15

Rectifiability of Singularities

There is a huge literature on singularities of solutions to geometric variational, and other, problems. Here we only briefly present some results and ideas related to rectifiability. We first discuss currents and varifolds and then harmonic maps. The methods for both are rather similar and we give a few more details in the latter case. These topics have a good bit in common with the other topics we then discuss.

15.1 Mass Minimizing Currents and Stationary Varifolds

Recall Chapter 13 for the notation and terminology. De Lellis's survey [161] gives an excellent up-to-date view of this wide topic. Currents give a very convenient setting for the Plateau problem for orientable surfaces. Let $B \in \mathcal{R}_{m-1}(\mathbb{R}^n)$ with $\partial B = 0$ and with finite mass. Then there are currents $T \in \mathcal{I}_m(\mathbb{R}^n)$ with $\partial T = B$, for example, cones over B. We say that such a T is *mass minimizing* if $M(T) \leq M(S)$ for all $S \in \mathcal{I}_m(\mathbb{R}^n)$ with $\partial S = \partial T$. For the important local and homological minimizers, see [203, 297, 397]. The existence of mass minimizing currents follows from the compactness Theorem 13.5 together with the easy facts that mass is lower semicontinuous and the boundary operator is continuous. How much can we say about their regularity?

Definition 15.1 Let $T \in \mathcal{D}_m(\mathbb{R}^n)$. A point $x \in \operatorname{spt} T \setminus \operatorname{spt} \partial T$ is called a *regular* point of T if it has a neighbourhood U such that $\operatorname{spt} T \cap U$ is an m-dimensional smooth submanifold of $\mathbb{R}^n$. Otherwise x is called a *singular* point of T. The set of singular points of T is denoted by $\operatorname{Sing}(T)$.

Often the singular set is empty, but not always:

Theorem 15.2 *Let $T \in \mathcal{I}_m(\mathbb{R}^n)$ be mass minimizing.*

(1) *If* $m = 1$ *or* $m = n - 1 \leq 6$*, then* $\operatorname{Sing}(T) = \emptyset$.

(2) *If* $m = n - 1 \geq 7$*, then* $\dim \operatorname{Sing}(T) \leq m - 7$.

(3) *If* $m \geq 2$*, then* $\dim \operatorname{Sing}(T) \leq m - 2$.

The bounds in (2) and (3) are sharp.

(1) was proved by Simons in [403]. The cases $m = 2$, by Fleming, and $m = 3$, by Almgren, were done earlier. An example showing the sharpness in (2) in $\mathbb{R}^8$ is the current induced by the cone $\left\{x \in \mathbb{R}^8 : x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\right\}$. It has a singularity at the origin and it was shown to be mass minimizing (locally) by Bombieri, De Giorgi and Giusti in [70], see also [227]. Cartesian products with $\mathbb{R}^{n-8}$ give higher-dimensional examples. Examples for (3) are obtained by complex analytic varieties, which also minimize area. For example, $\left\{(w,z) \in \mathbb{C}^2 : w^2 = z^3\right\}$ has a genuine branch point at the origin. The estimate (2) was proved by Federer in [204] and (3) by Almgren in his massive more than 1,000-page paper, which appeared as a Princeton University preprint in the early 1980s and was published in [10]. One of the key tools, due to De Giorgi and Reifenberg, in codimension 1 regularity theory is the approximation by graphs of harmonic functions. In higher codimensions this is not possible, for example, because of the complex analytic varieties. To overcome this, Almgren developed a theory of multivalued functions minimizing the Dirichlet integral. Much of Almgren's work has been simplified and extended by De Lellis and Spadaro, see the surveys [160, 161] and the references given there.

So the singular sets are small but their structure is a big open question. As above, in all known examples they are quite nice, but in general it is not known if they could be some kind of fractals. However, see the comment at the end of this section. Simon proved the following in [399]:

Theorem 15.3 *If* $n \geq 8$ *and* $T \in \mathcal{I}_{n-1}(\mathbb{R}^n)$ *is mass minimizing, then* $\operatorname{Sing}(T)$ *is* $(n-8)$*-rectifiable.*

The proof has similar ingredients as the proof of Simon's Theorem 15.6 for harmonic maps, which we shall discuss soon. In particular, the monotonicity formula, tangent cones and the general form of Theorem 4.22 play decisive roles.

De Lellis, Hirsch, Marchese, Spolaor and Stuvard thoroughly investigated the rectifiability and more detailed structure of the singular sets of mass minimizing currents mod p. See [163] and the references given there.

Much less is known about the singular sets of stationary m-varifolds, not even if they have $\mathcal{H}^m$ measure zero. The best known result is still due to Allard

[7] saying that the regular set is an open dense subset of the support. However, the stratification can be used to obtain relevant information about the singularities, for both currents and varifolds. Let v be an integer multiplicity rectifiable m-varifold in $\mathbb{R}^n$. A varifold cone C is said to be k-symmetric if there exists $V \in G(n,k)$ such that C is $V \times C'$ for some cone C'. Then the kth stratum of v is

$$S^k(v) = \{x \in \operatorname{spt} v \colon \text{no tangent cone of } v \text{ at } x \text{ is } (k+1)\text{-symmetric}\}.$$

So $S^0(v) \subset \cdots \subset S^{m-1}(v) \subset \mathrm{Sing}(v)$. Almgren [10] had proved in the 1980s that $\dim S^k(v) \leq k$. Naber and Valtorta [359] proved much more:

Theorem 15.4 *Let v be an integer multiplicity rectifiable m-varifold in $\mathbb{R}^n$ whose first variation is a Radon measure. Then for $k = 0, \dots, m-1$, $S^k(v)$ is k-rectifiable. Moreover, for $\mathcal{H}^k$ almost all $x \in S^k(v)$ there exists a unique $V \in G(n,k)$ such that every tangent cone of v at x is of the form $V \times C$ for some cone C.*

Note that the last statement does not mean that the tangent cones at x would be unique.

For mass minimizing integer multiplicity rectifiable $(n-1)$-currents T in $\mathbb{R}^n$, Simon proved, in addition to Theorem 15.3, that the whole singular set agrees with the top stratum: $\mathrm{Sing}(T) = S^{n-8}(T)$. We shall see other results like this later in this chapter. For mass minimizing currents of codimension bigger than 1, $S^{m-1}(T)$ need not be the whole singular set. For example, for $\left\{(w,z) \in \mathbb{C}^2 \colon w^2 = z^3\right\}$ the origin is a singular point but there is a unique 2-plane tangent cone. However, with multiplicity $2 > 1$, which is the reason why this can happen.

As for harmonic maps (see below) a key feature in Naber and Valtorta's method is to consider quantitative stratifications; no tangent cone in a ball is ε close to a symmetric cone. The proof is rather similar to that of Theorem 15.7. Their method also gives a new proof for Theorem 15.3.

If the Euclidean metric is perturbed to a suitable C^∞ metric d, the singular sets can be quite wild, as shown by Simon in [401] and [402]: if $N \geq 7, l \geq 1$ and $K \subset \mathbb{R}^l$ is compact, he then constructed minimal hypersurfaces in $\left(\mathbb{R}^{N+1+l}, d\right)$ with singular set $\{0\} \times K$. The very complicated and technical proof is mainly based on PDE methods; singular solutions of the symmetric minimal surface equation, see [208], are the building blocks of the construction.

15.2 Energy Minimizing Maps

Let Ω be an open subset of $\mathbb{R}^n$ and N a Riemannian submanifold of $\mathbb{R}^p$.

Definition 15.5 A map $u\colon \Omega \to N$ in the Sobolev space $W^{1,2}(\Omega, N)$ (that is, $u \in W^{1,2}(\Omega, \mathbb{R}^p)$ and $u(x) \in N$ for almost all $x \in \Omega$) is *energy minimizing* if for every ball $B(x,r) \subset \Omega$,

$$\int_{B(x,r)} |Du|^2 \leq \int_{B(x,r)} |Dw|^2 \tag{15.1}$$

for all $w \in W^{1,2}(\Omega, N)$ such that $w = u$ in a neighbourhood of $\partial B(x,r)$.

The *regular set*, $\mathrm{Reg}(u)$, of u is the set of points $x \in \Omega$ such that u is C^∞ in some neighbourhood of x. The *singular set*, $\mathrm{Sing}(u)$, of u is $\Omega \setminus \mathrm{Reg}(u)$.

Then $x \in \mathrm{Reg}(u)$ if u is continuous in some neighbourhood of x, cf. [393, p. 309].

Energy minimizing maps satisfy a Laplace-type equation generated by the restriction that the target is N. If $n \leq 2$, these maps have no singularities, but already when $n = 3$ and N is a two-sphere they occur: $u\colon \mathbb{R}^3 \to S^2, u(x) = x/|x|$ is energy minimizing and it has a singularity at the origin. Then $u_n\colon \mathbb{R}^n \to S^2, n > 3, u_n(x) = u(x_1, x_2, x_3)$ is energy minimizing in $\mathbb{R}^n$ with an $(n-3)$-dimensional singular set. This is sharp: Schoen and Uhlenbeck [393] showed that the Hausdorff dimension of $\mathrm{Sing}(u)$ is at most $n-3$. Simon proved in [398], see also [400], the following much stronger result:

Theorem 15.6 *If $n \geq 3$, N is real analytic and $u\colon \Omega \to N$ is energy minimizing, then* $\mathrm{Sing}(u)$ *is $(n-3)$-rectifiable.*

The proof is ingenious and very complicated. We try to give some flavour of it. The book [400] gives a detailed exposition, also of the background material. In the end, the rectifiability is obtained by showing that essential parts of the singular set satisfy the assumptions of Theorem 4.22, that is, the ε approximation by $(n-3)$-planes and the gap condition. Or rather, the assumptions of a generalization of Theorem 4.22 where additional exceptional sets are allowed. But it is a long way to get there.

Let $u\colon \Omega \to N$ be energy minimizing. For much of what is said below the real analyticity of N is not needed and Simon proved some partial results in the general case. There is some PDE theory involved, which, in particular, gives that there is $\varepsilon(n,N) > 0$ such that if $u\colon \Omega \to N$ is energy minimizing, $B(x,r) \subset \Omega$ and $r^{2-n} \int_{B(x,r)} |Du|^2 < \varepsilon(n,N)$, then $x \in \mathrm{Reg}(u)$. This gives almost immediately that the singular set of u has locally finite $\mathcal{H}^{n-2}$ measure and with a little more work one finds that $\mathcal{H}^{n-2}(\mathrm{Sing}(u)) = 0$. But the bound $n-3$ requires further effort.

Two basic tools are analogous to those we met in Allard's rectifiability theorem for varifolds: the monotonicity formula and tangent maps. The monotonicity formula now takes the form

$$s^{2-n}\int_{B(x,s)}|Du|^2 - r^{2-n}\int_{B(x,r)}|Du|^2 = 2\int_{B(x,s)\setminus B(x,r)}|y-x|^{2-n}|\partial_{\frac{y-x}{|y-x|}}u(y)|^2\,dy \tag{15.2}$$

when $0 < r < s$ and $B(x,s) \subset \Omega$. In particular, the density

$$\Theta_u(x) = \lim_{r\to 0} r^{2-n}\int_{B(x,r)}|Du|^2 \tag{15.3}$$

exists. The proof of the monotonicity formula is based on the variational equation which u satisfies.

A useful property of the density is that it is upper semicontinuous. We have also that $x \in \mathrm{Reg}(u)$ if and only if $\Theta_u(x) = 0$. One direction follows from the above $\varepsilon(n,N)$-property and the other is trivial.

Set $u_{x,r}(y) = u(x+ry)$ when $x, x+ry \in \Omega$. A map $\varphi\colon \mathbb{R}^n \to N$ is called a *tangent map* of u at x if there is a sequence $r_i > 0$ tending to 0 such that $u_{x,r_i} \to \varphi$ locally in $W^{1,2}(\mathbb{R}^n)$. The boundedness of the density ratios, due to the monotonicity formula, and a compactness theorem yield that tangent maps always exist. White showed in [437] that in general they need not be unique, but it is an open question whether they are unique when the target N is real analytic. See [346] for a discussion on this and related issues.

The tangent maps are energy minimizing and they have other very useful properties. First,

$$\Theta_u(x) = \Theta_\varphi(0) = r^{2-n}\int_{B(0,r)}|D\varphi|^2 \text{ for all } r > 0.$$

From this, one concludes with the monotonicity formula for φ (since the left, and hence also the right, hand side now is 0) that

$$\varphi(\lambda x) = \varphi(x) \text{ for all } x \in \mathbb{R}^n, \lambda > 0, \tag{15.4}$$

and that $x \in \mathrm{Reg}(u)$ if and only if there is a constant tangent map at x. A little more calculus gives that $\Theta_\varphi(x) \le \Theta_\varphi(0)$ for all $x \in \mathbb{R}^n$ and that

$$S(\varphi) := \left\{x \in \mathbb{R}^n : \Theta_\varphi(x) = \Theta_\varphi(0)\right\}$$

is a linear subspace of $\mathbb{R}^n$ such that $\varphi(x+y) = \varphi(x)$ for $x \in \mathbb{R}^n, y \in S(\varphi)$. Moreover, $S(\varphi) = \mathbb{R}^n$ if and only if φ is constant; otherwise, $S(\varphi) \subset \mathrm{Sing}(\varphi)$.

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As $\mathcal{H}^{n-2}(\mathrm{Sing}(\varphi)) = 0$, we have $\dim S(\varphi) \le n-3$ for any non-constant tangent map φ.

Let $x \in \mathrm{Sing}(u)$ and $\delta > 0$. Then there is $0 < \varepsilon < \Theta_u(x)$ such that for every $0 < r < \varepsilon$ there is an $(n-3)$-dimensional affine plane V for which

$$\left\{y \in B(x,r) \colon \Theta_u(y) \ge \Theta_u(x) - \varepsilon\right\} \subset \left\{y \colon d(y,V) \le \delta r\right\}. \tag{15.5}$$

Recall that for any tangent map φ the set $S(\varphi)$ is contained in an $(n-3)$-plane. Inclusion (15.5) is not immediate from this, but not very difficult either. The upper semicontinuity of the density plays a role here.

Inclusion (15.5) gives an approximation of the singular set with $(n-3)$-planes and one can deduce from it that $\dim \mathrm{Sing}(u) \le n-3$. But to conclude rectifiability with something like Theorem 4.22 we would still need the gap condition. That is the hardest part of the proof. Roughly speaking (very roughly), suppose we start with (15.5) at some x at some scale r such that there are no gaps in a range of scales below r. Then many technical integral estimates on the derivatives of u imply that at these scales u is L^2 close to a map φ as in (15.4), which gives the required approximation at smaller scales with planes parallel to V.

Naber and Valtorta proved in [356] with different methods further deep results on the structure of the singular sets, recall their similar results for varifolds from the previous section. Define

$$S^k(u) = \left\{x \in \mathrm{Sing}(u) \colon \ \dim S(\varphi) \le k \text{ for all tangent maps } \varphi \text{ of } u \text{ at } x\right\}.$$

Then we have the stratification of the singular set

$$S^0(u) \subset S^1(u) \subset \cdots \subset S^{n-3}(u) = S^{n-2}(u) = S^{n-1}(u) = \mathrm{Sing}(u).$$

The equalities follow from the fact, which is essentially mentioned above, that $\dim S(\varphi) \le n-3$ if and only if $x \in \mathrm{Sing}(u)$.

By what was said about $S(\varphi)$, $x \in S^k(u)$ means that no tangent map φ of u at x is $(k+1)$-symmetric. A tangent map φ is k-symmetric if there is $V(\varphi) \in G(n,k)$ such that $\varphi(x+y) = \varphi(x)$ for $x \in \mathbb{R}^n$, $y \in V(\varphi)$.

Schoen and Uhlenbck proved in [393] that $\dim S^k(u) \le k$; similar arguments that gave $\dim \mathrm{Sing}(u) \le n-3$ apply. Naber and Valtorta proved in [356], without assuming real analyticity,

Theorem 15.7 *If $u \colon \Omega \to N$ is energy minimizing, then $S^k(u)$ is k-rectifiable for $k = 0, 1, \ldots, n-3$.*

This gives a new proof for Simon's Theorem 15.6. They also prove the result more generally for stationary maps, and in [357] for a larger class of approximate harmonic maps. The latter paper has some simplifications of the proof

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of Theorem 15.7. In addition, these papers contain many other significant results. A key feature in their method is to consider quantitative stratifications; no tangent map in a ball is ε close to a symmetric map. For these, they proved volume estimates for the r-neighbourhoods of $S^k(u)$ which in particular yield the bound k for the Minkowski dimension. They also showed that the plane $V(\varphi)$, mentioned above, is independent of φ.

The density ratios are again one of the key factors. Setting $\Theta_u(x,r) = r^{2-n}\int_{B(x,r)}|Du|^2$, it is shown for any finite Borel measure μ that if u is not ε close to any $(k+1)$-symmetric map in $B(0,8)$, then

$$\inf_{V\ k-\text{plane}} \int d(x,V)^2\,d\mu x \lesssim \int (\Theta_u(x,8) - \Theta_u(x,1))\,d\mu x. \tag{15.6}$$

Scaled versions of this are applied with μ a discrete approximation of the Hausdorff measure. They give volume estimates. Based on (15.6), rectifiability is derived from Reifenberg-type theorems, recall Section 4.7 and Theorem 6.4. The proofs involve complicated covering and induction arguments.

Defect measures provide one of the tools: by compactness, any bounded sequence (u_i) in $W^{1,2}$ has a subsequence (u_{i_j}) converging weakly in $W^{1,2}$ to u such that $|\nabla u_{i_j}|^2$ converges weakly to a measure $|\nabla u|^2 + \nu$.

Theorem 15.8 *If (u_i) is a sequence of stationary maps in $\mathbb{R}^n$ converging weakly in $W^{1,2}$ to u and $|\nabla u_i|^2$ converges weakly to $|\nabla u|^2 + \nu$, then u is stationary and the defect measure ν is $(n-2)$-rectifiable.*

This was proved by Lin in [295], see also [296]. The methods have some similar ingredients as those in Sections 4.3 and 4.4, but Lin gives independent proofs relying on facts at hand.

De Lellis, Marchese, Spadaro and Valtorta [164] and Hirsch, Stuvard and Valtorta [234] proved for Almgren's multivalued functions results analogous to Theorem 15.6. Alper proved in [11] that the zero sets of harmonic maps from three-dimensional domains into a cone over the real projective plane are 1-rectifiable. In [12], he showed that the singular set of the free interface in an optimal partition problem for the first Dirichlet eigenvalue in $\mathbb{R}^n$ is $(n-2)$-rectifiable.

15.3 Mean Curvature Flow

A one-parameter family $\{M_t, t \geq 0\}$ of compact surfaces moves by mean curvature if the normal velocity of M_t equals the mean curvature vector at each

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point of M_t. We first discuss the case where the initial surface is a smooth hypersurface M_0 in $\mathbb{R}^n$. Then the flow is governed by a heat equation-type partial differential equation: if $F\colon M_0 \times [0, T] \to \mathbb{R}^n$ parametrizes part of the motion, $M_t = F(M_0 \times \{t\})$, then $\partial_t F = H$. Here H is the mean curvature vector of M_t, which also is a Laplacian of F in the metric of M_t. Then the area of M_t is decreasing and singularities will appear. For instance, a sphere shrinks into a point and a cylinder into a line. See, for example, the survey [119] of Colding, Minicozzi and Pedersen and the book [304] of Mantegazza for many interesting phenomena. Here we only briefly discuss the result of Colding and Minicozzi [118] on the rectifiability of singularities.

A point (x, s), $x \in M_s$, $s > 0$, is a singular point of the flow (M_t) if $\{(y, t), y \in M_t, t > 0\}$ is not a smooth manifold in any neighbourhood of (x, s). Let $S \subset \mathbb{R}^n \times [0, \infty)$ be the set of singular points. The flow (M_t) is called mean convex if M_0, and hence every M_t, has non-negative mean curvature. White [440] showed that for them the singular set has parabolic (and so also Euclidean) Hausdorff dimension at most $n - 2$. Colding and Minicozzi proved for such, and more general, flows that the singular set is $(n - 2)$-rectifiable. In fact they proved much more:

Theorem 15.9 *If the flow (M_t) is mean convex, then the singular set S can be covered with finitely many bi-Lipschitz images of subsets of $\mathbb{R}^{n-2}$ together with a set of Hausdorff dimension $n - 3$.*

The paper [118] contains many other facts about the structure of S. For example, the bi-Lipschitz images can be taken to be Lipschitz graphs with respect to the parabolic distance on $\mathbb{R}^n \times \mathbb{R}$, so S is parabolic rectifiable, recall Section 5.6. Again, also the rectifiability of the stratification of S is established.

The proof uses similar tools that we have seen above: monotonicity formula, tangent flows and a parabolic Reifenberg theorem. The monotonicity formula is due to Huisken [243]; $t^{(1-n)/2} \int_{M_t} e^{-|y-x|^2/t}\, d\mathcal{H}^{n-1} y$ is non-decreasing. The tangent flows of (M_t) are the weak limits of $\delta_i^{-1} M_{\delta_i^2 t}$, $\delta_i \to 0$. By an earlier result of Colding and Minicozzi [117], the tangent flows are unique for the mean convex flows, which is crucial in the proof of Theorem 15.9. In full generality the uniqueness of tangent flows is open. The parabolic Reifenberg theorem is now rather simple because the approximating plane is assumed to be the same at all small scales. That this suffices is due to the strong information coming from the uniqueness of the tangent flows.

The tangent flows are not only unique but also cylindrical, that is, of the form $\mathbb{R}^k \times S^{n-k}$. The uniqueness means that k and the direction of the $\mathbb{R}^k$ factor are unique. In fact, Colding and Minicozzi proved their results for all motions that have only cylindrical singularities. For $k < n$, the kth stratum S_k consists of those points of S where the Euclidean factor has dimension at most k. Then $S_0 \subset S_1 \subset \cdots \subset S_{n-2} = S$ and, by [118], each S_k is k-rectifiable.

Much of the basics of the theory with smooth initial surfaces was established only in the 1980s and later. Surprisingly early, in the 1970s, Brakke [80] developed a very general theory with rectifiable m-varifolds $v_t, 0 < m < n$. There are at least two good reasons to do this. Quite often the classical solutions develop singularities and the evolution in the classical sense stops. For varifolds, singularities are allowed and the flow exists for all $t > 0$. In many applications singularities are unavoidable. Brakke himself applied his flow to grain boundaries. For many further developments with a number of variants, connections and applications, see, for example, [422], [233] and the references given there.

In this general case, the equation for the flow is more complicated and in fact an inequality rather than an equality is needed. This is necessary to have useful compactness. As mentioned in Section 13.3, the first variation of a varifold leads to a concept of mean curvature. One also needs a weak formulation of the velocity. The first variations δ_{v_t} are assumed to be Radon measures and the curvatures are assumed to be in $L^2(\mu_{v_t})$. I skip the precise definitions.

The construction of Brakke's flow is a highly non-trivial matter. Brakke used a complicated approximation procedure. Ilmanen [250] showed that sequences of energy densities of the Allen–Cahn equation converge to rectifiable measures leading to a Brakke flow. In [251] he used elliptic regularization. For other methods, also with prescribed boundary conditions, see [233, 386, 405, 422, 441] and their references.

15.4 Gromov–Hausdorff Limits and Related Matters

This section deals with one small but important part of the theory of Riemannian manifolds with lower bounded Ricci curvature: the structure of Gromov–Hausdorff limits of such manifolds. Since Gromov's fundamental work in the 1980s, see, for example, [228], this topic has been intensively studied by many authors. Below we shall briefly discuss only matters directly related to rectifiability. For many other aspects, see, for example, the introductions of [262] and [93] and the survey [355]. There is a lot of similarity in results and some similarity in methods with those for harmonic maps discussed in Section 15.2.

Recall the definitions related to Gromov–Hausdorff convergence from Section 7.7. In addition, a *tangent cone* of a metric space X at a point $x \in X$ is any pointed limit $(X_\infty, d_\infty, x_\infty)$ of $(X, r_i^{-i}d, x)$ where $r_i \to 0$.

Let (M_i) be a sequence of n-dimensional Riemannian manifolds and $m_i \in M_i$. We shall always assume that the Ricci curvatures and the volumes of the

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unit balls are bounded below:

$$\mathrm{Ric}_{M_i} \geq -(n-1) \ \text{ and } \mathcal{H}^n(B(m_i,1)) > c > 0 \text{ for all } i. \tag{15.7}$$

Let (X, d, x) be a pointed limit of (M_i, m_i). Then, under the above conditions on M_i, tangent cones Y exist at every point of X by a compactness theorem of Gromov and they are metric cones, $Y = C(W)$, by a result of Cheeger and Colding [91]. By definition, the metric cone $C(W)$ over a metric space W is the completion of $(0, \infty) \times W$ with a particular metric, see [92]. They are also metric measure spaces where the measure is a limit of volume measures. A point $x \in X$ is *regular* if every tangent cone at x is isometric to $\mathbb{R}^n$. Otherwise, x is *singular*. We denote the set of singular points by $\mathrm{Sing}(X)$.

Colding and Naber [120] showed that the tangent cones need not be unique under the above assumptions.

Cheeger and Colding proved in [91] that $\dim \mathrm{Sing}(X) \leq n-2$. Assuming the two-sided bound $|\,\mathrm{Ric}_{M_i}| \leq n-1$, Cheeger and Naber proved in [94] that $\dim \mathrm{Sing}(X) \leq n-4$. But we have more:

Theorem 15.10 *Let* $n \geq 4$. *Then* $\mathrm{Sing}(X)$ *is* $(n-4)$*-rectifiable if* $|\,\mathrm{Ric}_{M_i}| \leq n-1$.

This was proved by Jiang and Naber in [262]. The proof is quite involved with many different kinds of techniques. Again monotonicity formulas play a decisive role, as they do for many other results mentioned below. Neck decompositions, recall Section 6.2, are central. Now instead of (15.6) the L^2 integral of curvature is dominated by monotonic entities.

Cheeger and Colding introduced in [91] the stratification of the singular set. Let $S^k \subset \mathrm{Sing}(X)$ be the set of points $x \in X$ for which no tangent cone at x is of the form $C(W) \times \mathbb{R}^{k+1}$. Then

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} = \mathrm{Sing}(X),$$

and, by [91], $\dim S^k \leq k$, $S^{n-2} = \mathrm{Sing}(X)$, and, by [94], $S^{n-4} = \mathrm{Sing}(X)$ if the Ricci curvatures are also bounded above; $|\,\mathrm{Ric}_{M_i}| \leq (n-1)$. Cheeger, Jiang and Naber proved the rectifiability in [93]:

Theorem 15.11 S^k *is* k*-rectifiable for all* $k = 0, 1, \ldots, n-2$. *In particular,* $\mathrm{Sing}(X)$ *is* $(n-2)$*-rectifiable.*

As for harmonic maps, quantitative stratification is considered in [93] and quantitative volume estimates are obtained for them. The proof again involves monotonicity and neck decompositions.

With only the lower bound for the Ricci curvature, rectifiability is essentially the best one can say; the singular sets can be Cantor sets, see [93, 292].

Recall from the end of Section 15.2 Lin's results on defect measures related to sequences of stationary harmonic maps. Similar defect measures occur for sequences of connections and their Yang–Mills energies on n-dimensional manifolds. Tian proved their $(n-4)$-rectifiability in [407]. See also [358] for further work involving neck-type decompositions.

There are also many rectifiability results on metric measure spaces. I don't go into any details here; I just mention some of them. Li and Naber proved in [292] results analogous to the above for Alexandrov spaces with curvature bounded below. A special case of the results of Mondino and Naber [349] shows that metric measure spaces $(X, d, \mathcal{H}^n)$ with lower bounds for the Ricci curvature are n-rectifiable, see also [84] for another proof. Brué, Naber and Semola showed in [82] that their boundaries are $(n-1)$-rectifiable. The boundary of X is the closure of $S^{n-1} \setminus S^{n-2}$, which in this setting need not be empty. Here S^k is a stratum as above. Brué, Pasqualetto and Semola [83] developed De Giorgi's theory for sets of finite perimeter, including their rectifiability, in these and more general spaces. Lee, Naber and Neumayer introduced in [287] rectifiable Riemann spaces to deal with some problems in Gromov–Hausdorff convergence. They are topological measure spaces, not necessarily metric. David [152] showed that the tangents of AD-regular Lipschitz differentiability spaces, recall Section 7.5, are uniformly rectifiable provided the space has charts of maximal dimension. Otherwise they are purely unrectifiable.

15.5 Measure Solutions of PDEs

Consider the system of constant coefficient linear partial differential equations on $\mathbb{R}^n$,

$$\mathcal{A}u = \sum_{|\alpha| \le k} A_\alpha \partial^\alpha u = 0, \ u \colon \mathbb{R}^n \to \mathbb{R}^l,$$

where $\partial^\alpha = \partial_{\alpha_1} \ldots \partial_{\alpha_n}, |\alpha| = \max\{\alpha_j \colon j = 1, \ldots, n\}$ for the multi-index $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and the $A_\alpha \colon \mathbb{R}^l \to \mathbb{R}^p$ are linear maps. An $\mathbb{R}^l$-valued Radon measure on $\mathbb{R}^n$, $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^l)$, is said to be *$\mathcal{A}$-free* if $\mathcal{A}\mu = 0$ in the distributional sense.

Examples of curl-free measures are the derivatives Du of BV-maps $u \colon \mathbb{R}^n \to \mathbb{R}^l$, see below.

By the Radon–Nikodym theorem any $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^l)$ can be written as $\mu = \mu_a + \mu_s = \mu_a + D(\mu, |\mu|)|\mu|_s$, where μ_a is absolutely continuous with respect to $\mathcal{L}^n$, $D(\mu, |\mu|)$ is the Radon–Nikodym derivative of μ with respect to the

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total variation measure $|\mu|$ and $|\mu|_s$ is the singular part of $|\mu|$ in its Lebesgue decomposition.

The structure of the singular parts μ_s is a much studied question with many applications, see [29, 175] and the ICM survey [176]. The wave cone

$$\Lambda_{\mathcal{A}} = \bigcup_{\xi\in\mathbb{R}^n\setminus\{0\}} \ker A^k(\xi) \subset \mathbb{R}^l, \text{ where } A^k(\xi) = \sum_{|\alpha|=k} \xi^\alpha A_\alpha,\ \xi^\alpha = \xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$$

is central in this investigation, as well as in many other topics, see [175, 176] and the references given there. We shall look at some examples below.

If $\mathcal{A}$ is homogeneous, $\mathcal{A} = \sum_{|\alpha|=k} A_\alpha\partial^\alpha$, then $\lambda \in \mathbb{R}^l$ belongs to $\Lambda_{\mathcal{A}}$ if and only if there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $x \mapsto \lambda h(x\cdot\xi)$ is $\mathcal{A}$-free for all smooth functions $h\colon \mathbb{R} \to \mathbb{R}$. That is, in the words of De Philippis and Rindler in [176]: 'Roughly speaking, $\Lambda_{\mathcal{A}}$ contains all the amplitudes along which the system is not elliptic' and "'one-dimensional" oscillations and concentrations are possible only if the amplitude (direction) belongs to the wave cone'.

The following theorem of De Philippis and Rindler gives a new proof of Alberti's rank one Theorem 12.15 and extends it to BD-maps. It has many other consequences too, see [175].

Theorem 15.12 *If $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^l)$ is $\mathcal{A}$-free, then $D(\mu, |\mu|)(x) \in \Lambda_{\mathcal{A}}$ for $|\mu|_s$ almost all $x \in \mathbb{R}^n$.*

The proof uses tangent measures and pseudo-differential calculus. Partially based on similar ideas, Arroyo-Rabasa, De Philippis, Hirsch and Rindler proved in [29] several interesting rectifiability results. They are formulated in terms of m-dimensional wave cones

$$\Lambda_{\mathcal{A}}^m = \bigcap_{V\in G(n,m)} \bigcup_{\xi\in V\setminus\{0\}} \ker A^k(\xi).$$

Clearly, these sets increase when m increases.

Examples (1) The divergence operator on $\mathbb{R}^n$ is $\mathcal{A} = \sum_{i=1}^n A_i\partial_i$, where $A_i x = e_i\cdot x$, with e_i the standard basis vectors. Then $A^1(\xi)x = \xi\cdot x$, so $\Lambda_{\mathcal{A}} = \Lambda_{\mathcal{A}}^m = \mathbb{R}^n$ for $2 \le m \le n$ and $\Lambda_{\mathcal{A}}^1 = \{0\}$.

(2) The curl is defined for $m\times n$ – matrix-valued measures μ on $\mathbb{R}^n$ by

$$\operatorname{curl}\mu = (\partial_i\mu_j^k - \partial_j\mu_i^k), i, j = 1,\dots,n, k = 1,\dots,m.$$

This can be written in the form $\operatorname{curl}\mu = \sum_{i=1}^n A_i\partial_i\mu$ for some A_i. Further, it can be checked that for every $\xi \in \mathbb{R}^n\setminus\{0\}$ the kernel of $\operatorname{curl}(\xi)$ consists of the matrices $a\otimes\xi,\ a \in \mathbb{R}^m$, see [207], Remark 3.3(iii), for these facts. It follows that $\Lambda_{\mathrm{curl}}^{n-1} = \{0\}$.

(3) Also the second-order operator curl curl on $n\times n$ – matrix-valued measures is of the above type with $\Lambda_{\mathrm{curl\,curl}}^{n-1} = \{0\}$.

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Recall from Section 4.5 the integral-geometric measure $\mathcal{I}_1^m$, whose null-sets are those that project to measure zero on almost all m-planes. By [29]

Theorem 15.13 *Let $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^l)$ be $\mathcal{A}$-free and let $E \subset \mathbb{R}^n$ be a Borel set with $\mathcal{I}_1^m(E) = 0$. Then $D(\mu, |\mu|)(x) \in \Lambda_{\mathcal{A}}^m$ for $|\mu|$ almost all $x \in E$. In particular, if $\Lambda_{\mathcal{A}}^m = \{0\}$, then $|\mu|(E) = 0$, whence $|\mu| \ll \mathcal{I}_1^m \ll \mathcal{H}^m$.*

Notice first that if $m = n$, and so $\mathcal{I}_1^m = \mathcal{L}^m$, this is the same as Theorem 15.12. In particular, if for $|\mu|$ almost all $x \in \mathbb{R}^m$, $A^k(\xi)(D(\mu, |\mu|)(x)) \neq 0$ for $\xi \in \mathbb{R}^m \setminus \{0\}$, then $|\mu|$ is absolutely continuous.

The proof of Theorem 15.13 is by contradiction; the following sketch is rather imprecise and very incomplete. The counter-assumption leads to a subset F of E with $|\mu|(F) > 0$, a point $x \in F$ and a plane $V \in G(n, m)$ such that $\mathcal{H}^m(P_V(F)) = 0$ and

$$A^k(\xi)\Big(D(\mu, |\mu|)(x)\Big) \neq 0 \text{ for } \xi \in V \setminus \{0\}. \tag{15.8}$$

Moreover, x can be chosen so that for a sequence $\mu_j \in \mathcal{M}(B(0,1))$ of normalized blow-ups at x of $\mu \llcorner F$ the total variations $|\mu_j|$ converge to a non-zero measure $\sigma \in \mathcal{M}(B(0,1))$. Setting $\nu_j = P_{V\#}(\mu_j)$ and $F_j = \operatorname{spt}|\nu_j|$, one has $\mathcal{H}^m(F_j) = 0$. By delicate analysis based on (15.8), there is $\theta \in L^1(V \cap B(0,1))$ such that $\lim_{j\to\infty} \||\nu_j| - \theta\mathcal{H}^m|(V \cap B(0,1)) = 0$. This leads to the contradiction

$$\begin{aligned} 0 &< \sigma(B(0,1)) \leq \liminf_{j\to\infty} |\nu_j|(B(0,1)) \\ &\leq \liminf_{j\to\infty} \left(\int_{F_j} \theta \, d\mathcal{H}^m + \||\nu_j| - \theta\mathcal{H}^m|\Big(F_j \cap B(0,1)\Big) \right) = 0. \end{aligned}$$

As a corollary to Theorem 15.13, we have

Theorem 15.14 *Let $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^l)$ be $\mathcal{A}$-free and suppose that $\Lambda_{\mathcal{A}}^m = \{0\}$. Then $\Theta^{*m}(|\mu|, x) < \infty$ for $|\mu|$ almost all $x \in \mathbb{R}^n$ and $|\mu| \llcorner \{x \in \mathbb{R}^n : \Theta^{*m}(|\mu|, x) > 0\}$ is m-rectifiable.*

By Theorem 1.3 and the last statement of Theorem 15.13, the upper density is finite $|\mu|$ almost everywhere. The set A, where $0 < \Theta^{*m}(|\mu|, x) < \infty$, has σ-finite $\mathcal{H}^m$ measure, and $|\mu|$ and $\mathcal{H}^m$ are mutually absolutely continuous on A by Theorem 1.3. Hence the rectifiability follows from Theorem 15.13 and the Besicovitch–Federer projection Theorem 4.17.

Here are some of the applications. Let $u = (u^1, \dots, u^l) \in BV(\mathbb{R}^n, \mathbb{R}^l)$. Then $\mu = Du = (\mu_i^k) = (\partial_i u^k), i = 1, \dots, n, k = 1, \dots, l$, is curl-free. By example 2 above, $\Lambda_{\text{curl}}^{n-1} = \{0\}$ and one can use Theorem 15.14 to get a new proof for the fact that $|Du| \llcorner \{x : \Theta^{*m}(|Du|, x) > 0\}$ is $(n-1)$-rectifiable, recall Theorem 12.14. In the same way, example 3 yields Theorem 12.17 for BD-maps.

As another application of Theorem 15.14, the authors of [29] gave a new proof and extensions of Allard's rectifiability Theorem 13.10 and the related results of [20] and [173]. Then $\mathscr{A}$ is a divergence operator on matrix-valued measures. Lin's defect measure Theorem 15.8 also follows from Theorem 15.14.

15.6 A Free Boundary Problem

Recall from Section 14.3 the work of David, Engelstein and Toro [141]. Edelen and Engelstein [183] studied the analogous one-phase problem. Let q be positive and Hölder continuous in Ω. Minimize

$$J(u) = \int_\Omega (|\nabla u(x)|^2 + q(x)^2 \chi_{u>0}(x))\, dx$$

among $u \in W^{1,2}(\Omega)$ with given non-negative boundary values. This problem has an interesting history starting from Alt and Caffarelli in 1981 and there are similarities to codimension 1 minimal surface theory, see [183]. In particular, let k^* be the smallest k such that the above problem admits a non-linear, one-homogeneous solution with $\Omega = \mathbb{R}^k, q = 1$. Then k^* is 5, 6 or 7, but otherwise the value is unknown. By a result of Weiss, we have for a minimizer u, $\dim \mathrm{Sing}(u) \leq n - k^*$, meaning that $\mathrm{Sing}(u) = \emptyset$, if $n < k^*$. The singular set now is the subset of $\partial\{u > 0\} \cap \Omega$, where u fails to be $C^{1,\alpha}$ for some $\alpha > 0$. Edelen and Engelstein proved

Theorem 15.15 *If u is a minimizer for J, then* $\mathrm{Sing}(u)$ *is* $(n - k^*)$*-rectifiable.*

They also considered stratification in the spirit we have seen before and they proved the rectifiability of each stratum. Their methods are influenced by those of Naber and Valtorta [356]. They also apply to the two-phase problem and to almost minimizers.

For further work along these lines by De Philippis, Engelstein, Spolaor and Velichkov, see [174].