappa + 2^{m+2}$,

and λ is the root in $(0, 1)$ of $(\sqrt{2} + 1)\lambda^{m+1} - (2 + \sqrt{2})\lambda - 1$.

4.3 Dodecagons

We proceed the same as for the octagon case, for $m \geq 2$. The case of a dodecagon with $m = 1$ is somewhat special, and is calculated using the code provided at the author's home page [4]. Using Figure 16 for a representation of which each symbol represents we get the maps for $m \geq 2$:

$$
\begin{aligned}
&A \to 12 \times B,\\
&B \to 8 \times B, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&C_n \to 6 \times B, C_{n-1}, D_{n-1}, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1} \quad (n \neq 1),\\
&D_n \to 6 \times B, C_{n-1}, D_{n-1}, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1} \quad (n \neq 1),
\end{aligned}
$$

https://doi.org/10.4153/CJM-2010-036-3 Published online by Cambridge University Press

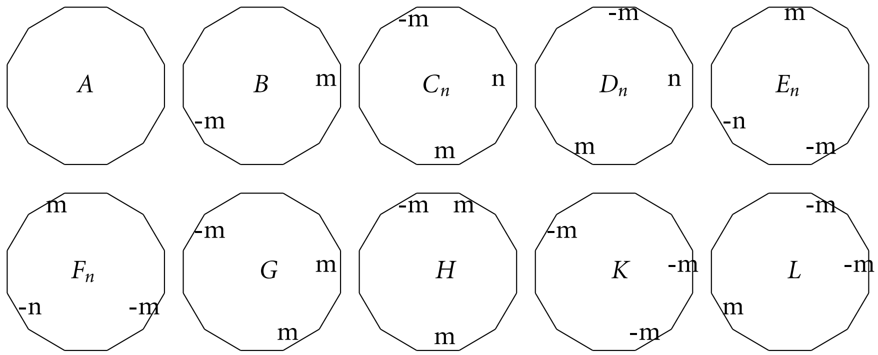


Figure 16: Pictorial Representations

$$\begin{aligned}
&E_n \to 6 \times B, E_{n-1}, F_{n-1}, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1} \quad (n \neq 1),\\
&F_n \to 6 \times B, E_{n-1}, F_{n-1}, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1} \quad (n \neq 1),\\
&C_1 \to 4 \times B, 2 \times G, 2 \times H, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&D_1 \to 4 \times B, 2 \times G, 2 \times H, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&E_1 \to 4 \times B, K, L, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&F_1 \to 4 \times B, K, L, C_{m-1}, D_{m-1}, E_{m-1}, F_{m-1},\\
&G \to 6 \times B, 2 \times C_{m-1}, 2 \times D_{m-1}, E_{m-1}, F_{m-1},\\
&H \to 6 \times B, 2 \times C_{m-1}, 2 \times D_{m-1}, E_{m-1}, F_{m-1},\\
&K \to 6 \times B, 3 \times E_{m-1}, 3 \times F_{m-1},\\
&L \to 6 \times B, C_{m-1}, D_{m-1}, 2 \times E_{m-1}, 2 \times F_{m-1}.
\end{aligned}$$

Using the same techniques as before, this gives us the theorem.

Theorem 4.3 *The 12-gasket of order m has dimension $\frac{\log(\kappa)}{-\log(\lambda)}$ where κ is the largest root of*

- $m = 1$, $\kappa^2 - 11\kappa + 16$,
- $m \geq 2$, $\kappa^{m+1} - 12\kappa^m - 2^{m-1}z + 18 \cdot 2^m$,

and λ is the root in $(0, 1)$ of $(\sqrt{3} + 2)\lambda^{m+1} - (3 + \sqrt{3})\lambda - 1$.

5 Conclusions

This paper answers a question of Broomhead, Montaldi and Sidorov as to the existence and construction of n-gaskets. It shows how to use the algorithm of Ngai and Wang to compute these gasket dimensions. There are a number of observations that are worth making at this point. First, the gaskets studied in this paper all have finite type. It can be shown that if $1/\lambda$ is a Pisot number, with $\mathbb{Q}(\lambda) \supseteq \mathbb{Q}(\cos(2\pi/n))$, then the n-gasket with contraction λ will also have finite type. This is a sufficient condition, but not necessary, as can be seen by noticing that most of the $\lambda_{n,m}$ in this paper are not in fact Pisot numbers. Examples would be $\lambda \approx 0.5698402911$, the root of $x^3 - x^2 + 2x - 1$, for $n = 3$ or $\lambda \approx 0.3819660113$, the root of $x^2 - 3x + 1$, for $n = 5$. These objects in general will be much more complicated than the n-gaskets of order m that we studied here. It is not immediately obvious whether the existence of such a λ in the relevant range is guaranteed for all n-gons. (If λ is too big, then the gasket will have dimension 2, and if it is too small it will satisfy the OSC.) Moreover, the computational aspects in even this simple case of the problem are immense. For example, the resulting minimal polynomial for κ, for the 3-gon with the contraction $\lambda \approx 0.5698402911$, is of degree 445. How these computations could be, or should be done for more complicated objects would require techniques from sparse matrix theory.

Variations of these sorts of objects can also be studied in higher dimensions. It is not clear if there is an equivalent trick to rotational precedence in higher dimensions. It is possible to ignore this, and simply do a larger computation, but that seems computationally inefficient.

Acknowledgments This work would not have been possible without the useful suggestions and comments of Nikita Sidorov. The calculation for the minimal polynomial of degree 445 mentioned in the conclusions was done by Clement Pernet and Arne Storjohann.

References

[1] D. Broomhead, J. Montaldi, and N. Sidorov, *Golden gaskets: variations on the Sierpiński sieve.* Nonlinearity **17**(2004), no. 4, 1455–1480. doi:10.1088/0951-7715/17/4/017
[2] K. Falconer, *Techniques in Fractal Geometry*. John Wiley & Sons, Chichester, 1997.
[3] ———, *Fractal Geometry. Mathematical Foundations and Applications.* Second edition. John Wiley & Sons, Hoboken, NJ, 2003.
[4] K. G. Hare. Home page. http://www.math.uwaterloo.ca/∼kghare, 2004.
[5] J. E. Hutchinson, *Fractals and self-similarity.* Indiana Univ. Math. J. **30**(1981), no. 5, 713–747. doi:10.1512/iumj.1981.30.30055
[6] S. P. Lalley, *β-expansions with deleted digits for Pisot numbers β.* Trans. Amer. Math. Soc. **349**(1997), no. 11, 4355–4365. doi:10.1090/S0002-9947-97-02069-2
[7] S.-M. Ngai and Y. Wang, *Hausdorff dimension of self-similar sets with overlaps.* J. London Math. Soc. **63**(2001), no. 3, 655–672. doi:10.1017/S0024610701001946

Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1
e-mail: kghare@math.uwaterloo.ca