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Products of Conjugacy Classes in $SU(2)$

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Abstract. We obtain a complete description of the conjugacy classes $C_1, \dots, C_n$ in $SU(2)$ with the property that $C_1 \cdots C_n = SU(2)$. The basic instrument is a characterization of the conjugacy classes $C_1, \dots, C_{n+1}$ in $SU(2)$ with $C_1 \cdots C_{n+1} \ni I$, which generalizes a result of [Je-We].

1 Introduction

The following problem was posed by D. Burago:

Problem Let G be a group. For which conjugacy classes $C_1, \dots, C_n$ of G is it true that the multiplication map

$$C_1 \times \cdots \times C_n \to G$$

is surjective?

We give a solution to this problem in the case $G = SU(2)$. In this case the conjugacy classes are parametrized by their eigenvalues

$$\operatorname{diag}(e^{i\lambda}, e^{-i\lambda})$$

so they are determined by one number $\lambda \in [0, \pi]$.

Burago's interest was primarily in discrete groups. The purpose of this note is to point out that the problem he posed is also of interest for Lie groups such as $SU(2)$, and to exhibit a solution in that case.

For more general Lie groups $G = SU(n)$ the problem could be studied by adapting results on the quantum cohomology of Grassmannians: see [Ag-Wo]. The problem is related to recent results described in the article [KLM].

2 Eigenvalues of a Multiple Product

For any $\lambda \in [0, \pi]$ we denote by $C(\lambda)$ the conjugacy class of the matrix

$$\operatorname{diag}(e^{i\lambda}, e^{-i\lambda})$$

in $SU(2)$. Note that any conjugacy class in $SU(2)$ is of the form $C(\lambda)$ for a unique $\lambda \in [0, \pi]$. The following result was proved in [Je-We, Proposition 3.1]:

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Proposition 2.1 *For $\lambda_1, \lambda_2, \lambda_3 \in [0, \pi]$ we have*

$$C(\lambda_1)C(\lambda_2)C(\lambda_3) \ni I$$

iff

$$|\lambda_1 - \lambda_2| \le \lambda_3 \le \min\{\lambda_1 + \lambda_2, 2\pi - (\lambda_1 + \lambda_2)\}. \tag{1}$$

Note that (1) is equivalent to

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &\le 2\pi \\ -\lambda_1 - \lambda_2 + \lambda_3 &\le 0 \\ -\lambda_1 + \lambda_2 - \lambda_3 &\le 0 \\ \lambda_1 - \lambda_2 - \lambda_3 &\le 0. \end{aligned}$$

The goal of this section is to prove the more general result:

Theorem 2.2 *For $n \ge 2$ an integer and $\lambda_1, \dots, \lambda_{n+1} \in [0, \pi]$ we have*

$$C(\lambda_1)\cdots C(\lambda_{n+1}) \ni I \tag{2}$$

iff the following system of inequalities holds:

(a) *If $n + 1 = 2k$ is an even number:*

$$S^1_{n+1}\left(\{\lambda_i\}\right) \le (n-1)\pi, \quad S^3_{n+1}\left(\{\lambda_i\}\right) \le (n-3)\pi, \quad \dots \quad S^{2k-1}_{n+1}\left(\{\lambda_i\}\right) \le 0 \tag{3}$$

where $S^j_{n+1}\left(\{\lambda_i\}\right)$ is any sum of the type $\sum_{i=1}^{n+1} \pm\lambda_i$ which contains exactly j minus signs.

(b) *If $n + 1 = 2k + 1$ is an odd number:*

$$S^0_{n+1}\left(\{\lambda_i\}\right) \le n\pi, \quad S^2_{n+1}\left(\{\lambda_i\}\right) \le (n-2)\pi, \quad \dots \quad S^{2k}_{n+1}(\{\lambda_i\}) \le 0 \tag{4}$$

where $S^j_{n+1}\left(\{\lambda_i\}\right)$ has the same meaning as before.

Remarks 1. A more concise way to express both (3) and (4) is

$$S^{n-2j}_{n+1}\left(\{\lambda_i\}\right) \le 2j\pi$$

for any $0 \le j \le n/2$ and any sum of the type S^{n-2j}_{n+1}.

2. An elementary computation involving the binomial formula shows that the number of inequalities in both (3) and (4) is

$$\binom{n+1}{0} + \binom{n+1}{2} + \cdots = \binom{n+1}{1} + \binom{n+1}{3} + \cdots = 2^n.$$

3. The result stated in Theorem 2.2 was obtained in the Ph.D. thesis by A. Galitzer. Her proof is described in [KM].

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We will use induction on n to prove this theorem. In order to make the induction step we will need the following result:

Lemma 2.3 *The condition* (2) *holds iff there exists* $\lambda \in [0, \pi]$ *such that*

$$C(\lambda_1) \cdots C(\lambda_{n-1}) C(\lambda) \ni I \tag{5}$$

and

$$C(\lambda) C(\lambda_n) C(\lambda_{n+1}) \ni I. \tag{6}$$

Proof The fundamental group of the $(n+1)$-punctured sphere Σ_{n+1} in two dimensions is the free group on n generators, or the group

$$\Pi_n = \langle x_1, \dots, x_{n+1} \mid x_1 \cdots x_{n+1} = 1 \rangle$$

with $n+1$ generators and one relation. We can form the $(n+1)$-punctured sphere by gluing an n-punctured sphere and a 3-punctured sphere along one of the boundary components of each. Call S the common boundary resulting from this construction and consider the fundamental groups of the two components as follows:

$$\Pi_{n-1} = \langle x_1, \dots, x_{n-1}, x \mid x_1 \cdots x_{n-1} x = 1 \rangle$$

and

$$\Pi_2 = \langle x', x_n, x_{n+1} \mid x' x_n x_{n+1} = 1 \rangle,$$

where x and x' represent the loop S in each of the two components. From the theorem of Seifert-van Kampen, we have that

$$\Pi_n = (\Pi_{n-1} \times \Pi_2) / \langle x x' = 1 \rangle \tag{7}$$

Now we consider representations of these groups into $G = SU(2)$. The condition (2) is equivalent to the existence of a representation ρ of Π_{n+1} such that

$$\rho(x_i) \in C(\lambda_i)$$

for any $1 \leq i \leq n+1$. From (7), this is equivalent to the existence of a representation ρ_{n-1} of Π_{n-1} which coincides with ρ on $x_1, \dots, x_{n-1}$, and a representation ρ_2 of Π_2 which coincides with ρ on x_n and x_{n+1}, and such that ρ_{n-1} and ρ_2 satisfy

$$\rho_{n-1}(x) \rho_2(x') = I.$$

The latter equality implies that the conjugacy classes of $\rho_{n-1}(x)$ and $\rho_2(x')$ are equal, call them $C(\lambda)$. (Note that in $SU(2)$ every element is conjugate to its inverse). The conditions (5) and (6) correspond respectively to the representations ρ_{n-1} and ρ_2. ∎

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Proof of Theorem 2.2 Just the induction step has to be performed. We want to prove that

$$C(\lambda_1)\cdots C(\lambda_{n+1}) \ni I$$

iff equation (3) or (4) holds. Suppose that $n = 2k$ is an even number. Condition (5) of Lemma 2.3 is equivalent to

$$\begin{aligned} S_n^1(\lambda_1,\dots,\lambda_{n-1},\lambda) &\le (n-2)\pi, \\ S_n^3(\lambda_1,\dots,\lambda_{n-1},\lambda) &\le (n-4)\pi,\dots, \\ S_n^{2k-1}(\lambda_1,\dots,\lambda_{n-1},\lambda) &\le 0 \end{aligned} \tag{8}$$

where we have used the induction hypothesis, and condition (6) is equivalent to

$$|\lambda_n - \lambda_{n+1}| \le \lambda \le \min\{\lambda_n + \lambda_{n+1}, 2\pi - (\lambda_n + \lambda_{n+1})\} \tag{9}$$

where we have used Proposition 2.1. By Lemma 2.3, condition (2) is equivalent to the system of inequalities obtained by considering each of the 2^{n-1} inequalities from (8) and deriving from it two inequalities, as follows:

(i) if λ occurs with a *plus* sign in that sum, replace it by $\lambda_n - \lambda_{n+1}$ and $-\lambda_n + \lambda_{n+1}$;
(ii) if λ occurs with a *minus* sign in that sum, replace it by $\lambda_n + \lambda_{n+1}$ and $-\lambda_n - \lambda_{n+1}$, but in the latter situation add 2π to the right hand side of the original inequality.

One sees that in the case (i) we replace an inequality of the type

$$S_n^j \le (n-j-1)\pi \tag{10}$$

by two different inequalities, both of the type

$$S_{n+1}^{j+1} \le (n-j-1)\pi. \tag{11}$$

In the case (ii) one again replaces an inequality of the type (10) by an inequality of the type (11) and an inequality of the type

$$S_{n+1}^{j-1} \le (n-j+1)\pi.$$

One obtains 2^n distinct inequalities of type (4), which means that (2) is really equivalent to (4).

A similar argument can be used when $n = 2k-1$ is an odd number. ■

Remark The result stated in Theorem 1.2 can also be obtained from [Ag-Wo, Theorem 3.1] by using the structure of the quantum cohomology ring of $\mathbb{C}P^1$. More precisely, let us consider the two Schubert classes in $H^*(\mathbb{C}P^1)$:

$$[\sigma_1] \in H^2(\mathbb{C}P^1) \text{ and } [\sigma_2] = 1 \in H^0(\mathbb{C}P^1).$$

The quantum cohomology ring of $\mathbb{C}P^1$ is

$$QH^*(\mathbb{C}P^1) = (H^*(\mathbb{C}P^1) \otimes \mathbb{R}[q], \star),$$

where q is a formal variable of degree 4 and $\star$ is an $\mathbb{R}[q]$-linear, commutative and associative product which satisfies

$$[\sigma_1] \star [\sigma_1] = q. \tag{12}$$

Each of the 2^n inequalities indicated in Theorem 2.2 can be obtained by choosing $i_1, \dots, i_n \in \{1, 2\}$ and evaluating the product

$$[\sigma_{i_1}] \star \cdots \star [\sigma_{i_n}]$$

in $QH^*(\mathbb{C}P^1)$. By the equation (12), this product is of the form $q^d \sigma_k$, where d is a positive integer and $k \in \{1, 2\}$. The inequality of the type (3) or (4) which corresponds to $i_1, \dots, i_n$ is

$$\sum_{j=1}^{n} (-1)^{i_j - 1} \lambda_j + (-1)^k \lambda_{n+1} \leq 2d\pi.$$

3 Surjectivity of a Multiple Product

Our main result is

Theorem 3.1 *We have*

$$C(\lambda_1) \cdots C(\lambda_n) = SU(2) \tag{13}$$

iff for any integer j with $0 \leq j \leq n/2$ and for any sum of the type $S_n^j = S_n^j\big(\{\lambda_i\}\big)$ (see Theorem 1.2) we have

$$-(j-1)\pi \leq S_n^j \leq (n - j - 1)\pi. \tag{14}$$

Proof The idea of the proof is that (13) holds iff (2) holds for any $\lambda_{n+1} \in [0, \pi]$. In turn, (2) is equivalent to (3) and (4). We just have to take each inequality from (3) (respectively (4)) and make the following formal replacements in its left-hand side:

(i) λ_{n+1} by π;
(ii) $-\lambda_{n+1}$ by 0.

Let us consider the case $n = 2k - 1$. We have to show that if we perform (i) and (ii) for each inequality contained in (2), we obtain exactly one of the following inequalities:

$$\pi \leq S_n^0 \leq (n-1)\pi \tag{15}$$

$$0 \leq S_n^1 \leq (n-2)\pi \tag{16}$$

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$$(17)\qquad -\pi \le S_n^2 \le (n-3)\pi$$

$$(18)\qquad -2\pi \le S_n^3 \le (n-4)\pi$$

We claim that if we label the inequalities given by (2) as [1], [3], ..., [2k−3], [2k−1] then each of [1] and [2k−1] gives exactly one of (15) and (16), each of [3] and [2k−3] gives exactly one of (17) and (18), ... and finally

- if $k = 2p$ is even, then each of $[2p-1]$ and $[2p+1]$ gives exactly one of
$$-(k-1)\pi \le S_n^{k-2} \le (k-2)\pi$$
$$-(k-2)\pi \le S_n^{k-1} \le (k-1)\pi.$$
- if $k = 2p+1$ is odd, then each of $[2p+1]$ gives exactly one of
$$-(k-2)\pi \le S_n^{k-1} \le (k-1)\pi.$$

Consider first [1] together with [2k−1]: the only S_{n+1}^1 which contains $-\lambda_{n+1}$ leads to

$$\lambda_1 + \cdots + \lambda_n \le (n-1)\pi$$

whereas the only S_{n+1}^{2k-1} which contains λ_{n+1} leads to

$$\lambda_1 + \cdots + \lambda_n \ge \pi.$$

The remaining inequalities of type $S_{n+1}^1 \le (n-1)\pi$ lead to all possible inequalities of the type

$$S_n^1 \le (n-2)\pi$$

and the remaining inequalities of the type $S_{n+1}^{2k-1} \le 0$ lead to all possible inequalities of the type

$$S_n^1 \ge 0.$$

The same idea applies[1] to each pair $[2j+1]$, $[2(k-j)-1]$, $0 \le j < k/2$ (if $k = 2p+1$ is an odd number, then for $j = p$ we have $2j+1 = 2(k-j)-1$ and the corresponding pair reduces to just one type of inequalities).

Similar ideas can be used in the case when $n = 2k$ is an even number. ■

Remark The system of inequalities (14) admit solutions for any $n \ge 2$. For $n = 2$ the *unique* solution is

$$(19)\qquad \lambda_1 = \lambda_2 = \frac{\pi}{2}.$$

For $n \ge 3$ there are several solutions, one of them consisting of λ_1, λ_2 given by (19) and

$$\lambda_3 = \cdots = \lambda_n = 0.$$

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[1] If we compare the total number of inequalities we start with to the number of inequalities obtained via (i) and (ii), we "deduce" that $\binom{n+1}{2j+1} + \binom{n+1}{n+1-(2j+1)} = 2\left(\binom{n}{2j+1} + \binom{n}{2j}\right)$. The latter equation is obviously true, by properties of Pascal's triangle.

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