CONSTRUCTION OF PLANETARY THEORY BY ITERATIVE PROCEDURE

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In this paper the method of determination of the planetary perturbations is proposed which is a modification of Dziobek-Brouwer's method [1,2]. For the simplicity the case of two mutually disturbing planets is considered. In the original version of the method the perturbations of rectangular planetary coordinates are presented by means of the formal integrals

$$\delta X_{ik} = \int \Big( \sum_{j=1}^{3} a_{ikj} \; G_{ij} \Big) dt \;+\; c_{ij} \iint \Big( \sum_{j=1}^{2} b_{ij} \; G_{ij} \Big) dt dt, \qquad i = 1,2; \quad k = 1,2,3 \tag{1}$$

where index i corresponds to the number of the planet; $\delta X_{ik}$ are perbations of $X_{ik}$ coordinates; $G_{ij}$ - components of the perturbating accelerations. The coefficients of $a_{ikj}$, $c_{ik}$, $b_{ij}$ are the well-known functions of the coordinates of the elliptic motion which can be developed as double Fourier series in mean longitudes. The denominators in Davis' formulas [3] for these coefficients contain the eccentricities. For this reason Musen [4] expressed an opinion that Brouwer's method would lose its effectiveness when small eccentricities are involved. These fictitious peculiarities are eliminated in the present paper by means of trivial transformations and the expressions for the coefficients are given in a simple symmetric form.

According to Brouwer's method for the determination of the mutual perturbations the equations of the motion for any planet are written down in a special coordinate system. Therefore the combined integration of the whole set of equations becomes difficult. The unified frame of reference connected with the ecliptic is taken now for all the planets under consideration and the formal integrals (1) are presented in the following vectorial form

$$\delta \bar{r}_i = \int [A_i + A_i'(\tilde{t}-t)] \; \bar{G}_i dt, \quad i = 1,2 \, . \tag{2}$$



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Here index i still corresponds to the number of the disturbed planet; $\delta\bar{t}_i$ is the perturbation of the radius-vector of the i-th planet; $\tilde{t}=t$ but it is considered as a constant while integrating; $A_i$ and $A_i'$ are linear operations in three-dimensional vector space to which the square third-order matrixes composed of perturbating function coefficients in Brouwer's equations (1) are related.

Brouwer expands the components of the vectors $\bar{G}_i$ in six-argument Taylor series with respect to the perturbations of the coordinates of both planets which is usually made while integrating the differential equations of the motion. The corresponding form of the integrands in (1) is very complicated, new terms being not described by the uniform formulas and added at each next step of the approximation. The most principal distinction of the proposed modification consists in new expansions of the functions $\bar{G}_i$. The expressions for $\bar{G}_i$ are independent on the order of the perturbations and well suited for the computer calculations. These are presented as functions symmetric with respect to the undisturbed radii-vectors and their perturbations

$$\bar{G}_i = \sum_{\ell=i}^{3} \kappa_{i\ell}\,\bar{\sigma}_{i\ell} - \frac{3}{2}\kappa_{ii}\, r_i^{-5}\,\delta\bar{r}_i\,[2\bar{r}_i\delta\bar{r}_i + (\delta\bar{r}_i)^2] - \frac{3}{2}\kappa_{ii}\, r_i^{-5}\, r_i(\delta\bar{r}_i)^2 , \quad i = 1,2 . \tag{3}$$

Here

$$\bar{\sigma}_{i\ell} = (\bar{r}_\ell + \delta\bar{r}_\ell) \sum_{j=s_{i\ell}}^{\infty} d_j r_\ell^{-2j-3}\,[2\bar{r}_\ell\delta\bar{r}_\ell + (\delta\bar{r}_\ell)^2]^j ,$$

$$d_j = (-1)^j \frac{(\frac{3}{2})_j}{(1)_j}\,; \quad s_{ii} = 2,\ s_{ip} = 0 \quad \text{when } p = 1, 2, 3,\ p \neq i; \tag{4}$$

$$\kappa_{ii} = -k^2(1 + m_i),\ \kappa_{ip} = -k^2 m_p \quad \text{when } p = 1,2,\ p \neq i;$$

$$\kappa_{13} = k^2 m_2,\ \kappa_{23} = k^2 m_1 ,$$

where k is the Gaussian constant, $m_i$ means the mass of the i-th planet $r_3$ is the distance between the planets.

The integration in (2) is fulfilled by iterations

$$\delta\bar{r}_i^{(n+1)} = \int [A_i + A_i'(\tilde{t} - t)]\,\bar{G}_i^{(n)}\,dt \tag{5}$$

which allows more complete automatisation of the calculations, some terms of order higher than n with respect to the disturbed mass being taken into account at every n-th approximation. As the first approximation the keplerian ellipse or the results of any earlier elaborated theory may be used.

The main difficulty in the present method as well as in any perturbation theory is the expansion of the reciprocal mutual planetary distance $r_3^{-1}$ involved in the expansions of the functions $\bar{G}_i$. Lately Newton's iteration algorithm is widely used in developing $r_3^{-1}$

$$\left(\frac{a}{r_3}\right)_{n+1} = \frac{1}{2}\left(\frac{a}{r_3}\right)_n\left[3-\left(\frac{r_3}{a}\right)^2\left(\frac{a}{r_3}\right)_n^2\right] \qquad (6)$$

where n is the iteration number, a is a major semi-axis of the external planetary orbit.

Newton's method could not be used as yet to solve the problem of Neptune-Pluto type because the known ways of choosing the first approximation failed to guarantee the convergence of the process of iterations.

In this paper Newton's algorithm is also used. But as the first approximation for $\left(\frac{a}{r_3}\right)_o$ the quantity $\frac{a}{M}$ is accepted where M is the maximum distance between the orbits of bodies considered. As was theoretically shown by Dr. M.S. Petrovskaya (Celestial Mechanics, in print) such a choice of the initial approximation can ensure the iteration convergence for any form and configuration of planetary orbits. We followed the above method in expanding the value $\frac{a}{r_3}$ into double Fourier series in mean longitudes for the problems of Jupiter-Saturn and Neptune-Pluto types. The calculations were made by BESM6 computer. In the Jupiter-Saturn case 8 iterations were necessary to get the precision of $10^{-8}$ and 9 iterations to provide the accuracy as high as $10^{-9}$. The number of terms in series was 916 and 940 respectively. As follows from the calculations the iterational procedure for the Neptune-Pluto problem is also convergent, though the convergence being rather slow. Thus, at an intermediate stage of calculations 14 iterations proved to be necessary to expand $\frac{a}{r_3}$ with the precision of $10^{-4}$.

The proposed modification of Brouwer's method may be applied to solve the classical problems of celestial mechanics in the cases of small eccentricities and inclinations of orbits. It may be used as well for the study of planetary and cometary motions when the orbits of the bodies intersect in projection, in particular when the mutual perturbations in the Neptune-Pluto system are determined. The method is suitable for the calculations of high-order planetary perturbations since any order perturbations may be represented by the same algorithm.

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