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On the entropy in II_1 von Neumann algebras

O. BESSON
Département de Mathématiques, Ecole Polytechnique Fédérale, Lausanne, Switzerland

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Abstract. Let α be an automorphism of a finite von Neumann algebra and let $H(\alpha)$ be its Connes–Størmer's entropy. We show that $H(\alpha)=0$ if α is the induced automorphism on the crossed product of a Lebesgue space by a pure point spectrum transformation. We also show that H is not continuous in α and we compute $H(\alpha)$ for some α.

0. *Introduction*

Let M be a finite von Neumann algebra with separable pre-dual and with faithful normal normalized trace τ, and let θ be an automorphism of M preserving the trace τ. In [**4**] Connes & Størmer have defined a notion of *entropy* $H(\theta)$ of θ. This notion extends the classical entropy of Kolmogorov in the sense that, if $(X, \mathscr{B}, \mu)$ is a probability space and T is an automorphism of this space with entropy $h(T)$ and if we also denote by T the automorphism induced on the abelian algebra $A=L^\infty(X, \mu)$, then

$$H(T)=h(T).$$

However, the following important question is still open. Let M be the crossed product of A by T and θ be the inner automorphism of M induced by T. Is it true that $H(\theta)=h(T)$? Our main result is a partial answer (see theorem 1.9):

THEOREM. *If T is ergodic and has pure point spectrum, then $H(\theta)=0$, so $H(\theta)=h(T)$.*

One of the ingredients of our proof is the following result (see proposition 1.7): endow the group Aut M of automorphisms of M with the topology of pointwise convergence in M_*.

PROPOSITION. *Let G be a compact subgroup of* Aut M. *Then, for all $g\in G$, $H(g)=0$.*

The compactness of G is easily seen to be essential.

In the second part of this paper we prove the following, which is a generalization of a result of Abramov (see theorem 2.1).

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PROPOSITION. *Let R be the injective factor of type* II_1 *and let* $\alpha:\mathbb{R}\to\operatorname{Aut} R$ *be a continuous homomorphism. Then*

$$H(\alpha_t)=|t|H(\alpha_1)\quad \text{for all } t\in\mathbb{R}.$$

This proposition might lead one to believe that the entropy is continuous (as a map from Aut M to $\bar{\mathbb{R}}_+$). Indeed, Connes has asked whether this is true for the norm topology on Aut M. The answer is that it is never continuous when M is of type II_1 (see corollary 3.2).

PROPOSITION. *The map* $H:\operatorname{Aut} M\to\overline{\mathbb{R}_+}$ *is not continuous for the norm topology on* Aut M.

This proposition and its proof remain true for the new notion of entropy introduced in [**5**].

As in the classical case, the notion of entropy is an invariant which is far from complete. At the end of this paper we give an example of an uncountable family $(\theta_\lambda)_{\lambda\in\mathbb{R}_+}$ of automorphisms of the factor R which have zero entropy, are all aperiodic [**3**, p. 293] (and hence are all outer conjugate [**3**, theorem 2]) but are not pairwise conjugate.

Throughout this paper we shall use the notation of [**4**] for entropy and relative entropy. If N is a finite-dimensional subalgebra of M, we denote by E_N the unique faithful normal conditional expectation of M on N which is τ-preserving.

1. *Entropy and compact groups*

Let M be a type II_1 von Neumann algebra with separable pre-dual and let τ be a faithful normal trace on M with $\tau(1)=1$.

LEMMA 1.1. *Let G be a topological group and $\alpha:G\to\operatorname{Aut} M$ be an action continuous for the topology of pointwise convergence in 2-norm on* Aut M *and such that $\tau(\alpha_g(x))=\tau(x)$ for all $x\in M$. Then, for all compact subsets K of M in the 2-norm topology, we have*

$$\sup_{x\in K}\|x-\alpha_g(x)\|_2\to 0\quad \text{if } g\to e,$$

where e is the neutral element of G.

Proof. Let $\varepsilon>0$ be given. For any $x\in K$ let

$$B(x,\varepsilon)=\{y\in M:\|x-y\|_2<\varepsilon\}.$$

Since K is compact, there exist $x_1,\ldots,x_m\in K$ such that

$$K\subset\bigcup_{i=1}^m B(x_i,\varepsilon).$$

By hypothesis on α, there exists a neighbourhood W_i of e in G such that, for all $g\in W_i$,

$$\|x_i-\alpha_g(x_i)\|_2<\varepsilon.$$

For all $x \in B(x_i, \varepsilon)$ we have:

$$\|x-\alpha_g(x)\|_2 \le 2\|x-x_i\|_2+\|x_i-\alpha_g(x_i)\|_2 < 3\varepsilon \quad \text{if } g\in W_i.$$

Let

$$W=\bigcap_{i=1}^{m} W_i.$$

We obtain

$$\|x-\alpha_g(x)\|_2<3\varepsilon$$

for all $x\in K$ and all $g\in W$. □

Remark 1.2. When M is a II_1 factor with separable pre-dual, the topology of pointwise convergence in 2-norm is equivalent to the p-topology, so to the u-topology [**7**, corollary 3.8] and to the pointwise strong convergence on $\operatorname{Aut} M$ [**2**, p. 541].

Let F be the set of all finite dimensional von Neumann subalgebras of M.

LEMMA 1.3. *Let N and P be in F, then $H(N|P)=0$ if and only if $N\subset P$.*

Proof. Let S_1 be the set

$$S_1=\{x=(x_i)_{i\in\mathbb{N}}: x_i\in M_+, \textstyle\sum x_i=1 \text{ and } x_i=0 \text{ for almost all } i\}.$$

By definition,

$$H(N|P)=\sup_{x\in S_1}\sum_i \tau\eta E_P(x_i)-\tau\eta E_N(x_i),$$

where η is the function $x\in[0,\infty[\to -x\log x\in\mathbb{R}$ (see [**4**]).

Assume that $H(N|P)=0$ and let $x=(x_i)\in S_1$, $x_i\in N$. By Jensen's inequality, we have

$$\tau\eta E_P(x_i)\ge\tau\eta(x_i) \quad \text{for all } i$$

[**4**, p. 293], so

$$\sum_i \tau\eta E_P(x_i)-\tau\eta(x_i)\ge 0.$$

As $H(N|P)=0$, we obtain

$$\sum_i \tau\eta E_P(x_i)-\tau\eta(x_i)=0,$$

hence

$$\tau\eta E_P(x_i)=\tau\eta(x_i) \quad \text{for all } i.$$

Let B_i be the abelian von Neumann subalgebra of P generated by $E_P(x_i)$ and 1. We have

$$E_{B_i}E_P(x_i)=E_P(x_i).$$

So

$$E_{B_i}(x_i)=E_P(x_i),$$

hence

$$\tau\eta E_{B_i}(x_i)=\tau\eta(x_i).$$

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By [**10**, inequality 9.5′, p. 84], we obtain $x_i \in B_i$, so $x_i \in P$ for all i and $N \subset P$.
The converse implication is clear. □

PROPOSITION 1.4. *The map* $d: F \times F \to \mathbb{R}$,

$$d(N, P) = H(N|P) + H(P|N)$$

is a distance on F.

Proof. It is clear that d is positive and symmetric; the triangular inequality follows from [**4**, property G]; and, if $d(N, P) = 0$, then $N = P$ by lemma 1.3. □

PROPOSITION 1.5. *Let K be a compact set in F. Then, for any sequence* $(N_j)_{j\in\mathbb{N}}$, $N_j \in K$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} H(N_1, \ldots, N_n) = 0.$$

Proof. Let $\varepsilon > 0$ be given. There exists an integer $m > 0$ such that, for all $n \geq m$, there exists $i < m$ with $d(N_n, N_i) < \varepsilon$; so

$$H(N_1, \ldots, N_n) - H(N_1, \ldots, N_{n-1}) \leq H(N_n|N_i) < \varepsilon$$

[**4**, property F]. Hence

$$\frac{1}{n} H(N_1, \ldots, N_n) = \frac{1}{n}\left[\sum_{i=m}^{n-1} (H(N_1, \ldots, N_{i+1}) - H(N_1, \ldots, N_i)) + H(N_1, \ldots, N_m)\right]$$

$$\leq \frac{1}{n}[(n-m)\varepsilon + H(N_1, \ldots, N_m)].$$

As ε is arbitrary, we obtain the conclusion. □

Let G be a subgroup of Aut M, compact for the topology of pointwise convergence in 2-norm on Aut M and such that $\tau(g(x)) = \tau(x)$ for all $x \in M$ and all $g \in G$.

LEMMA 1.6. *For all* $N \in F$, *the closure in F of the set* $\{g^n(N): n \in \mathbb{N}\}$ *is compact.*

Proof. Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of positive integers. There exists a subsequence of (n_k), still to be denoted by (n_k), such that (g^{n_k}) converges for the topology of uniform convergence in 2-norm on compact sets of M (lemma 1.1). Hence the sequence $(g^{n_k}(N))$ converges in F by [**4**, theorem 1]. □

The following proposition is an immediate consequence of proposition 1.5 and lemma 1.6.

PROPOSITION 1.7. *With the above assumptions we have* $H(g) = 0$ *for all* $g \in G$.

Let $(X, \mathcal{B}, \mu)$ be a standard Borel space with $\mu(X) = 1$ and let T be an ergodic automorphism of X preserving μ. Let

$$R = L^\infty(X, \mu) \times_T \mathbb{Z}$$

be the hyperfinite II_1 factor, the crossed product of X by T. Let A be the canonical image of $L^\infty(X, \mu)$ in R, E be the conditional expectation of R on A, and U be the unitary of R corresponding to the translation by 1 in $\mathbb{Z}$. Set

$$R_0 = \left\{y \in R: y = \sum_{n\in J} a_n U^n,\ a_n \in A,\ J \subset \mathbb{Z},\ J \text{ finite}\right\}.$$

For any $f \in L^1(X, \mu)$ and any $y \in R_0$, the map $\phi_{y,f}$ defined by

$$\phi_{y,f}(x) = \int_X E(y^*xy) f \, d\mu$$

is a σ-weakly continuous linear functional on R.

PROPOSITION 1.8. *The linear space generated by* $\phi_{y,f}$, $y \in R_0$, $f \in L^1(X, \mu)$ *is dense in* R_*.

Proof. See [**1**, § 1.2]. □

THEOREM 1.9. *Suppose that* T *has pure point spectrum. Then* $H(\operatorname{Ad} U) = 0$, *so* $H(\operatorname{Ad} U) = h(T)$.

Proof. By [**11**, theorem 3.4, p. 68] we can suppose that X is a compact abelian group and T is a rotation on X; i.e. there exists $g \in X$ with $T = T_g$, where

$$T_g(h) = g \cdot h \quad \text{for all } h \in X.$$

As X is abelian, we have

$$T_g T_k = T_k T_g \quad \text{for all } k \in X.$$

Hence T_k extends to an automorphism θ_k of R with

$$\theta_k(a) = T_k(a) \quad \text{for all } a \in A$$

and

$$\theta_k(U) = U.$$

We shall show that the action of X on R given by $k \in X \to \theta_k \in \operatorname{Aut} R$ is continuous for the p-topology.

Let

$$y = \sum_{i=1}^{r} b_i U^{n_i} \in R_0$$

and

$$x = \sum_{n=-\infty}^{\infty} a_n U^n \in R, \quad a_n \in A.$$

Then

$$y^*xy = \sum_{i,j,n} U^{-n_i} b_i^* a_n U^n b_j U^{n_j}$$

$$= \sum_{i,j,n} T_g^{-n_i}(b_i^* a_n) T_g^{n-n_i}(b_j) U^{n-n_i+n_j}.$$

Thus

$$E(y^*xy) = \sum_{i,j} T_g^{-n_i}(b_i^*) T_g^{-n_j}(b_j) T_g^{-n_i}(a_{n_i-n_j})$$

and

$$E(y^*\theta_k(x)y) = \sum_{i,j} T_g^{-n_i}(b_i^*) T_g^{-n_j}(b_j) T_g^{-n_i}(T_k(a_{n_i-n_j})).$$

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For $f \in L^1(X, \mu)$ we obtain

$$\phi_{y,f}(x-\theta_k(x)) = \int_X \left[\sum_{i,j} T_g^{-n_i}(b_i^*) T_g^{-n_j}(b_j) T_g^{-n_i}(a_{n_i-n_j} - T_k(a_{n_i-n_j}))\right] f\, d\mu.$$

Hence

$$|\phi_{y,f}(x-\theta_k(x))| \le \sum_{i,j} \|b_i\|\,\|b_j\| \int_X |T_g^{-n_i}(a_{n_i-n_j} - T_k(a_{n_i-n_j}))|\,|f|\, d\mu$$

and

$$|\phi_{y,f}(x-\theta_k(x))| \to 0 \quad \text{when } k \to e,$$

where e is the neutral element of X.

Clearly, the same result remains true for all finite linear combinations of $\phi_{y,f}$. So, by proposition 1.8, the action $k \to \theta_k$ is continuous for the p-topology.

Hence, from remark 1.2 and proposition 1.7, we have that

$$H(\theta_k) = 0 \quad \text{for all } k \in X$$

and, as $\theta_g = \operatorname{Ad} U$, we obtain the conclusion. □

2. *Entropy of a flow*

In this section we prove the following theorem:

THEOREM 2.1. *Let $(\alpha_t)_{t\in\mathbb{R}}$ be a one-parameter group of automorphisms of the hyperfinite* II_1 *factor, continuous for the u-topology. Then*

$$H(\alpha_t) = |t| H(\alpha_1) \quad \text{for all } t \in \mathbb{R}.$$

Proof. As $H(\theta) = H(\theta^{-1})$, for all $\theta \in \operatorname{Aut} R$, we can suppose that $t > 0$. As in [**8**, p. 127], we shall prove that, for $0 < s < t$,

$$H(\alpha_t) \le (t/s) H(\alpha_s).$$

Let m be a positive integer and let N be a finite-dimensional von Neumann subalgebra of R. We denote by $k(n)$ a positive integer such that

$$nt \le k(n)s < (n+1)t$$

and by $r(p)$ the integer such that

$$r(p) \cdot s/m \le pt < (r(p)+1)s/m.$$

For $k = k(n)$ we see that

$$H(N, \alpha_t(N), \ldots, \alpha_{nt}(N)) \le H(N, \alpha_{s/m}(N), \ldots, \alpha_{(km+m-1)s/m}(N)) + \sum_{p=1}^{n} H(\alpha_{pt}(N) | \alpha_{r(p)s/m}(N)).$$

But

$$H(\alpha_{pt}(N) | \alpha_{r(p)s/m}(N)) = H(\alpha_\lambda(N) | N),$$

where

$$\lambda = pt - r(p)s/m \quad \text{and} \quad 0 \le \lambda < s/m.$$

By remark 1.2 and lemma 1.1, we can suppose that (α_t) is continuous for the topology of uniform convergence on compact sets of R in the 2-norm topology.

So, for any $\varepsilon > 0$, there exists m sufficiently large such that

$$H(\alpha_\lambda(N)|N) < \varepsilon$$

[4, theorem 1]. Hence

$$\frac{1}{n} H(N, \alpha_t(N), \ldots, \alpha_{nt}(N)) \leq \frac{km+m-1}{n} \frac{1}{km+m-1} \times H(N, \ldots, \alpha_{(km+m-1)s/m}(N)) + \varepsilon.$$

Moreover, if $n \to \infty$, $k(n) \to \infty$ and $k(n)/n \to t/s$. Hence

$$H(N, \alpha_t) \leq \frac{t}{s} mH(N, \alpha_{s/m}) + \varepsilon.$$

Let N be such that

$$H(\alpha_t) \leq H(N, \alpha_t) + \varepsilon.$$

Then

$$\begin{aligned} H(\alpha_t) \leq H(N, \alpha_t) + \varepsilon &\leq \frac{t}{s} mH(N, \alpha_{s/m}) + 2\varepsilon \\ &\leq \frac{t}{s} mH(\alpha_{s/m}) + 2\varepsilon \\ &= \frac{t}{s} H(\alpha_s) + 2\varepsilon, \end{aligned}$$

because R is hyperfinite [4, remark 6]. Since ε is arbitrary, we obtain

$$H(\alpha_t) \leq \frac{t}{s} H(\alpha_s).$$

Let q be an integer such that $0 < t/q < s$. By the above statement we have

$$H(\alpha_s) \leq \frac{s}{t} qH(\alpha_{t/q}) = \frac{s}{t} H(\alpha_t).$$

Hence

$$H(\alpha_t) = \frac{t}{s} H(\alpha_s). \qquad \square$$

3. *Non-continuity of the entropy*

Here we prove that the map

$$\theta \in \mathrm{Aut}\, M \to H(\theta) \in \overline{\mathbb{R}_+}$$

is not norm continuous.

PROPOSITION 3.1. *The set of periodic unitaries of M is dense in the group of all unitaries of M in the norm topology.*

Proof. If n is a positive integer, write

$$\omega_{n,k} = \exp(2ik\pi/n)$$

and

$$\Omega_{n,k} = \{\omega \in \mathbb{C} : \omega = \exp(it) \text{ with } t \in \mathbb{R} \text{ and } 2ik\pi/n \leq t < 2i(k+1)\pi/n\}$$
$$(k = 0, \ldots, n-1).$$

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Define a Borel function F_n on the unit circle S^1 of $\mathbb{C}$ by

$$F_n(z) = \omega_{n,k} \quad \text{if } z \in \Omega_{n,k}.$$

Then

$$|F_n(z) - z| \le \varepsilon_n = |\omega_{n,1} - \omega_{n,0}|$$

for each $z \in S^1$. Obviously,

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

Let u be a unitary in M. For each integer n, let

$$u_n = F_n(u).$$

Then u_n is a periodic unitary of M and

$$\lim_{n\to\infty} \|u_n - u\| \le \lim_{n\to\infty} \varepsilon_n = 0. \qquad \square$$

COROLLARY 3.2. *If M is a type* II_1 *von Neumann algebra, the map H* : Aut $M \to \overline{\mathbb{R}_+}$, *$\theta \to H(\theta)$ is not norm continuous.*

Proof. Since M is of type II_1, it contains the hyperfinite II_1 factor R. Let T and U be as defined just above proposition 1.8 and suppose that $h(T) > 0$. Then

$$H(\mathrm{Ad}\, U) \ge h(T) > 0.$$

Hence there exists U unitary of M with $H(\mathrm{Ad}\, U) > 0$. By proposition 3.1, there is a sequence (v_n) of periodic unitaries of M such that

$$\|U - v_n\| \to 0 \quad \text{as } n \to \infty,$$

so

$$\|\mathrm{Ad}\, U - \mathrm{Ad}\, v_n\| \to 0 \quad \text{as } n \to \infty.$$

As $H(\mathrm{Ad}\, v_n) = 0$ for all n, we obtain the conclusion. $\square$

4. *An uncountable family of automorphisms*

In this section we give an uncountable family of aperiodic automorphisms with zero entropy.

Let $(X, \mathcal{B}, \mu)$ be a standard Borel space with $\mu(X) = 1$ and let T be an ergodic automorphism of X preserving μ. Let $R = L^\infty(X, \mu) \times_T \mathbb{Z}$, U and A be as defined just above proposition 1.8.

For $t \in [0, 1[$, let

$$\chi_t = \exp(2i\pi t) \in S^1 = \hat{\mathbb{Z}}$$

and let V_t be the unitary operator on $L^2(\mathbb{Z}, L^2(X, \mu))$ defined by

$$V_t \xi(n) = \chi_t^{-n} \xi(n).$$

For any $a \in A$ we have

$$V_t a V_t^* = a$$

and

$$V_t U^n V_t^* = \chi_t^{-n} U^n$$

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for all $n \in \mathbb{Z}$, so the map

$$\theta_t(x) = V_t x V_t^*$$

is an automorphism of R. The action θ of S^1 on R so defined is called the dual action (see [**9**]). We note that θ_t is not ergodic and that the system (R, θ_t, τ) is not asymptotically abelian in mean for any t [**6**, definition 1, p. 12], where τ is the canonical trace on R.

PROPOSITION 4.1. *The dual action is continuous for the topology of pointwise strong convergence on* Aut R.

Proof. See [**9**, p. 257]. □

From this proposition, remark 1.2 and proposition 1.7 we deduce:

COROLLARY 4.2. *For all* $t \in [0, 1[$, $H(\theta_t) = 0$.

We shall denote by $P(T)$ the point spectrum of T.

PROPOSITION 4.3. *For* $t \in [0, 1[$, θ_t *is an inner automorphism of* R *if and only if* $\chi_t = \exp(2i\pi t) \in P(T)$.

Proof. Assume that θ_t is inner, i.e. there is v unitary in R such that

$$\theta_t = \mathrm{Ad}\, v.$$

As $\theta_t(a) = a$ for all $a \in A$, we have $v \in A$ because A is maximal abelian in R. Moreover,

$$vUv^* = \chi_t^{-1} U$$

so

$$T(v) = \chi_t v,$$

hence $\chi_t \in P(T)$.

Conversely, assume that $\chi_t \in P(t)$ and let $f \in L^2(X, \mu)$ be an eigenfunction corresponding to the eigenvalue χ_t. We have

$$|f(T\omega)| = |\chi_t||f(\omega)| = |f(\omega)|$$

for almost all $\omega \in X$. Since T is ergodic, $|f| = k$ constant almost everywhere and $f \in L^\infty(X, \mu)$. Let v be the canonical image of $k^{-1}f$ in A; v is unitary and is an eigenfunction of T. So

$$T(v) = \chi_t v = UvU^*$$

and

$$vUv^* = \chi_t^{-1} U = \theta_t(U).$$

Since $vav^* = a$ for all $a \in A$, we have

$$\theta_t = \mathrm{Ad}\, v. \qquad \square$$

COROLLARY 4.4. *If* T *is weak-mixing, then for any irrational number* $t \in [0, 1[$, θ_t *is aperiodic.*

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Proof. If T is weak-mixing, then $P(T)=\{1\}$. If t is an irrational number in $[0, 1[$, then

$$\chi_t^n \neq 1$$

for any integer $n \neq 0$. Hence θ_t^n is an outer automorphism for all $n \neq 0$; that is, θ_t is aperiodic. □

Now let $t \in [0, 1[$ be a fixed irrational number and let S and T be ergodic automorphisms of X preserving the measure μ. Let

$$R = L^\infty(X, \mu) \times_T \mathbb{Z} = L^\infty(X, \mu) \times_S \mathbb{Z}$$

and let U (resp. V) be the unitary operator in R corresponding to T (resp. S).

Let θ be the dual action for T and σ be the dual action for S. Suppose that there is $\psi \in \operatorname{Aut} R$ such that

$$\sigma_t = \psi\theta_t\psi^{-1}.$$

For all $a \in A$ we have

$$a = \sigma_t(a) = \psi\theta_t\psi^{-1}(a)$$

so $\psi^{-1}(a) \in A$, because t is irrational. Hence

$$\psi(A) = A.$$

Moreover,

$$\sigma_t(V) = \chi_t^{-1} V = \psi\theta_t\psi^{-1}(V).$$

Hence

$$\chi_t^{-1}\psi^{-1}(V) = \theta_t\psi^{-1}(V)$$

so

$$\theta_t(U^*\psi^{-1}(V)) = U^*\psi^{-1}(V)$$

and

$$U^*\psi^{-1}(V) = a \in A$$

because t is irrational. Hence

$$S(b) = VbV^* = \psi(U)\psi(a)b\psi(a)^*\psi(U)^* = \psi(U)b\psi(U)^*$$

so

$$S = \psi T\psi^{-1}.$$

Consequently, if σ_t and θ_t are conjugate in R, then S and T are conjugate in A.

Conversely, assume that there is $\psi \in \operatorname{Aut} A$ such that $S = \psi T\psi^{-1}$. We shall still denote by ψ its canonical extension to R ($\psi(U) = V$). We then have

$$\psi\theta_t\psi^{-1}(V) = \chi_t^{-1} V = \sigma_t(V)$$

and

$$\psi\theta_t\psi^{-1}(a) = a$$

for all $a \in A$. Hence

$$\sigma_t = \psi\theta_t\psi^{-1}.$$

We have proved the theorem:

THEOREM 4.5. *With the above notation, S and T are conjugate if and only if* σ_t *and* θ_t *are conjugate for some irrational number* $t \in [0, 1[$.

COROLLARY 4.6. *There is an uncountable family of aperiodic automorphisms in the hyperfinite* II_1 *factor with zero entropy.*

Proof. For $\lambda \in \mathbb{R}_+^*$ let S_λ be the Bernoulli shift with entropy λ on a Lebesgue space $(X, \mathcal{B}, \mu)$. Let θ^λ be the dual action for S^λ in $R = L^\infty(X, \mu) \times_{S_\lambda} \mathbb{Z}$ and let $t \in [0, 1[$ be an irrational number. For all $\lambda \in \mathbb{R}_+^*$, the action α_λ of $\mathbb{Z}$ on R given by

$$\alpha_\lambda(n) = (\theta_t^\lambda)^n$$

is outer by corollary 4.4, and

$$H(\alpha_\lambda(1)) = 0$$

by corollary 4.2. Moreover, if $\lambda \neq \lambda'$, then S_λ and $S_{\lambda'}$ are not conjugate because their entropies differ. Hence, by theorem 4.5, α_λ and $\alpha_{\lambda'}$ are not conjugate. □

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