A generalisation of Dirichlet's multiple integral

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The well-known multiple integral

$$\int \dots \int_{R_n} x_1^{a_1 - 1} x_2^{a_2 - 1} \dots x_n^{a_n - 1} (1 - x_1 - x_2 - \dots - x_n)^{a_0 - 1} dx_1 \dots dx_n,$$

where R_n is the region defined by $x_1 \geqslant 0$, $x_2 \geqslant 0$, ..., $x_n \geqslant 0$, $x_1 + x_2 + \dots + x_n \leqslant 1$, and where $a_0, a_1, \dots, a_n$ are positive constants, can be evaluated either in the classical way using the Dirichlet transformation or by the use of the Laplace transform [1]. I. J. Good [2] has considered a more general integral and has proved the following result by induction:—

If $f_1(t), f_2(t), \dots, f_n(t)$ are Lebesgue measurable for $0 \leqslant t \leqslant 1$, $m_1, m_2, \dots, m_n, m_{n+1} (= 0)$ are real numbers, $M_r = m_1 + m_2 + \dots + m_r$, $x_1, x_2, \dots, x_n$ are non-negative variables and $X_r = x_1 + x_2 + \dots + x_r$, then

$$\int \dots \int_{X_n \leqslant 1} \prod_{r=1}^{n-1} \left\{ x_r^{m_r} f_r\left(\frac{X_r}{X_{r+1}}\right) \right\} x_n^{m_n} f_n(X_n)\, dx_1 \dots dx_n$$

$$= \prod_{r=1}^{n} \int_0^1 f_r(x) (1 - x)^{m_{r+1}} x^{M_r + r - 1} dx. \qquad (1)$$

It does not seem to be possible to establish this relation by employing the Laplace transform, but we show below that it can be obtained using the Mellin transform.

The Mellin transform of a function $\phi(x)$, defined for $x > 0$, is defined by

$$\Phi(s) = M\phi(x) = \int_0^\infty \phi(x) x^{s-1} dx,$$

and the inverse relation is

$$\phi(x) = M^{-1}\Phi(s) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \Phi(s) x^{-s} ds.$$

We need below the convolution theorem for the Mellin transform, viz.,

$$M\phi_1(x).\ M\phi_2(x) \dots \dots M\phi_n(x) = M\psi(x),$$

where

$$\psi(x) = \int_0^\infty \dots \int_0^\infty \phi_2(u_2)\phi_3(u_3) \dots \phi_n(u_n)\phi_1\left(\frac{x}{u_2 u_3 \dots u_n}\right) \frac{du_2 \dots du_n}{u_2 \dots u_n}. \qquad (2)$$

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Proof of (1)

Let $f_1(x), \ldots, f_n(x)$ be functions defined for $0 \leqslant x \leqslant 1$, and consider the functions $\phi_i(x)$ defined by

$$\phi_i(x) = \begin{cases} f_i(x)\,(1-x)^{m_{i+1}}\,x^{N_i}, & 0 \leqslant x \leqslant 1, \\ 0, & x > 1, \end{cases}$$

where $i = 1, 2 \ldots, n$, the real numbers $m_1, \ldots, m_n$ and $N_1, \ldots, N_n$ are such that the functions $\phi_i(x)$ possess Mellin transforms, and $m_{n+1} = 0$. Then, by the convolution theorem (2), we obtain

$$\left(\int_0^1 f_1(x)\,(1-x)^{m_2}\,x^{N_1+s-1}\,dx\right) \ldots \left(\int_0^1 f_{n-1}(x)\,(1-x)^{m_n}\,x^{N_{n-1}+s-1}dx\right) \left(\int_0^1 f_n(x)\,x^{N_n+s-1}dx\right)$$

$$= M\int_0^1 \ldots\ldots \int_0^1 f_n(u_n)u_n^{N_n} f_{n-1}(u_{n-1})\,(1-u_{n-1})^{m_n}\,u_{n-1}^{N_{n-1}} \ldots f_2(u_2)$$
$$(1-u_2)^{m_3}\,u_2^{N_2} f_1\left(\frac{x}{u_2\ldots u_n}\right)\left(1-\frac{x}{u_2\ldots u_n}\right)^{m_2}\left(\frac{x}{u_2\ldots u_n}\right)^{N_1} \frac{du_2\ldots du_n}{u_2\ldots u_n}$$
$$= \int_0^1 f_n(u_n)\,u_n^{N_n-N_1-1}\,du_n \int_0^1 f_{n-1}(u_{n-1})(1-u_{n-1})^{m_n}\,u_{n-1}^{N_{n-1}-N_1-1}\,du_{n-1} \ldots$$
$$\int_0^1 f_2(u_2)\,(1-u_2)^{m}\,u_2^{N_2-N_1-1}\,du_2 \int_0^{u_2\ldots u_n} f_1\left(\frac{x}{u_2\ldots u_n}\right)\left(1-\frac{x}{u_2\ldots u_n}\right)^{m_2} x^{N_1+s-1}\,dx. \qquad (3)$$

We now introduce the variables $X_1, \ldots, X_n$ where

$$x = X_1,\; u_2 = X_2/X_3,\; u_3 = X_3/X_4,\; \ldots,\; u_{n-1} = X_{n-1}/X_n,\; u_n = X_n.$$

Then

$$0 \leqslant X_1 \leqslant X_2 \leqslant \ldots \leqslant X_n \leqslant 1,\; dx\,du_2 \ldots du_n = (X_3X_4\ldots X_n)^{-1}\,dX_1 \ldots dX_n,$$

and the right-hand side of (3) becomes

$$\underset{(0\leqslant X_1\leqslant\ldots\leqslant X_n\leqslant 1)}{\int\cdots\int} f_1\left(\frac{X_1}{X_2}\right) X_1^{N_1+s-1} f_2\left(\frac{X_2}{X_3}\right)(X_2-X_1)^{m_2}\,X_2^{N_2-N_1-m_2-1} \ldots$$
$$f_{n-1}\left(\frac{X_{n-1}}{X_n}\right)(X_n-X_{n-1})^{m_n}\,X_{n-1}^{N_{n-1}-N_{n-2}-m_{n-1}-1}$$
$$f_n(X_n)X_n^{N_n-N_{n-1}-m_n-1}\,dX_1 \ldots dX_n.$$

Finally, writing $X_1 = x_1,\; X_2 - X_1 = x_2,\; \ldots,\; X_n - X_{n-1} = x_n$, we have

$$\int_0^1 f_1(x)(1-x)^{m_2} x^{N_1+s-1}\,dx.\ \int_0^1 f_2(x)(1-x)^{m_3} x^{N_2+s-1}\,dx \ldots$$
$$\int_0^1 f_{n-1}(x)(1-x)^{m_n} x^{N_{n-1}+s-1}\,dx.\ \int_0^1 f_n(x)\, x^{N_n+s-1}\,dx$$
$$=\int_{X_n\leqslant 1}\!\!\ldots\int x_1^{N_1+s-1} f_1\left(\frac{X_1}{X_2}\right) x_2^{m_2} f_2\left(\frac{X_2}{X_3}\right) x_3^{m_3} f_3\left(\frac{X_3}{X_4}\right)\ldots x_n^{m_n} f_n(X_n).$$
$$(x_1+x_2)^{N_2-N_1-m_2-1}(x_1+x_2+x_3)^{N_3-N_2-m_3-1}\ldots$$
$$(x_1+\ldots+x_n)^{N_n-N_{n-1}-m_n-1}\,dx_1\ldots dx_n. \quad (4)$$

This reduces to (1) if we choose the numbers $N_1, \ldots, N_n$ such that

$$N_i = N_{i-1} + m_i + 1$$

and take $N_0 = -s$.

REFERENCES.

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