

## DISTANCE OF A BLOCH FUNCTION TO THE LITTLE BLOCH SPACE

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Motivated by a formula of P. Jones that gives the distance of a Bloch function to BMOA, the space of bounded mean oscillations, we obtain several formulas for the distance of a Bloch function to the little Bloch space,  $\mathcal{B}_0$ . Immediate consequences are equivalent expressions for functions in  $\mathcal{B}_0$ . We also give several examples of distances of specific functions to  $\mathcal{B}_0$ . We comment on connections between distance to  $\mathcal{B}_0$  and the essential norm of some composition operators on the Bloch space,  $\mathcal{B}$ . Finally we show that the distance formulas in  $\mathcal{B}$  have Bloch type spaces analogues.

### 1. INTRODUCTION

Let  $U$  denote the open unit disk and  $\partial U$  the unit circle in the complex plane. The Bloch space  $\mathcal{B}$  of  $U$  is the space of holomorphic functions  $f$  on  $U$  such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in U} (1 - |z|^2) |f'(z)| < \infty.$$

It is easy to see that  $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{\mathcal{B}}$  defines a norm that makes  $\mathcal{B}$  a Banach space that is invariant under Möbius transformations and in fact for all  $f \in \mathcal{B}$

$$\|f \circ \omega \varphi_a\|_{\mathcal{B}} = \|f\|_{\mathcal{B}},$$

where  $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ ,  $a \in U$  and  $\omega \in \partial U$ .

The little Bloch space  $\mathcal{B}_0$  of  $U$  is the closed subspace of  $\mathcal{B}$  consisting of functions  $f$  with

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |f'(z)| = 0.$$

Examples of functions in  $\mathcal{B}$  include all bounded holomorphic functions on  $U$ ; but  $\mathcal{B}$  contains unbounded functions ( $\log(1 - z) \in \mathcal{B}$ ). Other examples include certain lacunary series.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n},$$

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where  $(\lambda_n)$  is a sequence of integers satisfying

$$\frac{\lambda_{n+1}}{\lambda_n} \geq \lambda > 1,$$

$\lambda$  is a constant and  $n \geq 1$ . Then  $f \in \mathcal{B}$  if and only if  $a_n$  is a bounded sequence and  $f \in \mathcal{B}_0$  if and only if  $a_n \rightarrow 0$ , as  $n \rightarrow \infty$ . See [1, 7, 12] for more information on  $\mathcal{B}$ .

The motivation for this paper is a formula of Jones, (see [1, Theorem 9] and [4, p. 503] for a proof), that gives the distance of a Bloch function to BMOA the space of bounded mean oscillations.

We obtain the following formulas for the distance of a Bloch function to  $\mathcal{B}_0$ :

**THEOREM 3.5.** For  $f \in \mathcal{B}$ ,  $p \geq 2$ ,

$$\inf A_p(f) \leq \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leq 6 \inf A_p(f)$$

and

$$\inf A(f) \leq \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leq 6 \inf A(f).$$

See (8), (9) below for the definitions of the sets  $A_p(f)$ ,  $A(f)$  respectively.

We obtain as a corollary equivalent expressions for functions in  $\mathcal{B}_0$ :

**COROLLARY 3.6.** For  $f \in \mathcal{B}$ , the following are equivalent:

1.  $f \in \mathcal{B}_0$ .
2. For all  $\varepsilon > 0$ ,  $\Omega_\varepsilon(f)$  is a compact subset of  $U$ .
3. For some  $p \geq 2$  and all  $\varepsilon > 0$ ,  $\int_{\Omega_\varepsilon(f)} 1/(1 - |w|^2)^p dA(w) < \infty$ .
4. For any  $p \geq 2$  and  $\varepsilon > 0$ , there is a constant  $c$  so that

$$\int_{\Omega_\varepsilon(f)} |g'(z)|^p dA(z) \leq c \|g\|_{\mathcal{B}}^p, \quad \text{for all } g \in \mathcal{B}.$$

See (1) below for the definition of the set  $\Omega_\varepsilon(f)$ .

We also show that

**THEOREM 3.9.** For  $f \in \mathcal{B}$ ,

$$\limsup_{|z| \rightarrow 1} |f'(z)| (1 - |z|^2) \leq \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leq 2 \limsup_{|z| \rightarrow 1} |f'(z)| (1 - |z|^2).$$

In Section 4 we give several examples of distances of specific functions to  $\mathcal{B}_0$  for example,

$$\text{dist}_{\mathcal{B}}(\log(1 - z), \mathcal{B}_0) = 2 = \|\log(1 - z)\|_{\mathcal{B}}$$

and

$$\text{dist}_{\mathcal{B}}(e^{(z+1)/(z-1)}, \mathcal{B}_0) = \frac{2}{e} = \|e^{(z+1)/(z-1)}\|_{\mathcal{B}}.$$

Let  $\text{const.}$  denote a positive and finite constant which may change from one occurrence to the next but will not depend on the functions involved.

In Section 5 we make a connection between distance to  $\mathcal{B}_0$  and the essential norm of some composition operators  $C_\phi$  on  $\mathcal{B}$ . We show the following:

**COROLLARY 5.1.** *Let  $\phi$  be a univalent self-map of  $U$  so that  $\phi(U)$  has pointwise order of contact 1 with  $\partial U$  at the point 1. Then,*

$$\frac{1}{4} \text{dist}_{\mathcal{B}}(\log(1 - \phi), \mathcal{B}_0) \leq \|C_\phi\|_e \leq \text{const. dist}_{\mathcal{B}}(\log(1 - \phi), \mathcal{B}_0).$$

In the last section we define the Bloch type spaces  $\mathcal{B}_\alpha$  and the little Bloch type spaces  $\mathcal{B}_{\alpha,0}$  and we show that the distance formulas in  $\mathcal{B}$  have  $\mathcal{B}_\alpha$  analogues. The main results in the section are Theorem 6.1 and Corollary 6.4. Finally, we obtain as corollaries equivalent expressions for functions in  $\mathcal{B}_{\alpha,0}$  (see Corollary 6.3).

### 2. PRELIMINARIES

For  $f \in \mathcal{B}$  [12, Lemma 4.2.8] gives

$$f(z) = f(0) + f'(0)z + \int_U \frac{(1 - |w|^2) f'(w)}{\bar{w}(1 - \bar{w}z)^2} dA(w), \quad z \in U,$$

where  $dA(w)$  is the normalised area measure on  $U$ . For any  $\varepsilon > 0$  let  $\Omega_\varepsilon(f)$  be

$$(1) \quad \Omega_\varepsilon(f) = \left\{ z \in U : (1 - |z|^2) |f'(z)| \geq \varepsilon \right\}.$$

Then write

$$\begin{aligned} f(z) &= f(0) + f'(0)z + \int_{\Omega_\varepsilon(f)} \frac{(1 - |w|^2) f'(w)}{\bar{w}(1 - \bar{w}z)^2} dA(w) \\ &\quad + \int_{U \setminus \Omega_\varepsilon(f)} \frac{(1 - |w|^2) f'(w)}{\bar{w}(1 - \bar{w}z)^2} dA(w) \\ (2) \quad &= f(0) + f'(0)z + f_1(z) + f_2(z). \end{aligned}$$

The result of Lemma A below is part of the proof of Jones' Theorem that Ghatage and Zheng give in [4, p. 512] but we include it for completeness.

**LEMMA A.** *Given  $f \in \mathcal{B}$  and  $\varepsilon > 0$ , then  $f_2 \in \mathcal{B}$  and*

$$\|f_2 - f_2(0) - f_2'(0)z\|_{\mathcal{B}} \leq 6\varepsilon.$$

**PROOF:** Since  $f_2$  is holomorphic on  $U$ ,

$$f_2'(z) - f_2'(0) = z \int_0^1 f_2''(tz) dt$$

for all  $z \in U$ . Thus,

$$\begin{aligned}
 (1 - |z|^2) |f'_2(z) - f'_2(0)| &\leq (1 - |z|^2) |z| \int_0^1 \frac{1}{(1 - |z|^2 t^2)^2} dt \sup_{w \in U} (1 - |w|^2)^2 |f''_2(w)| \\
 &\leq (1 - |z|^2) |z| \int_0^1 \frac{1}{(1 - |z|^2 t)^2} dt \sup_{w \in U} (1 - |w|^2)^2 |f''_2(w)| \\
 &= |z| \sup_{w \in U} (1 - |w|^2)^2 |f''_2(w)| \\
 (3) \qquad \qquad \qquad &\leq \sup_{w \in U} (1 - |w|^2)^2 |f''_2(w)|.
 \end{aligned}$$

Now for each  $w \in U$ ,

$$\begin{aligned}
 (1 - |w|^2)^2 |f''_2(w)| &= (1 - |w|^2)^2 \left| \int_{U \setminus \Omega_\varepsilon(f)} \frac{6\bar{u} (1 - |u|^2) f'(u)}{(1 - w\bar{u})^4} dA(u) \right| \\
 &\leq 6(1 - |w|^2)^2 \int_{U \setminus \Omega_\varepsilon(f)} \frac{(1 - |u|^2) |f'(u)|}{|1 - w\bar{u}|^4} dA(u) \\
 &\leq 6\varepsilon (1 - |w|^2)^2 \int_U \frac{1}{|1 - w\bar{u}|^4} dA(u) \\
 (4) \qquad \qquad \qquad &= 6\varepsilon.
 \end{aligned}$$

Therefore by (3), (4),

$$\|f_2 - f_2(0) - f'_2(0)z\|_B = \sup_{z \in U} (1 - |z|^2) |f'_2(z) - f'_2(0)| \leq 6\varepsilon. \quad \square$$

NOTE. Given  $f, g \in \mathcal{B}$  and  $z \in U$

$$(5) \qquad \qquad \qquad (1 - |z|^2) |f'(z)| \leq \|f - g\|_B + (1 - |z|^2) |g'(z)|.$$

The result of Lemma B below is part of the proof of Theorem 3 in [4, p. 512] but we include it for completeness.

**LEMMA B.** *If  $f \in \mathcal{B}_0$  then  $\Omega_\varepsilon(f)$  is a compact subset of  $U$  for all  $\varepsilon > 0$ .*

**PROOF:** Given  $f \in \mathcal{B}_0$  and  $\varepsilon > 0$ , since  $\mathcal{B}_0$  is the closure in  $\mathcal{B}$  of the polynomials ([13, p. 84]), choose a polynomial  $g$  so that  $\|f - g\|_B < \varepsilon/2$ . Then using (5) we obtain

$$(6) \qquad \qquad \qquad \Omega_\varepsilon(f) \subseteq \Omega_{\varepsilon/2}(g)$$

We shall show that

$$(7) \qquad \qquad \qquad \Omega_{\varepsilon/2}(g) \subseteq D_\varepsilon = \left\{ z \in U : \text{dist}(z, \partial U) \|g'\|_\infty \geq \frac{\varepsilon}{4} \right\}.$$

Let  $z \in \Omega_{\varepsilon/2}(g)$ ; then

$$(1 - |z|^2) \|g'\|_\infty \geq (1 - |z|^2) |g'(z)| \geq \frac{\varepsilon}{2}.$$

So,

$$(1 - |z|)\|g'\|_\infty \geq \frac{\varepsilon}{4}$$

and (7) follows. The set  $D_\varepsilon$  is a compact set. Indeed, if  $\|g'\|_\infty = 0$  then  $g$  is a constant function and  $\Omega_{\varepsilon/2}(g) = D_\varepsilon = \emptyset$ ; and if  $\|g'\|_\infty \neq 0$  then

$$D_\varepsilon = \left\{ z \in U : \text{dist}(z, \partial U) \geq \varepsilon / (4 \|g'\|_\infty) \right\},$$

which is clearly a compact subset of  $U$ . Therefore by (6) and (7)  $\Omega_\varepsilon(f)$  is a compact subset of  $U$  as well. □

### 3. DISTANCE FORMULAS

In this section given  $f \in \mathcal{B}$ ,  $f_1$  and  $f_2$  refers to the functions in (2). The distance in the Bloch norm of  $f$  to a subset of  $\mathcal{B}$ ,  $X$ , is denoted by  $\text{dist}_{\mathcal{B}}(f, X)$ .

**LEMMA 3.1.** *If  $f \in \mathcal{B}$  and there exists a function  $g \in \mathcal{B}_0$  so that  $\|f - g\|_{\mathcal{B}} \leq \alpha$  for some  $\alpha > 0$ , then  $\Omega_\varepsilon(f)$  is a compact set for all  $\varepsilon > \alpha$ .*

**PROOF:** Fix  $\alpha > 0$ , let  $\varepsilon > \alpha$  then using (5) we obtain  $\Omega_\varepsilon(f) \subseteq \Omega_{\varepsilon-\alpha}(g)$ . By Lemma B  $\Omega_{\varepsilon-\alpha}(g)$  is a compact subset of  $U$  therefore so is  $\Omega_\varepsilon(f)$ . □

For  $f \in \mathcal{B}$  and  $p > 0$ , define  $A_p(f)$  by

$$(8) \quad A_p(f) = \left\{ \varepsilon > 0 : \frac{\chi_{\Omega_\varepsilon(f)}(z)}{(1 - |z|^2)^p} dA(z) \text{ is a finite measure} \right\}.$$

And let  $A(f)$  be

$$(9) \quad A(f) = \left\{ \varepsilon > 0 : \Omega_\varepsilon(f) \text{ is a compact subset of } U \right\}.$$

**PROPOSITION 3.2.** *For  $f \in \mathcal{B}$  and any  $p > 0$ ,*

$$\inf A_p(f) \leq \inf A(f) \leq \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0).$$

**PROOF:** Suppose the right inequality is false; then there exist  $\varepsilon_1$  and  $\varepsilon_2 > 0$  so that

$$\text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) < \varepsilon_1 < \varepsilon_2 < \inf A(f).$$

Therefore there exists a function  $g \in \mathcal{B}_0$  so that  $\|f - g\|_{\mathcal{B}} < \varepsilon_1$  and  $\Omega_{\varepsilon_2}(f)$  is not a compact set. But by Lemma 3.1  $\Omega_{\varepsilon_2}(f)$  must be a compact set, so we arrive at a contradiction. Thus the right inequality holds.

The left inequality follows since  $A(f) \subseteq A_p(f)$ , for all  $p$ . □

**PROPOSITION 3.3.** *Let  $X$  be a subspace of  $\mathcal{B}_0$  that contains  $\{a + bz : a, b \in \mathbb{C}\}$  and  $f \in \mathcal{B}$ . If there is some  $p > 0$  such that  $f_1 \in X$  for all  $\varepsilon \in A_p(f)$ , then*

$$\inf A_p(f) \leq \text{dist}_{\mathcal{B}}(f, X) \leq 6 \inf A_p(f).$$

PROOF: By Lemma A

$$(10) \quad \|f - f_1 - f(0) - f'(0)z - f_2(0) - f_2'(0)z\|_{\mathcal{B}} \leq 6\epsilon.$$

Since  $X$  contains all linear functions,  $f_1 \in X$  for all  $\epsilon \in A_p(f)$ , (10) gives

$$\text{dist}_{\mathcal{B}}(f, X) \leq 6 \inf A_p(f).$$

The left inequality follows from Proposition 3.2. □

REMARK. In the proposition above the right inequality holds for any subspace  $X$  of  $\mathcal{B}$  that contains  $\{a + bz : a, b \in \mathbb{C}\}$ .

A function  $f$  holomorphic on  $U$  belongs to the *minimal Besov space*  $B_1$  if and only if  $\int_U |f''(z)| dA(z) < \infty$ .  $B_1$  is a subspace of  $\mathcal{B}_0$ , in fact  $B_1$  is a subspace of  $A(U)$  the Banach space of functions that are continuous on the closed unit disk and holomorphic on the open unit disk with the supremum norm. See [2] and [12] for more information on  $B_1$ .

PROPOSITION 3.4. For  $f \in \mathcal{B}$  and  $p \geq 2$ ,

$$\inf A_p(f) \leq \text{dist}_{\mathcal{B}}(f, B_1) \leq 6 \inf A_p(f).$$

PROOF: Let  $\epsilon \in A_p(f)$ . We shall show that  $f_1 \in B_1$ . Recall that

$$f_1(z) = \int_{\Omega_\epsilon(f)} \frac{(1 - |w|^2) f'(w)}{\bar{w}(1 - \bar{w}z)^2} dA(w).$$

Then,

$$|f_1''(z)| \leq \int_{\Omega_\epsilon(f)} \left| \frac{(1 - |w|^2) f'(w)}{\bar{w}(1 - \bar{w}z)^4} 6\bar{w}^2 \right| dA(w)$$

and

$$\begin{aligned} \int_U |f_1''(z)| dA(z) &\leq 6 \int_U \int_{\Omega_\epsilon(f)} \frac{(1 - |w|^2) |f'(w)|}{|1 - \bar{w}z|^4} dA(w) dA(z) \\ &= 6 \int_{\Omega_\epsilon(f)} (1 - |w|^2) |f'(w)| \int_U \frac{1}{|1 - \bar{w}z|^4} dA(z) dA(w) \\ &\leq 6 \|f\|_{\mathcal{B}} \int_{\Omega_\epsilon(f)} \frac{1}{(1 - |w|^2)^2} dA(w) \\ (11) \quad &\leq 6 \|f\|_{\mathcal{B}} \int_{\Omega_\epsilon(f)} \frac{1}{(1 - |w|^2)^p} dA(w) < \infty, \end{aligned}$$

for all  $p \geq 2$ . Therefore (11) and Proposition 3.3 imply the result. □

The Besov space  $B_1$  is a subspace of the little Bloch space  $\mathcal{B}_0$  that contains all polynomials. Thus, the closure of  $B_1$  in the Bloch norm is  $\mathcal{B}_0$ , since  $\mathcal{B}_0$  is the closure of all polynomials in the Bloch norm. The next theorem follows from Proposition 3.2 and Proposition 3.4. Recall the definitions of  $A_p(f)$  and  $A(f)$  in (8), (9) respectively.

**THEOREM 3.5.** For  $f \in \mathcal{B}$ ,  $p \geq 2$ ,

$$\inf A_p(f) \leq \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leq 6 \inf A_p(f)$$

and

$$\inf A(f) \leq \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leq 6 \inf A(f).$$

**REMARK.** The best bounds for  $\text{dist}_{\mathcal{B}}(f, \mathcal{B}_0)$  are  $\inf A(f)$  from below,  $6 \inf A_2(f)$  from above, as  $A(f) \subseteq A_p(f) \subseteq A_q(f) \subseteq A_2(f)$  for all  $p, q$  with  $2 \leq p \leq q$ .

The following theorem of Arazy, Fisher and Peetre is from [2, p. 132]. We give a different proof of (2) implies (1).

**THEOREM C.** Let  $\mu$  be a positive measure on  $U$  and let  $0 < p < \infty$ . Then,

$$(1) \int_U \frac{d\mu(z)}{(1 - |z|^2)^p} < \infty$$

if and only if there is a constant  $c$  with

$$(2) \int_U |f'(z)|^p d\mu(z) \leq c \|f\|_{\mathcal{B}}^p, \text{ for all } f \in \mathcal{B}.$$

**PROOF:** For any  $f \in \mathcal{B}$

$$\begin{aligned} \int_U |f'(z)|^p d\mu(z) &= \int_U \frac{|f'(z)|^p (1 - |z|^2)^p}{(1 - |z|^2)^p} d\mu(z) \\ &\leq \|f\|_{\mathcal{B}}^p \int_U \frac{1}{(1 - |z|^2)^p} d\mu(z). \end{aligned}$$

This proves the easy implication, (1) implies (2). Ramey and Ullrich proved in [8] that there exists functions  $f, g \in \mathcal{B}$  so that

$$(12) \quad |f'(z)| + |g'(z)| \geq \frac{\text{const.}}{(1 - |z|^2)}.$$

Therefore it is easy to see that

$$\text{const.} \left( |f'(z)|^p + |g'(z)|^p \right) \geq \frac{1}{(1 - |z|^2)^p}.$$

Integrating the above with respect to  $d\mu(z)$  shows that (2) implies (1). □

The following is an immediate consequence of Theorem 3.5 and Theorem C.

**COROLLARY 3.6.** For  $f \in \mathcal{B}$ , the following are equivalent:

1.  $f \in \mathcal{B}_0$ .
2. For all  $\varepsilon > 0$   $\Omega_{\varepsilon}(f)$  is a compact subset of  $U$ .
3. For some  $p \geq 2$  and all  $\varepsilon > 0$ ,

$$\int_{\Omega_{\varepsilon}(f)} \frac{1}{(1 - |w|^2)^p} dA(w) < \infty.$$

4. For any  $p \geq 2$  and  $\epsilon > 0$ , there is a constant  $c$  so that

$$\int_{\Omega_\epsilon(f)} |g'(z)|^p dA(z) \leq c \|g\|_{\mathcal{B}}^p, \quad \text{for all } g \in \mathcal{B}.$$

Let  $D(0, \alpha)$  denote the disk centred at 0 of radius  $\alpha$ . A *nontangential approach region*  $\Omega_\alpha$  ( $0 < \alpha < 1$ ) in  $U$  with vertex  $\zeta \in \partial U$  is the convex hull of  $D(0, \alpha) \cup \{\zeta\}$  minus the point  $\zeta$ .

For any region  $G$  in the complex plane let  $\partial G$  denote the boundary of the region. For an open subset  $G$  of  $U$  with  $\zeta \in \partial G \cap \partial U$  we say that it has *pointwise order of contact (at most)  $b$*  ( $b > 0$ ) with  $\partial U$  at  $\zeta$  if

$$(13) \quad \frac{1 - |z|}{|\zeta - z|^b} \geq \text{const.}$$

as  $z$  approaches  $\zeta$  within  $G$ . So if  $\phi$  is a holomorphic self-map of  $U$  such that  $\phi(U) = \Omega_\alpha$  ( $0 < \alpha < 1$ ) with vertex  $\zeta$  then  $\phi(U)$  has pointwise order of contact 1 with  $\partial U$  at  $\zeta$ ; if  $\phi(U)$  is a disk inside  $U$  whose boundary makes tangential contact with  $\partial U$  at the point  $\zeta$  then  $\phi(U)$  has pointwise order of contact 2 with  $\partial U$  at the point  $\zeta$ .

In [3, p. 2191] Bourdon, Cima, and Matheson introduced the notion of mean order of contact. An open subset  $G$  of  $U$  has *mean order of contact (at most)  $b$*  ( $b > 0$ ) with  $\partial U$  if

$$(14) \quad \int_0^{2\pi} \chi_G(re^{i\theta}) d\theta = O(1 - r)^{1/b}$$

as  $r \rightarrow 1^-$ . The integral on the left side of (14) represents the angular measure of  $G$  intersected with the circle  $\{z \in U : |z| = r\}$ .

Recall the definition of  $\Omega_\epsilon(f)$  in (1) for any  $f \in \mathcal{B}$ .

**PROPOSITION 3.7.** *Let  $f \in \mathcal{B}$  so that for all  $\epsilon > 0$  the mean order of contact of  $\Omega_\epsilon(f)$  with  $\partial U$  is  $\alpha_\epsilon < 1$ . Then  $f \in \mathcal{B}_0$ .*

PROOF: For a fixed  $\epsilon > 0$ , (14) gives

$$\begin{aligned} \int_{\Omega_\epsilon(f)} \frac{1}{(1 - |w|^2)^2} dA(w) &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \chi_{\Omega_\epsilon(f)}(re^{i\theta}) d\theta \frac{r}{(1 - r^2)^2} dr \\ &\leq \text{const.} \int_0^1 \frac{(1 - r)^{1/\alpha_\epsilon}}{(1 - r)^2} dr < \infty \end{aligned}$$

since  $(1/\alpha_\epsilon) - 2 > -1$ . Thus by Corollary 3.6  $f \in \mathcal{B}_0$ . □

**PROPOSITION 3.8.** *Let  $f \in \mathcal{B}$  be so that for all  $\epsilon > 0$  the mean order of contact of  $\Omega_\epsilon(f)$  with  $\partial U$  is 1. Then for all  $\epsilon > 0$  and all  $\beta > 0$ ,*

$$\int_{\Omega_\epsilon(f)} \frac{1}{(1 - |w|^2)^{2-\beta}} dA(w) < \infty.$$



We omit the proof as it is similar to the one of Proposition 3.7.

REMARK. For the function  $f(z) = \log(1 - z)$  the mean order of contact of  $\Omega_\epsilon(f)$  with  $\partial U$  is 1 for all  $\epsilon > 0$  (see the remark after example 4.1 for a proof). This shows that the conclusion of the above proposition is valid for the function  $\log(1 - z)$ . Therefore  $p \geq 2$  in Theorem 3.5 is best possible since  $\inf A_p(f) = 0$  for all  $p < 2$ . Similarly in Corollary 3.6  $p \geq 2$  is best possible. If  $g \in \mathcal{B}_0$  then by condition (2) of Corollary 3.6

$$\int_{\Omega_\epsilon(g)} \frac{1}{(1 - |w|^2)^p} dA(w) < \infty$$

for all  $p$ . But the converse is valid only for  $p \geq 2$ .

It is well known that the Bloch space  $\mathcal{B}$  can be thought of as the area version of the space of bounded mean oscillations  $BMOA$ . Thus, motivated by a formula for the distance of a  $BMOA$  function to  $VMOA$ , the space of vanishing mean oscillations, given by Stegenga and Stephenson in [10] we prove the following theorem. In the proof we use a modified version of an argument of Montes-Rodriguez, given in [6, p. 346].

**THEOREM 3.9.** For  $f \in \mathcal{B}$ ,

$$\limsup_{|z| \rightarrow 1} |f'(z)| (1 - |z|^2) \leq \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leq 2 \limsup_{|z| \rightarrow 1} |f'(z)| (1 - |z|^2).$$

PROOF: For a given  $f \in \mathcal{B}$  and  $g \in \mathcal{B}_0$  (5) gives

$$\limsup_{|z| \rightarrow 1} (1 - |z|^2) |f'(z)| \leq \|f - g\|_{\mathcal{B}},$$

from which the left inequality follows.

For the right inequality fix  $f \in \mathcal{B}$ ,  $M > 0$ ,  $0 < \rho < 1$  so that for all  $z \in U$  with  $|z| \geq 1 - \rho$ ,  $|f'(z)| (1 - |z|^2) \leq M$ . We shall show that  $\text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \leq 2M$ . Also fix for the moment  $r$  with  $1 > r > 1 - \rho$ , let  $f_r(z) = f(rz)$  for  $z \in U$ . Then,

$$\begin{aligned} \|f - f_r\|_{\mathcal{B}} &= \sup_{z \in U} |(f - f_r)'(z)| (1 - |z|^2) \\ &\leq \sup_{|z| \leq (1-\rho)/r} |(f - f_r)'(z)| (1 - |z|^2) + \sup_{|z| > (1-\rho)/r} |(f - f_r)'(z)| (1 - |z|^2) \\ (15) \quad &= I + II. \end{aligned}$$

Then

$$\begin{aligned} II &\leq \sup_{|z| > (1-\rho)/r} |f'(z)| (1 - |z|^2) + \sup_{|z| > (1-\rho)/r} |f'_r(z)| (1 - |z|^2) \\ (16) \quad &\leq \sup_{|z| > 1-\rho} |f'(z)| (1 - |z|^2) + \sup_{|z| > 1-\rho} |f'(z)| (1 - |z|^2) \leq 2M. \end{aligned}$$

Also

$$\begin{aligned} I &= \sup_{|z| \leq (1-\rho)/r} |(f - f_r)'(z)| (1 - |z|^2) \\ &\leq \sup_{|z| \leq (1-\rho)/r} |f'(z) - f'(rz)| (1 - |z|^2) + (1 - r) \sup_{|z| \leq (1-\rho)/r} |f'(rz)| (1 - |z|^2) \\ (17) \quad &\leq A + (1 - r) \|f\|_{\mathcal{B}}. \end{aligned}$$

By taking the line integral of  $f''$  from  $rz$  to  $z$  we get

$$(18) \quad \begin{aligned} A &= \sup_{|z| \leq (1-\rho)/r} |f'(z) - f'(rz)|(1 - |z|^2) \\ &\leq (1 - r) \sup_{|z| \leq (1-\rho)/r} |f''(\xi(z))| |z| (1 - |z|^2), \end{aligned}$$

where  $\xi(z)$  is a point in the closed disk of radius  $(1 - \rho)/r$ . Using the Maximum Modulus Theorem and Cauchy's Estimates for  $f'$  on the circle centred at  $\xi(z)$  with radius  $R = (1 - (1 - \rho)/r)/2$  in (18), we obtain

$$(19) \quad \begin{aligned} A &\leq \frac{1 - r}{R} \max_{|z|=R+(1-\rho)/r} |f'(z)| \sup_{|z| \leq (1-\rho)/r} |z|(1 - |z|^2) \\ &\leq \frac{(1 - r)}{R} \|f\|_B \frac{(1 - \rho)/r}{1 - ((1 - \rho)/r + R)^2}, \end{aligned}$$

since

$$\max_{|z|=(1-\rho)/r+R} |f'(z)| \leq \|f\|_B \frac{1}{1 - ((1 - \rho)/(r) + R)^2}.$$

Therefore by (15), (16), (17),

$$(20) \quad \begin{aligned} \text{dist}_B(f, B_0) &\leq \|f - f_r\|_B \\ &\leq A + (1 - r)\|f\|_B + 2M. \end{aligned}$$

Notice that

$$(21) \quad \lim_{r \rightarrow 1} \frac{1}{R} \frac{(1 - \rho)/r}{1 - ((1 - \rho)/(r) + R)^2} = \frac{8(1 - \rho)}{\rho^2(4 - \rho)}$$

thus letting  $r \rightarrow 1$  in (19) and in (20), we obtain

$$\text{dist}_B(f, B_0) \leq 2M,$$

which proves the right inequality. □

Recall the Koebe Distortion Theorem (see for example [9, p. 156]) which asserts that if  $\psi$  is a univalent function on  $U$  then for any  $z \in U$ ,

$$(22) \quad \frac{1}{4} |\psi'(z)| (1 - |z|^2) \leq \text{dist}(\psi(z), \partial\psi(U)) \leq |\psi'(z)| (1 - |z|^2),$$

where  $\text{dist}(\psi(z), \partial\psi(U))$  is the Euclidean distance from  $\psi(z)$  to  $\partial\psi(U)$ . By (22), an immediate corollary of Theorem 3.9 is as follows.

**COROLLARY 3.10.** *For any univalent function  $f \in B$ ,*

$$\limsup_{|z| \rightarrow 1} \text{dist}(f(z), \partial f(U)) \leq \text{dist}_B(f, B_0) \leq 8 \limsup_{|z| \rightarrow 1} \text{dist}(f(z), \partial f(U)).$$

4. EXAMPLES

It is easy to see that for any constant  $c$  and any  $f \in \mathcal{B}$

$$\text{dist}_{\mathcal{B}}(f + c, \mathcal{B}_0) = \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0) = \inf_{g \in \mathcal{B}_0} \|f - g\|_{\mathcal{B}}.$$

Also because of the Möbius invariance of  $\mathcal{B}$

$$(23) \quad \text{dist}_{\mathcal{B}}(f \circ \omega \varphi_a, \mathcal{B}_0) = \text{dist}_{\mathcal{B}}(f, \mathcal{B}_0),$$

for all  $a \in U, \omega \in \partial U$ .

EXAMPLE 4.1. The function  $f(z) = \log(1 - z)$  belongs to the Bloch space but not to the little Bloch space. We shall show that

$$(24) \quad \text{dist}(\log(1 - z), \mathcal{B}_0) = 2 = \|\log(1 - z)\|_{\mathcal{B}}.$$

First

$$\frac{1 - |z|^2}{|1 - z|} \leq \frac{1 - |z|^2}{1 - |z|} \leq 2$$

for all  $z \in U$ , therefore  $\|\log(1 - z)\|_{\mathcal{B}} \leq 2$ ; on the other hand

$$\|\log(1 - z)\|_{\mathcal{B}} \geq \frac{1 - (1 - (1/n))^2}{|1 - (1 - (1/n))|}$$

or  $\|\log(1 - z)\|_{\mathcal{B}} \geq 2 - (1/n)$  for all nonnegative integers  $n$  and so  $\|\log(1 - z)\|_{\mathcal{B}} = 2$ . By (1) the set

$$(25) \quad \Omega_{\varepsilon}(f) = \left\{ z \in U : \frac{1 - |z|^2}{|1 - z|} \geq \varepsilon \right\}$$

behaves like a nontangential approach region in  $U$  that touches the unit circle only at the point 1; so  $\Omega_{\varepsilon}(f)$  is never a compact subset of  $U$  unless it is the empty set, thus trivially compact. If  $\Omega_{\varepsilon}(f) \neq \emptyset$  then  $\varepsilon < 2$  since for any  $z_0 \in \Omega_{\varepsilon}(f)$ ,

$$\varepsilon \leq \frac{1 - |z_0|^2}{|1 - z_0|} \leq 1 + |z_0| < 2;$$

on the other hand if  $\varepsilon < 2$  then  $\Omega_{\varepsilon}(f) \neq \emptyset$ , since it is easy to see that  $z_n = 1 - (1/n) \in \Omega_{\varepsilon}(f)$  for all  $n$  large enough. Therefore  $\Omega_{\varepsilon}(f) = \emptyset$  if and only if  $\varepsilon \geq 2$  if and only if  $\Omega_{\varepsilon}(f)$  is a compact subset of  $U$ . Thus

$$(26) \quad \inf A(\log(1 - z)) = 2 = \|\log(1 - z)\|_{\mathcal{B}}.$$

Since

$$\text{dist}_{\mathcal{B}}(\log(1 - z), \mathcal{B}_0) \leq \|\log(1 - z)\|_{\mathcal{B}}$$

and by Theorem 3.5

$$\inf A(\log(1 - z)) \leq \text{dist}_{\mathcal{B}}(\log(1 - z), \mathcal{B}_0),$$

using (26) we obtain (24) and by (23) for all  $a \in U, \omega \in \partial U$

$$\text{dist}_{\mathcal{B}}\left(\log\left(1 - \omega \frac{a - z}{1 - \bar{a}z}\right), \mathcal{B}_0\right) = 2.$$

Similarly to the proof of (24) one can show that

$$\text{dist}_{\mathcal{B}}(\log(k'), \mathcal{B}_0) = 6 = \|\log(k')\|_{\mathcal{B}},$$

where  $k$  is the Koebe function  $k(z) = z/((1 - z)^2)$  and that

$$\text{dist}_{\mathcal{B}}\left(\log\left(\frac{z}{1 - z^2}\right)', \mathcal{B}_0\right) = 4 = \left\|\log\left(\frac{z}{1 - z^2}\right)'\right\|_{\mathcal{B}}.$$

REMARK. By (25) for the function  $f(z) = \log(1 - z)$  and each  $\varepsilon > 0, \Omega_{\varepsilon}(f)$  has pointwise order of contact 1 with  $\partial U$  at the point 1. In [3, p. 2192] is shown that for domains whose boundary touches the unit circle at exactly one point  $\zeta$ , if  $G$  has pointwise order of contact 1 with  $\partial U$  at  $\zeta$  then it has mean order of contact 1 with  $\partial U$ . Therefore for all  $\varepsilon > 0, \Omega_{\varepsilon}(f)$  has mean order of contact 1 with  $\partial U$ .

For the rest of this section let  $\phi$  be a univalent self-map of the unit disk; then  $\phi \in \mathcal{B}_0$  as for example  $\phi(U)$  has a finite area and all such functions lie in  $\mathcal{B}_0$ . Below we shall describe the distance of  $\log(1 - \phi)$  to  $\mathcal{B}_0$  for several  $\phi$ . First notice that if  $\phi(U) \subseteq U \setminus D(1, r)$ , where  $D(1, r)$  is the disk centred at 1 of radius  $r$ , then  $\log(1 - \phi) \in \mathcal{B}_0$ .

EXAMPLE 4.2. Let  $\phi$  be such that  $\phi(U) = \Omega_{\alpha}$  a nontangential approach region in  $U$  with vertex 1, for some  $0 < \alpha < 1$ . We shall show that

$$(27) \quad \frac{1}{4} \|\log(1 - \phi)\|_{\mathcal{B}} \leq \text{dist}_{\mathcal{B}}(\log(1 - \phi), \mathcal{B}_0) \leq \|\log(1 - \phi)\|_{\mathcal{B}}.$$

Using the Schwarz–Pick Lemma it is easy to see that  $\log(1 - \phi) \in \mathcal{B}$ . Then by (1) and (22) for all  $\varepsilon > 0$  the set

$$\Omega_{\varepsilon}(\log(1 - \phi)) = \left\{z \in U : \frac{|\phi'(z)|(1 - |z|^2)}{|1 - \phi(z)|} \geq \varepsilon\right\}$$

contains the set

$$G = \left\{z \in U : \frac{\text{dist}(\phi(z), \partial\phi(U))}{|1 - \phi(z)|} \geq \varepsilon\right\}.$$

Then  $\phi(G)$  behaves like a nontangential approach region contained in  $\phi(U)$ . Therefore, unless  $G = \emptyset$ , it is not a compact subset of  $U$ . If  $\Omega_{\varepsilon}(\log(1 - \phi))$  is a compact subset of  $U$  then so is  $G$  and therefore  $G = \emptyset$  and for all  $z \in U$  we have

$$\frac{\text{dist}(\phi(z), \partial\phi(U))}{|1 - \phi(z)|} < \varepsilon.$$

By (22)  $\|\log(1 - \phi)\|_{\mathcal{B}}/4 < \varepsilon$  so we obtain

$$\frac{1}{4} \|\log(1 - \phi)\|_{\mathcal{B}} \leq \inf A(\log(1 - \phi)).$$

Thus Theorem 3.5 gives (27).

Next we give an example where equality holds in the right inequality of (27) for a function  $\phi$  with  $\phi(U)$  inside a nontangential approach region.

EXAMPLE 4.3. For  $0 < \alpha < 1$  define

$$\phi_{\alpha}(z) = \frac{\sigma(z)^{\alpha} - 1}{\sigma(z)^{\alpha} + 1}$$

where

$$\sigma(z) = \frac{1+z}{1-z}, \quad z \in U.$$

The map  $\phi_{\alpha}$  is a holomorphic self-map of  $U$  whose image is a lens-shaped region thus it is called a “lens map” ([9, p. 27]). We shall show that

$$(28) \quad \inf A(\log(1 - \phi_{\alpha})) = \text{dist}(\log(1 - \phi_{\alpha}), \mathcal{B}_0) = \|\log(1 - \phi_{\alpha})\|_{\mathcal{B}} = 2\alpha.$$

By (1) for each  $\varepsilon > 0$

$$(29) \quad \begin{aligned} \Omega_{\varepsilon}(\log(1 - \phi_{\alpha})) &= \left\{ z \in U : \frac{|\phi'_{\alpha}(z)| (1 - |z|^2)}{|1 - \phi_{\alpha}(z)|} \geq \varepsilon \right\} \\ &= \left\{ z \in U : \frac{2\alpha (1 - |z|^2)}{|1 - z^2| |1 + ((1 - z)/(1 + z))^{\alpha}|} \geq \varepsilon \right\}. \end{aligned}$$

Fix  $\varepsilon < 2$ , let  $z_n = 1 - 1/n$ ; then by (29) we can easily see that  $z_n \in \Omega_{\varepsilon}(\log(1 - \phi_{\alpha}))$  if and only if

$$\frac{2\alpha}{1 + (1/(2n - 1))^{\alpha}} \geq \varepsilon,$$

which is true for all  $n$  large enough thus  $\Omega_{\varepsilon}(\log(1 - \phi_{\alpha}))$  is not a compact subset of  $U$ . This shows that

$$(30) \quad 2\alpha \leq \inf A(\log(1 - \phi_{\alpha})).$$

For each  $0 < \alpha < 1$

$$(31) \quad \begin{aligned} \|\log(1 - \phi_{\alpha})\|_{\mathcal{B}} &= \sup_{z \in U} \frac{2\alpha (1 - |z|^2)}{|1 - z^2| |1 + ((1 - z)/(1 + z))^{\alpha}|} \\ &\leq 2\alpha \sup_{z \in U} \left| \frac{1}{1 + ((1 - z)/(1 + z))^{\alpha}} \right| \\ &\leq 2\alpha \end{aligned}$$

since

$$1 + ((1 - z)/(1 + z))^{\alpha}$$

maps  $U$  onto the sector

$$\left\{ w : |\arg(w - 1)| < (\alpha\pi)/2 \right\}.$$

Thus Theorem 3.5, (30) and (31) give (28).

REMARK. A formula like in (27) is not true for all functions of the form  $\log(1 - \phi)$  where  $\phi(U)$  has mean order of contact  $b$  ( $0 < b \leq 1$ ) with  $\partial U$ , as there are such  $\phi$  with  $\log(1 - \phi) \in \mathcal{B}_0$ . Below we give examples of this.

(i) For each nonnegative integer  $n$  let

$$\Delta_n = \left\{ z \in U : |\arg(1 - z)| < \theta_n, \rho_n < |z| \leq \rho_{n+1} \right\},$$

where  $\rho_n = 1 - (1/2^n)$ ,  $0 < \theta_n < \pi/7$  such that  $\sum_{n=1}^\infty \theta_n < \infty$ . Let  $\Delta = \bigcup_{n=1}^\infty \Delta_n$ . It follows that  $\Delta$ , as it clearly has pointwise order of contact 1 with  $\partial U$  at the point 1, has mean order of contact 1 (see the remark after example 4.1). We would like to give a direct proof of this for our domain  $\Delta$  using the definition of mean order of contact; since

$$(32) \quad \int_0^{2\pi} \chi_\Delta(re^{i\theta}) d\theta \leq \sum_{n=1}^\infty \int_0^{2\pi} \chi_{\Delta_n}(re^{i\theta}) d\theta \leq \sum_{n=1}^\infty 2\psi r,$$

where  $\psi = \arg(z)$ ,  $z$  is a point, on the first quadrant, on the boundary of the arc  $\Delta \cap \{z \in U : |z| = r\}$ . By the Law of Cosines and the Law of Sines

$$\tan \psi = \frac{1 - r}{r} \tan \theta_n$$

therefore (32) gives

$$\begin{aligned} \int_0^{2\pi} \chi_\Delta(re^{i\theta}) d\theta &\leq \sum_{n=1}^\infty 2r \tan^{-1} \left( \frac{1 - r}{r} \tan \theta_n \right) \\ &\leq \text{const. } \sum_{n=1}^\infty \theta_n (1 - r) \leq \text{const. } (1 - r). \end{aligned}$$

Thus (14) shows that  $\Delta$  has mean order of contact with  $\partial U$  1. Let  $\phi$  be the Riemann map from  $U$  onto  $\Delta$ . It is easy to see, using (22), that  $\log(1 - \phi) \in \mathcal{B}_0$ , thus (27) is not valid for such a  $\phi$ .

(ii) If  $\phi$  is such that  $\phi(U)$  has mean order of contact  $b$  ( $0 < b < 1$ ) with  $\partial U$  and  $\phi(U)$  lies inside a polygon inscribed in the unit circle then  $\log(1 - \phi) \in \mathcal{B}_0$ ; this follows from Corollary 5.2 in [3] and Theorem 5.3 in [5]. Thus (27) is not valid for such a  $\phi$ .

EXAMPLE 4.4. If  $\phi(U)$  is a polygon inscribed in the unit circle and one of the vertices of the polygon is the point 1 then it is easy to see that (27) is valid, as in the proof of the formula in the case where  $\phi(U) = \Omega_\alpha$  ( $0 < \alpha < 1$ ).

EXAMPLE 4.5. If  $\phi(U)$  is a disk tangent to the unit circle at the point 1 then it is easy to see that  $\phi$  has the form  $\psi \circ M$  where  $\psi(z) = \lambda z + (1 - \lambda)$  ( $0 < \lambda < 1$ ) and  $M$  is a

Möbius transformation. Using the Möbius invariance of the Bloch seminorm and (24) we obtain

$$\begin{aligned} \text{dist}_{\mathcal{B}}(\log(1 - \phi), \mathcal{B}_0) &= \text{dist}_{\mathcal{B}}(\log(1 - \psi), \mathcal{B}_0) \\ &= \text{dist}_{\mathcal{B}}(\log \lambda(1 - z), \mathcal{B}_0) \\ &= \text{dist}_{\mathcal{B}}(\log(1 - z), \mathcal{B}_0) = 2. \end{aligned}$$

EXAMPLE 4.6. The function  $S(z) = e^{(z+1)/(z-1)}$  belongs to the Bloch space but not to the little Bloch space; actually  $S \in H^\infty \setminus \mathcal{B}_0$ . We shall show that

$$(33) \quad \text{dist}_{\mathcal{B}}(S, \mathcal{B}_0) = \frac{2}{e} = \|S\|_{\mathcal{B}}.$$

For  $\alpha > 0$ , the function  $S$  maps the circle

$$C_\alpha = \left\{ z \in U : \frac{1 - |z|^2}{|1 - z|^2} = \alpha \right\}$$

(centre  $\alpha/(1 + \alpha)$ , radius  $1/(1 + \alpha)$ ) internally tangent to the unit circle at the point 1) to the circle  $\{\zeta \in U : |\zeta| = e^{-\alpha}\}$ . By (1)

$$\begin{aligned} (34) \quad \Omega_\varepsilon(S) &= \left\{ z \in U : (1 - |z|^2) |S'(z)| \geq \varepsilon \right\} \\ &= \left\{ z \in U : \frac{2|S(z)|(1 - |z|^2)}{|1 - z|^2} \geq \varepsilon \right\}. \end{aligned}$$

On  $C_\alpha$  the left hand side of the inequality in (34) equals  $2\alpha e^{-\alpha}$ . Thus,

$$\Omega_\varepsilon(S) = \left\{ z \in U : 2\alpha e^{-\alpha} \geq \varepsilon, \quad \alpha = \frac{1 - |z|^2}{|1 - z|^2} \right\}.$$

Also,

$$(35) \quad \|S\|_{\mathcal{B}} = \sup_{\alpha > 0} 2e^{-\alpha}\alpha = \frac{2}{e}.$$

The set  $\Omega_\varepsilon(S)$  is either the empty set, when  $\varepsilon > 2/e$  or a circle internally tangent to the unit circle at  $z = 1$  when  $\varepsilon = 2/e$ , or it is the area between two circles both internally tangent to the unit circle at  $z = 1$  when  $\varepsilon < 2/e$ . Therefore  $\Omega_\varepsilon(f)$  is never a compact subset of  $U$  unless it is the empty set thus trivially compact. Thus, Theorem 3.5 and (35) show that

$$\begin{aligned} (36) \quad \inf A(S) = \|S\|_{\mathcal{B}} &= \frac{2}{e} \leq \text{dist}_{\mathcal{B}}(S, \mathcal{B}_0) \\ &= \text{dist}_{\mathcal{B}}(S - S(0), \mathcal{B}_0) \\ &\leq \|S - S(0)\|_{\mathcal{B}} = \|S\|_{\mathcal{B}} = \frac{2}{e}. \end{aligned}$$

Therefore (33) follows.

5. DISTANCE TO  $\mathcal{B}_0$  AND THE ESSENTIAL NORM OF A COMPOSITION OPERATOR

If  $\phi$  is a holomorphic self-map of  $U$ , then the *composition operator*  $C_\phi$

$$C_\phi f = f \circ \phi$$

maps holomorphic functions  $f$  to holomorphic functions. It is a bounded operator on  $\mathcal{B}$ , and if  $\phi \in \mathcal{B}_0$  it is bounded on  $\mathcal{B}_0$  as well ([2, Theorem 12]). The essential norm of  $C_\phi$ ,  $\|C_\phi\|_e$ , is the distance in the operator norm from  $C_\phi$  to the compact operators. In [6, Theorem 2.1, Proposition 2.2] Montes-Rodriguez showed that for  $\phi$  a univalent self-map of  $U$

$$(37) \quad \|C_\phi\|_e = \limsup_{|z| \rightarrow 1} \frac{|\phi'(z)|(1 - |z|^2)}{1 - |\phi(z)|^2}.$$

**COROLLARY 5.1.** *Let  $\phi$  be a univalent self-map of  $U$  so that  $\phi(U)$  has pointwise order of contact 1 with  $\partial U$  at the point 1. Then,*

$$\frac{1}{4} \text{dist}_{\mathcal{B}}(\log(1 - \phi), \mathcal{B}_0) \leq \|C_\phi\|_e \leq \text{const.} \text{dist}_{\mathcal{B}}(\log(1 - \phi), \mathcal{B}_0).$$

PROOF: By (13), (37) and Theorem 3.9

$$\begin{aligned} \text{dist}_{\mathcal{B}}(\log(1 - \phi), \mathcal{B}_0) &\leq 2 \limsup_{|z| \rightarrow 1} \frac{|\phi'(z)|(1 - |z|^2)}{|1 - \phi(z)|} \\ &\leq 4 \limsup_{|z| \rightarrow 1} \frac{|\phi'(z)|(1 - |z|^2)}{1 - |\phi(z)|^2} = 4 \|C_\phi\|_e \\ &\leq \text{const.} \limsup_{|z| \rightarrow 1} \frac{|\phi'(z)|(1 - |z|^2)}{|1 - \phi(z)|} \\ &\leq \text{const.} \text{dist}_{\mathcal{B}}(\log(1 - \phi), \mathcal{B}_0). \end{aligned}$$

Therefore the result follows. □

6. DISTANCE IN BLOCH TYPE SPACES

In [13] Zhu defined the *Bloch type spaces* of holomorphic functions on  $U$ , which are generalisations of the Bloch space  $\mathcal{B}$ . For each  $\alpha > 0$ ,  $\mathcal{B}_\alpha$  denotes the space of holomorphic functions  $f$  on  $U$  for which

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z \in U} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

The *little Bloch type spaces* are generalisations of the little Bloch space  $\mathcal{B}_0$ . For each  $\alpha > 0$  let  $\mathcal{B}_{\alpha,0}$  denote the space of all functions in  $\mathcal{B}_\alpha$  so that

$$\lim_{|z| \rightarrow 1} |f'(z)|(1 - |z|^2)^\alpha = 0.$$



Both  $\mathcal{B}_\alpha$  and  $\mathcal{B}_{\alpha,0}$  are Banach spaces with norm

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \|f\|_{\mathcal{B}_\alpha},$$

for all  $f \in \mathcal{B}_\alpha$ . As Zhu notes these spaces are not new. When  $0 < \alpha < 1$  then  $\mathcal{B}_\alpha$  can be identified with the holomorphic Lipschitz space  $\text{Lip}_{1-\alpha}$ , the space of all holomorphic functions on  $U$  with  $|f(z) - f(w)| \leq c|z - w|^{1-\alpha}$ , for some constant  $c > 0$  (depending on  $f$ ) and all  $z, w \in U$ . And when  $\alpha > 1$ ,  $\mathcal{B}_\alpha$  can be identified with the space of holomorphic functions  $f$  with

$$\sup_{z \in U} (1 - |z|^2)^{\alpha-1} |f(z)| < \infty.$$

Below we shall describe how the distance formulas in  $\mathcal{B}$  have Bloch type spaces analogues.

One can define analogues of  $\Omega_\varepsilon(f)$ ,  $A_p(f)$  and  $A(f)$  for Bloch type spaces, just by replacing, on the definition of these sets, (see (1), (8), (9)), the term  $1 - |z|^2$  with  $(1 - |z|^2)^\alpha$ . Call these new sets  $\Omega_{\varepsilon,\alpha}(f)$ ,  $A_{p,\alpha}(f)$  and  $A_\alpha(f)$  respectively. Then by Corollary 4 in [13] the analogue of (2) holds in  $\mathcal{B}_\alpha$  (change powers 2 to  $1 + \alpha$ ). Thus Lemma A (with 6 replaced with a constant,  $\text{const.}$ , that depends on  $\alpha$ ) holds. Lemma B follows from Proposition 2 in [13], which says that  $\mathcal{B}_{\alpha,0}$  is the closure of all polynomials in the  $\mathcal{B}_\alpha$  norm. Lemma 3.1, Proposition 3.2, Proposition 3.3 (with 6 replaced with a constant,  $\text{const.}$ , that depends on  $\alpha$ ) hold. We omit the details as the proofs are similar to the corresponding results in  $\mathcal{B}$ .

Let  $A$  and  $B$  be two quantities that depend on a holomorphic function  $f$  on  $U$ . We say that  $A$  is *equivalent* to  $B$ , we write  $A \sim B$ , if

$$\text{const. } A \leq B \leq \text{const. } A.$$

We shall show that the analogue of Theorem 3.5 holds for  $p > 2$ .

**THEOREM 6.1.** For  $f \in \mathcal{B}_\alpha$  ( $\alpha > 0$ ),  $p > 2$ ,

$$\text{dist}_{\mathcal{B}_\alpha}(f, \mathcal{B}_{\alpha,0}) \sim \inf A_{p,\alpha}(f) \sim \inf A_\alpha(f).$$

**PROOF:** By the  $\mathcal{B}_\alpha$  versions of Propositions 3.2 and 3.3 it suffices to show that for all  $\varepsilon \in A_{p,\alpha}(f)$ ,  $f_1 \in \mathcal{B}_{\alpha,0}$ . The function  $f_1$  in the  $\mathcal{B}_\alpha$  setting is

$$f_1(z) = \int_{\Omega_\varepsilon(f)} \frac{(1 - |w|^2)^\alpha f'(w)}{\bar{w}(1 - \bar{w}z)^{1+\alpha}} dA(w).$$

Thus for all  $\delta > 0$

$$\begin{aligned} (1 - |z|^2)^\alpha |f_1'(z)| &\leq (1 - |z|^2)^\alpha \|f\|_{\mathcal{B}_\alpha} \int_{\Omega_\varepsilon(f)} \frac{1}{(1 - |w||z|)^{\alpha+2}} dA(w) \\ &\leq \text{const. } (1 - |z|)^\delta \int_{\Omega_\varepsilon(f)} \frac{1}{(1 - |w|^2)^{2+\delta}} dA(w). \end{aligned}$$

It follows that  $f_1 \in \mathcal{B}_{\alpha,0}$  for all  $\varepsilon \in A_{p,\alpha}(f)$ ,  $p = 2 + \delta$  ( $\delta > 0$ ). Therefore the result follows. □

The analogue of Theorem C holds. In [11, Theorem 2.1.1] is the  $\mathcal{B}_\alpha$  analogue of (12) (replace the term  $1 - |z|^2$  in (12) with  $(1 - |z|^2)^\alpha$ ). Thus a corollary of the proof of Theorem C is as follows.

**THEOREM 6.2.** *Let  $\mu$  be a positive measure on  $U$ , let  $0 < p < \infty$  and  $\alpha > 0$ . Then,*

$$\int_U \frac{d\mu(z)}{(1 - |z|^2)^{\alpha p}} < \infty$$

*if and only if there is a constant  $c$  with*

$$\int_U |f'(z)|^p d\mu(z) \leq c \|f\|_{\mathcal{B}_\alpha}^p, \quad \text{for all } f \in \mathcal{B}_\alpha.$$

Therefore we obtain the analogue of Corollary 3.6 for  $p > 2$ :

**COROLLARY 6.3.** *For  $f \in \mathcal{B}_\alpha$  ( $\alpha > 0$ ), the following are equivalent:*

1.  $f \in \mathcal{B}_{\alpha,0}$ .
2. For all  $\varepsilon > 0$   $\Omega_{\varepsilon,\alpha}(f)$  is a compact subset of  $U$ .
3. For some  $p > 2$  and all  $\varepsilon > 0$ ,

$$\int_{\Omega_{\varepsilon,\alpha}(f)} 1 / \left( (1 - |w|^2)^p \right) dA(w) < \infty.$$

4. For any  $p > 2$  and  $\varepsilon > 0$ , there is a constant  $c$  so that

$$\int_{\Omega_{\varepsilon,\alpha}(f)} |g'(z)|^p dA(z) \leq c \|g\|_{\mathcal{B}_\alpha}^p, \quad \text{for all } g \in \mathcal{B}_\alpha.$$

The analogue of Theorem 3.9 for all spaces  $\mathcal{B}_\alpha$  holds. The left inequality holds as the analogue of (5) is easily seen to be true in  $\mathcal{B}_\alpha$ . For the right inequality everything works through with the appropriate change of each  $1 - |z|^2$  term to  $(1 - |z|^2)^\alpha$  and  $\mathcal{B}, \mathcal{B}_0$  replaced with  $\mathcal{B}_\alpha, \mathcal{B}_{\alpha,0}$  respectively. In (19)  $1 / \left( 1 - (R + (1 - \rho)/r) \right)^2$  needs to be replaced by its  $\alpha$  power then the corresponding limit in (21) equals  $(2^{2\alpha+1}(1 - \rho)) / (\rho^{\alpha+1}(4 - \rho)^\alpha)$ . Therefore we obtain the following corollary.

**COROLLARY 6.4.** *For  $f \in \mathcal{B}_\alpha$  ( $\alpha > 0$ ),*

$$\limsup_{|z| \rightarrow 1} |f'(z)| (1 - |z|^2)^\alpha \leq \text{dist}_{\mathcal{B}_\alpha}(f, \mathcal{B}_{\alpha,0}) \leq 2 \limsup_{|z| \rightarrow 1} |f'(z)| (1 - |z|^2)^\alpha.$$

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