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Hausdorff dimension of Dirichlet non-improvable set versus well-approximable set

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Abstract. Dirichlet's theorem, including the uniform setting and asymptotic setting, is one of the most fundamental results in Diophantine approximation. The improvement of the asymptotic setting leads to the well-approximable set (in words of continued fractions)

$$\mathcal{K}(\Phi) := \{x : a_{n+1}(x) \geq \Phi(q_n(x)) \text{ for infinitely many } n \in \mathbb{N}\};$$

the improvement of the uniform setting leads to the Dirichlet non-improvable set

$$\mathcal{G}(\Phi) := \{x : a_n(x)a_{n+1}(x) \geq \Phi(q_n(x)) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Surprisingly, as a proper subset of Dirichlet non-improvable set, the well-approximable set has the same s-Hausdorff measure as the Dirichlet non-improvable set. Nevertheless, one can imagine that these two sets should be very different from each other. Therefore, this paper is aimed at a detailed analysis on how the growth speed of the product of two-termed partial quotients affects the Hausdorff dimension compared with that of single-termed partial quotients. More precisely, let $\Phi_1, \Phi_2 : [1, +\infty) \to \mathbb{R}^+$ be two non-decreasing positive functions. We focus on the Hausdorff dimension of the set $\mathcal{G}(\Phi_1)\backslash\mathcal{K}(\Phi_2)$. It is known that the dimensions of $\mathcal{G}(\Phi)$ and $\mathcal{K}(\Phi)$ depend only on the growth exponent of Φ. However, rather different from the current knowledge, it will be seen in some cases that the dimension of $\mathcal{G}(\Phi_1)\backslash\mathcal{K}(\Phi_2)$ will change greatly even slightly modifying Φ_1 by a constant.

Key words: Dirichlet improvable set, well-approximable set, continued fractions, Hausdorff dimension
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1. *Introduction*

Diophantine approximation aims at quantitative analysis on how well irrational numbers can be approximated by rational numbers. Dirichlet's theorem is the first non-trivial quantitative result in this aspect and is the starting point of metric Diophantine approximation.

THEOREM 1.1. (Dirichlet [19]) *Let $x \in \mathbb{R}$. For any positive number $Q > 1$, there exists an integer q with $1 \le q < Q$, such that*

$$\|qx\| \le \frac{1}{Q}, \quad \text{i.e.} \quad \min_{1 \le q < Q, q \in \mathbb{N}} \|qx\| \le \frac{1}{Q},$$

where $\|\cdot\|$ denotes the distance to integers $\mathbb{Z}$.

As a corollary, one has the following.

COROLLARY 1.2. *For any real number x, there are infinitely many integers $q \in \mathbb{N}$, such that*

$$\|qx\| < 1/q.$$

The result in Theorem 1.1 is called the *uniform Dirichlet theorem* and the result in Corollary 1.2 is called the *asymptotic Dirichlet theorem*. The study of the improvability of Dirichlet's theorem opens up the metric theory in Diophantine approximation.

- The improvability of the asymptotic theorem leads to the ψ well-approximable set

$$\mathcal{W}(\psi) = \{x \in [0, 1) : \|qx\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}.$$

  The metric theory of $\mathcal{W}(\psi)$ and its variants constitute the major topic in metric Diophantine approximation [20]. For examples, Khintchine's theorem [10], Jarník's theorem [9], the mass transference principle [2], the Duffin–Schaeffer conjecture [15] etc.
- The improvability of the asymptotic theorem leads to the Dirichlet improvable set

$$\mathcal{D}(\psi) = \{x \in [0, 1] : \min_{1 \le q < Q} \|qx\| \le \psi(Q) \text{ for all } Q \gg 1\}.$$

  The work of Davenport and Schmidt [4] draw one's attention to the improvability of Dirichlet's theorem itself instead of its corollary. For examples, uniformly well approximable sets [12], uniform Diophantine exponent [3], homogeneous and inhomogeneous Dirichlet improvability [13, 14] etc.

As far as one-dimensional Diophantine approximation is concerned, the continued fraction expansion plays a significant role. Indeed, the metric theories, including Lebesgue measure and Hausdorff dimension, of the sets $\mathcal{W}(\psi)$ and $\mathcal{D}(\psi)$ are both studied via continued fractions at the very beginning.

Let $x = [a_1(x), a_2(x), \ldots]$ be the continued fraction of x, and $p_n(x)/q_n(x)$ be the nth convergent of x. Then by the best rational approximation of the convergents, more precisely,

$$\min_{1 \le q < q_{n+1}(x)} \|qx\| = \|q_n(x) \cdot x\|,$$

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the sets $\mathcal{W}(\psi)$ and $\mathcal{D}(\psi)$ can be rewritten by changing q to $q_n(x)$ and Q to $q_{n+1}(x)$. Easy calculation leads to the following sets:

$$\mathcal{K}(\Phi_2) = \{x \in [0,1) : a_{n+1}(x) \geq \Phi_2(q_n(x)) \text{ for infinitely many } n \in \mathbb{N}\},$$
$$\mathcal{G}(\Phi_1) = \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Phi_1(q_n(x)) \text{ for infinitely many } n \in \mathbb{N}\}.$$

(Later we use *i.m.* for *infinitely many*.) By taking

$$\Phi_2(q) = \frac{1}{\psi(q)q} \quad \text{and} \quad \Phi_1(q) = \frac{\psi(q)q}{1-\psi(q)q},$$

one has the inclusions

$$\mathcal{K}(\Phi_2) \subset \mathcal{W}(\psi) \subset \mathcal{K}\big(\tfrac{1}{2}\Phi_2\big) \quad \text{and} \quad \mathcal{G}(\Phi_1) \subset \mathcal{D}^c(\psi) \subset \mathcal{G}\big(\tfrac{1}{4}\Phi_1\big),$$

where $\mathcal{D}^c$ means the complement set of $\mathcal{D}$.

Based on these relations, Khintchine [10] (or see his monograph [11]) presented the Lebesgue measure of $\mathcal{W}(\psi)$ and Jarník [9] showed its Hausdorff measure; for $\mathcal{D}^c(\psi)$, its Lebesgue measure is given by Kleinbock and Wadleigh [13] and the Hausdorff measure and dimension result is given by Hussain *et al* [7].

The close relation between the sets $\mathcal{K}(\Phi_2)$ and $\mathcal{G}(\Phi_1)$ is disclosed in proving the Hausdorff measure theory of $\mathcal{D}^c(\psi)$.

THEOREM 1.3. (Hussain *et al* [7]) *Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Then for any* $0 \leq s < 1$,

$$\mathcal{H}^s(\mathcal{D}^c(\psi)) = \begin{cases} 0 & \text{if } \sum_t t\left(\dfrac{1}{t^2\Phi_1(t)}\right)^s < \infty; \\ \infty & \text{if } \sum_t t\left(\dfrac{1}{t^2\Phi_1(t)}\right)^s = \infty. \end{cases}$$

More precisely, the divergence theory is followed by just using the simple fact that

$$\mathcal{K}(\Phi) \subset \mathcal{G}(\Phi)$$

and the following Jarník's theorem.

THEOREM 1.4. (Jarník [9]) *Let $\Phi : \mathbb{N} \to \mathbb{R}^+$ be a non-decreasing positive function. Then for any* $0 \leq s < 1$,

$$\mathcal{H}^s(\mathcal{K}(\Phi)) = \begin{cases} 0 & \text{if } \sum_t t\left(\dfrac{1}{t^2\Phi(t)}\right)^s < \infty; \\ \infty & \text{if } \sum_t t\left(\dfrac{1}{t^2\Phi(t)}\right)^s = \infty. \end{cases}$$

So $\dim_{\mathrm{H}}(\mathcal{G}(\Phi)) = \dim_{\mathrm{H}}(\mathcal{K}(\Phi))$. It is surprising that the subset $\mathcal{K}(\Phi)$ can give the right dimension of $\mathcal{G}(\Phi)$ from below. So it is desirable to know how much is the difference between $\mathcal{K}(\Phi)$ and $\mathcal{G}(\Phi)$.

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THEOREM 1.5. (Bakhtawar, Bos and Hussain [1]) *Let $\Phi : \mathbb{N} \to \mathbb{R}^+$ be a non-decreasing function. Then*

$$\dim_{\mathrm{H}}(\mathcal{G}(\Phi)\backslash\mathcal{K}(\Phi)) = \dim_{\mathrm{H}}(\mathcal{K}(\Phi)). \tag{1.1}$$

To prove the equality in equation (1.1), the $\leq$ direction is trivial since $\dim_{\mathrm{H}}(\mathcal{G}(\Phi)) = \dim_{\mathrm{H}}(\mathcal{K}(\Phi))$; for the $\geq$ direction, one considers the following subset:

$$\bigg\{x \in [0, 1) : a_n(x) = 4,\ a_{n+1}(x) \geq \frac{\Phi(q_n(x))}{4}, \text{ i.m. } n \in \mathbb{N};$$
$$\text{and } a_{n+1}(x) < \Phi(q_n(x)) \text{ for all } n \in \mathbb{N}\bigg\}.$$

Since there is already enough room for the choice of $a_{n+1}(x)$ and such a room is almost the same as in finding the lower bound of the dimension of $\mathcal{K}(\Phi)$ (see for example [22]), it should be imagined that this subset should have the same dimension as $\mathcal{K}(\Phi)$.

Roughly speaking, only the term $a_{n+1}(x)$ contributes the dimension of $\mathcal{G}(\Phi)$ while $a_n(x)$ does not. One main reason is that the restriction $a_{n+1}(x) \leq \Phi(q_n(x))$ is too loose that it is already sufficient to ask that $a_{n+1}(x)$ is large and $a_n(x)$ behaves almost freely.

However, if $a_{n+1}(x)$ cannot be very large, then $a_n(x)$ must contribute to realize that $a_n(x)a_{n+1}(x)$ is large enough. So to have a better understanding about how $a_n(x)$ and $a_{n+1}(x)$ contribute to the dimension of $\mathcal{G}(\Phi)$, we consider the following difference set:

$$\mathcal{G}(\Phi_1)\backslash\mathcal{K}(\Phi_2) = \{x \in [0, 1) : a_n(x)a_{n+1}(x) \geq \Phi_1(q_n(x)), \text{ i.m. } n \in \mathbb{N};$$
$$\text{and } a_{n+1}(x) < \Phi_2(q_n(x)) \text{ for all } n \in \mathbb{N} \text{ large}\}.$$

When $\Phi_2 \leq \Phi_1$, both $a_n(x)$ and $a_{n+1}(x)$ have to contribute to realize $a_n(x)a_{n+1}(x) \geq \Phi_1(q_n(x))$. Then there will be a selection about how to choose $a_n(x)$ and $a_{n+1}(x)$ separately: equal or non-equal growth rate, which would be the optimal choice? The general principle of how $a_n(x)$ and $a_{n+1}(x)$ are chosen will be explained in detail in the proof. Moreover, one will see that a minor change on Φ will cause a big difference on the dimension.

We ask Φ_1 and Φ_2 to take the form as Jarník's original theorem, that is, $\Phi_i(q) = q^{t_i}$ and write $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ for the set $\mathcal{G}(\Phi_1)\backslash\mathcal{K}(\Phi_2)$.

THEOREM 1.6. *For any $t_1, t_2 > 0$:*

- *when $t_1 > t_2 + t_2/(1 + t_2)$,*
$$\mathcal{G}(t_1)\backslash\mathcal{K}(t_2) = \emptyset;$$
- *when $t_1 = t_2 + t_2/(1 + t_2)$,*
$$\mathcal{G}(t_1)\backslash\mathcal{K}(t_2) = \emptyset;$$
- *when $t_2 < t_1 < t_2 + t_2/(1 + t_2)$,*
$$\dim_{\mathrm{H}}(\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)) = 1 - \frac{t_1}{2 + t_2};$$
- *when $t_1 \leq t_2$,*
$$\dim_{\mathrm{H}}(\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)) = \frac{2}{2 + t_1}.$$

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We separate the case $t_1 = t_2 + t_2/(1+t_2)$ from the others, mainly because a different situation will happen for this case. We give two examples to illustrate this. Denote

$$E_1 = \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq q_n(x)^{t_1}, \text{ i.m. } n \in \mathbb{N},$$
$$a_{n+1}(x) < q_n(x)^{t_2} \text{ for all } n \in \mathbb{N} \text{ large}\},$$
$$E_2 = \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq 4^{-t_1} q_n(x)^{t_1}, \text{ i.m. } n \in \mathbb{N},$$
$$a_{n+1}(x) < 3q_n(x)^{t_2} \text{ for all } n \in \mathbb{N} \text{ large}\}.$$

The first set E_1 is nothing but $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$. We duplicate it here mainly for comparison.

PROPOSITION 1.7. *If $t_1 = t_2 + t_2/(1+t_2)$, then*

$$E_1 = \emptyset, \quad \dim_H E_2 = 1 - \frac{t_1}{2+t_2}.$$

These two examples illustrate that as far as the general functions Φ_i are concerned, minor change on the function will lead to a big difference between the dimensions. So it is almost hopeless to give a unified formula for the dimension of the set $\mathcal{G}(\Phi_1)\backslash\mathcal{K}(\Phi_2)$ (the formula is hopeful only when Φ_2 is good). Therefore for simplicity, we ask Φ_i to behave regularly instead of arbitrarily.

THEOREM 1.8. *Let Φ_1, Φ_2 be two non-decreasing functions. Assume that*

$$\lim_{q\to\infty} \frac{\log \Phi_1(q)}{\log q} = t_1, \quad \lim_{q\to\infty} \frac{\log \Phi_2(q)}{\log q} = t_2.$$

Then the following:

- *when $t_1 > t_2 + t_2/(1+t_2)$,*
$$\mathcal{G}(\Phi_1)\backslash\mathcal{K}(\Phi_2) = \emptyset;$$
- *when $t_2 < t_1 < t_2 + t_2/(1+t_2)$,*
$$\dim_H(\mathcal{G}(\Phi_1)\backslash\mathcal{K}(\Phi_2)) = 1 - \frac{t_1}{2+t_2};$$
- *when $t_1 \leq t_2$,*
$$\dim_H(\mathcal{G}(\Phi_1)\backslash\mathcal{K}(\Phi_2)) = \frac{2}{2+t_1}.$$

Even though only special functions are considered here, the proof below will be sufficient to illustrate how the partial quotients $a_n(x)$ and $a_{n+1}(x)$ contribute to the dimension of $\mathcal{G}(\Phi)$.

Throughout the paper, denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure, $\dim_H$ the Hausdorff dimension and 'cl' the closure of a set. We use $a \ll b$, $a \gg b$ and $a \asymp b$ respectively to mean that $0 < a/b \leq e_1$, $a/b \geq e_2 > 0$ and $e_2 \leq a/b \leq e_1$ for unspecified positive constants e_1, e_2.

2. *Preliminaries*

In this section, we shall collect some basic properties about continued fractions for later use. For more properties, one is referred to the monographs [8, 11].

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Continued fraction expansion is induced by the Gauss transformation $T : [0, 1) \to [0, 1)$ given by

$$T(0) := 0, \quad T(x) = \frac{1}{x} \ (\mathrm{mod}\ 1), \quad x \in (0, 1).$$

Then every irrational number $x \in [0, 1)$ can be uniquely expanded into an infinite continued fraction:

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots}} := [a_1(x), a_2(x), \ldots],$$

where $a_1(x) = \lfloor 1/x \rfloor$ and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$ are called the partial quotients of x. The finite truncation

$$\frac{p_n(x)}{q_n(x)} = [a_1(x), \ldots, a_n(x)]$$

is called the nth convergent of x.

The numerator and denominator of a convergent can be determined by the recursive relation: for any $k \geq 1$,

$$p_k(x) = a_k(x)p_{k-1}(x) + p_{k-2}(x), \quad q_k(x) = a_k(x)q_{k-1}(x) + q_{k-2}(x), \tag{2.1}$$

with the conventions $p_0 = 0, q_0 = 1, p_{-1} = 1, q_{-1} = 0$.

For simplicity, we write

$$p_n(x) = p_n(a_1, \ldots, a_n) = p_n, \quad q_n(x) = q_n(a_1, \ldots, a_n) = q_n \tag{2.2}$$

when the partial quotients $a_1, \ldots, a_n$ are clear.

LEMMA 2.1. *Let $a_1, \ldots, a_n, b_1, \ldots, b_m$ be integers in $\mathbb{N}$. For any $1 \leq k \leq n$, one has*

$$q_n \geq 2^{(n-1)/2}, \quad \textit{and} \quad p_{n-1}q_n - p_n q_{n-1} = (-1)^n, \tag{2.3}$$

$$1 \leq \frac{q_{n+m}(a_1, \ldots, a_n, b_1, \ldots, b_m)}{q_n(a_1, \ldots, a_n) \cdot q_m(b_1, \ldots, b_m)} \leq 2. \tag{2.4}$$

For any positive integers $a_1, \ldots, a_n$, define

$$I_n(a_1, \ldots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}$$

and call it *a cylinder of order n*. The length of a cylinder and its position in $[0, 1)$ is demonstrated in the following propositions.

PROPOSITION 2.2. (Khintchine [11]) *For any $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, p_k, q_k are defined recursively by equation (2.1) for $0 \leq k \leq n$. Then*

$$I_n(a_1, \ldots, a_n) = \begin{cases} \left[\dfrac{p_n}{q_n}, \dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}\right) & \textit{if } n \textit{ is even}, \\ \left(\dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}, \dfrac{p_n}{q_n}\right] & \textit{if } n \textit{ is odd}. \end{cases} \tag{2.5}$$

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Therefore, the length of a cylinder of order n is given by

$$|I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}.$$

Since every number in [0, 1) has continued fraction expansion, then

$$[0, 1) = \bigcup_{a_1,\ldots,a_n} I_n(a_1, \ldots, a_n).$$

Thus,

$$1 \le \sum_{a_1,\ldots,a_n} \frac{1}{q_n^2(a_1, \ldots, a_n)} \le 2. \tag{2.6}$$

PROPOSITION 2.3. (Khintchine [11]) *Let $I_n = I_n(a_1, \ldots, a_n)$ be a cylinder of order n, which is partitioned into sub-cylinders $\{I_{n+1}(a_1, \ldots, a_n, a_{n+1}) : a_{n+1} \in \mathbb{N}\}$. When n is odd, these sub-cylinders are positioned from left to right, as a_{n+1} increases from 1 to ∞; when n is even, they are positioned from right to left.*

Next, we introduce the mass distribution principle which is the classic method in estimating the Hausdorff dimension of a set from below.

PROPOSITION 2.4. [5] *Let E be a Borel set and μ be a measure with $\mu(E) > 0$. Suppose that for some $s > 0$, there exist constants $c > 0, r_o > 0$ such that for any $x \in E$ and $r < r_o$,*

$$\mu(B(x, r)) \le cr^s, \tag{2.7}$$

where $B(x, r)$ denotes an open ball centered at x and radius r, then $\dim_{\mathrm{H}} E \ge s$.

At the end, we give some dimensional numbers which are related to the dimension of the set of points with bounded partial quotients.

For any integer M, define

$$E_M = \{x \in [0, 1) : 1 \le a_n(x) \le M \text{ for all } n \ge 1\}.$$

For each integer N, define $\tilde{s}_N(M)$ to be the solution to the equation

$$\sum_{1\le a_1,\ldots,a_N\le M} \left(\frac{1}{q_N^2(a_1, \ldots, a_N)}\right)^s = 1.$$

PROPOSITION 2.5. (Good [6]) *The limit of $\tilde{s}_N(M)$ as $N \to \infty$ exists and*

$$\dim_{\mathrm{H}} E_M = \lim_{N\to\infty} \tilde{s}_N(M) := \tilde{s}(M).$$

It is well known that the set of points with bounded partial quotients (that is, the set of badly approximable points) is of Hausdorff dimension 1 (see [18]). Thus,

$$\lim_{M\to\infty} \dim_{\mathrm{H}} E_M = 1, \quad \text{i.e.} \quad \lim_{M\to\infty} \tilde{s}_M = 1.$$

These two results can also be seen by using the words from dynamical systems. More precisely, a pressure function with a continuous potential can be approximated by the

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pressure functions restricted to the sub-systems in continued fractions (see for example Mauldin and Urbański [16] or their monograph [17]).

3. *A Cantor set*

This section is devoted to dealing with the dimension of a Cantor set which is highly related to the dimension of $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ and also may have its own interest and applications to other problems in continued fractions. Bear in mind the notation in equation (2.2).

Let $\alpha_1, \alpha_2 > 0$ be two positive numbers. Denote by $E(\alpha_1, \alpha_2)$ the set

$$\{x \in [0, 1) : c_1 q_{n-1}^{\alpha_1}(x) \le a_n(x) < 2c_1 q_{n-1}^{\alpha_1}(x), c_2 q_n^{\alpha_2}(x) \le a_{n+1}(x) < 2c_2 q_n^{\alpha_2}(x), \text{i.m. } n \in \mathbb{N}\}$$

where c_1, c_2 are positive constants.

One will see how the growth of $a_n(x)$ and $a_{n+1}(x)$ affects the dimension of $E(\alpha_1, \alpha_2)$. For notational simplicity, we take $c_1 = c_2 = 1$ and the other case can be done with verbal modifications; if an integer n is assumed to be a real number ξ, we mean $n = \lfloor \xi \rfloor$; in the definition of $E(\alpha_1, \alpha_2)$, there are $q_{n-1}^{\alpha_1}$ many choices of $a_n(x)$.

THEOREM 3.1. *For any* $\alpha_1, \alpha_2 > 0$,

$$\dim_H E(\alpha_1, \alpha_2) = \min\left\{\frac{2}{\alpha_1 + 2}, \frac{\alpha_1 + 2}{(\alpha_1 + 1)(\alpha_2 + 2)}\right\}.$$

The proof of Theorem 3.1 is split into two parts: upper bound and lower bound.

3.1. *Upper bound.*

Because of the limsup nature, there are natural coverings for $E(\alpha_1, \alpha_2)$. For each $n \ge 1$, define

$$E_n = \{x \in [0, 1) : q_{n-1}^{\alpha_1}(x) \le a_n(x) < 2q_{n-1}^{\alpha_1}(x), q_n^{\alpha_2}(x) \le a_{n+1}(x) < 2q_n^{\alpha_2}(x)\}.$$

Then

$$E(\alpha_1, \alpha_2) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n \subset \bigcup_{n=N}^{\infty} E_n.$$

So in the following, we search for the potential optimal cover of E_n for each $n \ge N$.

By decomposing the unit interval into the collection of $(n-1)$th order cylinders, one has

$$E_n = \bigcup_{a_1,\ldots,a_{n-1}\in\mathbb{N}} \{x \in [0, 1) : a_i(x) = a_i, 1 \le i < n, q_{n-1}^{\alpha_1} \le a_n(x) < 2q_{n-1}^{\alpha_1}, q_n^{\alpha_2} \le a_{n+1}(x) < 2q_n^{\alpha_2}\}.$$

Then there are two potential optimal covers.

- Cover type I. For any integers $a_1, \ldots, a_{n-1} \in \mathbb{N}$, define

$$J_{n-1}(a_1, \ldots, a_{n-1}) = \bigcup_{q_{n-1}^{\alpha_1} \le a_n < 2q_{n-1}^{\alpha_1}} I_n(a_1, \ldots, a_n),$$

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which is an interval of length

$$|J_{n-1}(a_1, \ldots, a_{n-1})| = \sum_{q_{n-1}^{\alpha_1} \le a_n < 2q_{n-1}^{\alpha_1}} \left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| \asymp \frac{1}{q_{n-1}^{\alpha_1+2}}.$$

Then,

$$E_n \subset \bigcup_{a_1,\ldots,a_{n-1}} J_{n-1}(a_1, \ldots, a_{n-1}).$$

Therefore, an s-dimensional Hausdorff measure of $E(\alpha_1, \alpha_2)$ can be estimated as

$$\begin{aligned}\mathcal{H}^s(E(\alpha_1, \alpha_2)) &\le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_{n-1}} |J_{n-1}(a_1, \ldots, a_{n-1})|^s \\ &\le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_{n-1}} \frac{1}{q_{n-1}^{(\alpha_1+2)s}}.\end{aligned}$$

Recall equation (2.6) where

$$\sum_{a_1,\ldots,a_{n-1}} \frac{1}{q_{n-1}^2} \le 2, \quad \text{and } q_{n-1} \ge 2^{(n-2)/2}.$$

Thus for any $\epsilon > 0$ and by taking $s = (2 + 2\epsilon)/(\alpha_1 + 2)$, it follows that

$$\begin{aligned}\mathcal{H}^s(E(\alpha_1, \alpha_2)) &\le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_{n-1}} \left(\frac{1}{q_{n-1}^2} \cdot \frac{1}{2^{(n-2)\epsilon}} \right) \\ &\le 2 \liminf_{N\to\infty} \sum_{n=N}^{\infty} \frac{1}{2^{(n-2)\epsilon}} < \infty.\end{aligned}$$

This shows that

$$\dim_{\mathrm{H}} E(\alpha_1, \alpha_2) \le \frac{2}{\alpha_1 + 2}.$$

- Cover type II. For any integers $a_1, \ldots, a_{n-1} \in \mathbb{N}$ and $q_{n-1}^{\alpha_1} \le a_n < 2q_{n-1}^{\alpha_1}$, define

$$J_n(a_1, \ldots, a_n) = \bigcup_{q_n^{\alpha_2} \le a_{n+1} < 2q_n^{\alpha_2}} I_{n+1}(a_1, \ldots, a_{n+1}),$$

which is an interval of length

$$|J_n(a_1, \ldots, a_n)| \asymp \frac{1}{q_n^{\alpha_2+2}}.$$

Then,

$$E_n \subset \bigcup_{a_1,\ldots,a_{n-1}} \bigcup_{q_{n-1}^{\alpha_1} \le a_n < 2q_{n-1}^{\alpha_1}} J_n(a_1, \ldots, a_n).$$

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Therefore, an s-dimensional Hausdorff measure of $E(\alpha_1, \alpha_2)$ can be estimated as

$$\mathcal{H}^s(E(\alpha_1,\alpha_2)) \le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_{n-1}} \sum_{q_{n-1}^{\alpha_1}\le a_n<2q_{n-1}^{\alpha_1}} |J_n(a_1,\ldots,a_n)|^s$$
$$\le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_{n-1}} \sum_{q_{n-1}^{\alpha_1}\le a_n<2q_{n-1}^{\alpha_1}} \frac{1}{q_n^{(\alpha_2+2)s}}.$$

Recall that

$$q_n = a_n q_{n-1} + q_{n-2} \ge a_n q_{n-1}.$$

Thus it follows that

$$\mathcal{H}^s(E(\alpha_1,\alpha_2)) \le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_{n-1}} \frac{q_{n-1}^{\alpha_1}}{q_{n-1}^{(1+\alpha_1)(\alpha_2+2)s}}.$$

Then with a similar choice of s and the argument as in the first case, one has

$$\dim_{\mathrm{H}} E(\alpha_1,\alpha_2) \le \frac{2+\alpha_1}{(1+\alpha_1)(2+\alpha_2)}.$$

In summary, we have shown that

$$\dim_H E(\alpha_1,\alpha_2) \le \min\left\{\frac{2}{\alpha_1+2}, \frac{\alpha_1+2}{(\alpha_1+1)(\alpha_2+2)}\right\}.$$

3.2. *Lower bound.* We use the mass distribution principle (Proposition 2.4) to search for the lower bound of the dimension of $E(\alpha_1, \alpha_2)$: define a measure supported on $E(\alpha_1, \alpha_2)$ and then estimate the Hölder exponent of μ.

Recall $\alpha_1 > 0$. For any integers N, M, define the dimensional number $s = s_N(M)$ as the solution to

$$\sum_{1\le a_1,\ldots,a_N\le M} \frac{1}{q_N^{(2+\alpha_1)s}} = 1. \tag{3.1}$$

Then by Proposition 2.5, one has

$$\lim_{M\to\infty} \lim_{N\to\infty} s_N(M) = \frac{2}{\alpha_1+2}. \tag{3.2}$$

So fix $\epsilon > 0$ and then choose integers M, N sufficiently large such that

$$s > \frac{2}{\alpha_1+2} - \epsilon, \quad (2^{(N-1)/2})^{\epsilon/2} \ge 2^{100}.$$

Fix a sequence of largely sparse integers $\{l_k\}_{k\ge1}$, say,

$$l_k \gg e^{l_1+\cdots+l_{k-1}}, \text{ and take } n_k - n_{k-1} = l_k N + 1 \quad \text{for all } k \ge 1,$$

such that

$$(2^{\ell_k(N-1)/2})^{\epsilon/2} \ge \prod_{t=1}^{k-1} (M+1)^{\ell_t N(1+\alpha_2)^{k-t}(1+\alpha_1)^{k-t}}. \tag{3.3}$$

Then define a subset of $E(\alpha_1, \alpha_2)$ as

$$E = \{x \in [0, 1) : q_{n_k-1}(x)^{\alpha_1} \le a_{n_k}(x) < 2q_{n_k-1}(x)^{\alpha_1}, q_{n_k}(x)^{\alpha_2} \le a_{n_k+1}(x) < 2q_{n_k}(x)^{\alpha_2} \text{ for all } k \ge 1; \text{ and } a_n(x) \in \{1, \ldots, M\} \text{ for other } n \in \mathbb{N}\}. \tag{3.4}$$

For ease of notation, we perform the following.

- Use a symbolic space defined as $D_0 = \{\emptyset\}$, and for any $n \ge 1$,

$$D_n = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : q_{n_k-1}^{\alpha_1} \le a_{n_k} < 2q_{n_k-1}^{\alpha_1}, q_{n_k}^{\alpha_2} \le a_{n_k+1} < 2q_{n_k}^{\alpha_2} \text{ for all } k \ge 1 \text{ with } n_k, n_k+1 \le n; \text{ and } a_j \in \{1, \ldots, M\} \text{ for other } j \le n\},$$

which is just the collection of the prefix of the points in E.

- Use $\mathcal{U}$ to denote the following collection of finite words of length N:

$$\mathcal{U} = \{w = (\sigma_1, \ldots, \sigma_N) : 1 \le \sigma_i \le M, 1 \le i \le N\}.$$

In the following, we always use w to denote a generic word in $\mathcal{U}$.

3.2.1. *Cantor structure of E.* For any $(a_1, \ldots, a_n) \in D_n$, define

$$J_n(a_1, \ldots, a_n) = \bigcup_{a_{n+1}:(a_1,\ldots,a_n,a_{n+1})\in D_{n+1}} I_{n+1}(a_1, \ldots, a_n, a_{n+1})$$

and call it a *basic cylinder* of order n. More precisely, for each $k \ge 0$:

- when $n_{k-1} + 1 \le n < n_k - 1$ (by viewing $n_0 = 0$),

$$J_n(a_1, \ldots, a_n) = \bigcup_{1\le a_{n+1}\le M} I_{n+1}(a_1, \ldots, a_n, a_{n+1});$$

- when $n = n_k - 1$ or $n = n_k$,

$$J_{n_k-1}(a_1, \ldots, a_{n_k-1}) = \bigcup_{q_{n_k-1}^{\alpha_1}\le a_{n_k}<2q_{n_k-1}^{\alpha_1}} I_{n_k}(a_1, \ldots, a_n, a_{n_k}),$$

$$J_{n_k}(a_1, \ldots, a_{n_k}) = \bigcup_{q_{n_k}^{\alpha_2}\le a_{n_k+1}<2q_{n_k}^{\alpha_2}} I_{n_k+1}(a_1, \ldots, a_n, a_{n_k+1}).$$

Then define

$$\mathcal{F}_n = \bigcup_{(a_1,\ldots,a_n)\in D_n} J_n(a_1, \ldots, a_n)$$

and call it level n of the Cantor set E. It is clear that

$$E = \bigcap_{n=1}^{\infty} \mathcal{F}_n = \bigcap_{n=1}^{\infty} \bigcup_{(a_1,\ldots,a_n)\in D_n} J_n(a_1, \ldots, a_n).$$

We have the following observations about the length and gaps of the basic cylinders.

LEMMA 3.2. (Gap estimation) *Denote by $G_n(a_1, \ldots, a_n)$ the gap between $J_n(a_1, \ldots, a_n)$ and other basic cylinders of order n. Then*

$$G_n(a_1, \ldots, a_n) \ge \frac{1}{M} \cdot |J_n(a_1, \ldots, a_n)|.$$

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Proof. This can be observed from the positions of the cylinders in Proposition 2.3. Recall the definition of J_n given above and note that different cylinders I_n are disjoint. When $n = n_k - 1$ or $n = n_k$, the basic cylinder J_n lies in the middle part of I_n, so there are large gaps between J_n with other basic cylinders of order n. For other n, note that

$$\bigcup_{a>M} I_{n+1}(a_1, \ldots, a_n, a)$$

falls in the gap of $J_n(a_1, \ldots, a_n)$ and other basic cylinders in its right/left side (when n is odd/even). Then one needs only estimate the length of these gaps. A detailed proof can be found in [21] or [22]. □

Recall the definition of $\mathcal{U}$. Every element $x \in E$ can be written as the form

$$x = [w_1^{(1)}, \ldots, w_{\ell_1}^{(1)}, a_{n_1}, a_{n_1+1}, w_1^{(2)}, \ldots, w_{\ell_2}^{(2)}, a_{n_2}, a_{n_2+1}, \\ \ldots, w_1^{(k)}, \ldots, w_{\ell_k}^{(k)}, a_{n_k}, a_{n_k+1}, \ldots],$$

where $w_i^{(k)} \in \mathcal{U}$ for all $1 \le i \le \ell_k$, $k \ge 1$, and

$$q_{n_t-1}^{\alpha_1} \le a_{n_t} < 2q_{n_t-1}^{\alpha_1}, \quad q_{n_t}^{\alpha_2} \le a_{n_t+1} < 2q_{n_t}^{\alpha_2} \text{ for all } t \ge 1.$$

We estimate the length of basic cylinders $J_n(x)$ for all $n \ge 1$. For $n_k + 1 \le n < n_{k+1} - 1$, we have

$$|J_n(x)| = \left| \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{(M+1)p_n + p_{n-1}}{(M+1)q_n + q_{n-1}} \right| = \frac{M}{(q_n + q_{n-1})((M+1)q_n + q_{n-1})} \ge \frac{1}{8q_n^2},$$

and similarly,

$$|J_{n_k-1}(x)| = \frac{q_{n_k-1}^{\alpha_1}}{(q_{n_k-1}^{\alpha_1} q_{n_k-1} + q_{n_k-2})(2q_{n_k-1}^{\alpha_1} q_{n_k-1} + q_{n_k-2})},$$

so

$$\left(\frac{1}{q_{n_k-1}(x)}\right)^{\alpha_1+2} > |J_{n_k-1}(x)| \ge \frac{1}{8} \cdot \left(\frac{1}{q_{n_k-1}(x)}\right)^{\alpha_1+2},$$

$$\left(\frac{1}{q_{n_k-1}}\right)^{(a_1+1)(\alpha_2+2)} \ge \left(\frac{1}{q_{n_k}(x)}\right)^{\alpha_2+2} > |J_{n_k}(x)| \\ \ge \frac{1}{8} \cdot \left(\frac{1}{q_{n_k}(x)}\right)^{\alpha_2+2} \ge \frac{1}{2^{7+2\alpha_2}} \left(\frac{1}{q_{n_k-1}}\right)^{(a_1+1)(\alpha_2+2)}.$$

Here for the last inequality, we used $q_{n_k-1}^{\alpha_1} \le a_{n_k} < 2q_{n_k-1}^{\alpha_1}$.

Recall equation (3.3) for the choice of the largely sparse sequence $\{\ell_k\}$. Consequently, we have the following lemma.

LEMMA 3.3. (Length estimation) *Let $x \in E$ and an integer n with $n_k - 1 \le n < n_{k+1} - 1$.*

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- $n = n_k - 1$,

$$|J_{n_k-1}(x)| \geq \frac{1}{2^3} \cdot \frac{1}{q_{n_k-1}^{\alpha_1+2}} \geq \frac{1}{2^3} \cdot \bigg(\frac{1}{2^{\ell_k}} \cdot \prod_{i=1}^{\ell_k} \frac{1}{q_N(w_i^{(k)})} \cdot \frac{1}{q_{n_{k-1}+1}} \bigg)^{\alpha_1+2}$$
$$\geq \bigg(\prod_{i=1}^{\ell_k} \frac{1}{q_N(w_i^{(k)})} \bigg)^{(\alpha_1+2)(1+\epsilon)}. \tag{3.5}$$

- $n = n_k$,

$$|J_{n_k}(x)| \geq \frac{1}{2^3} \frac{1}{q_{n_k}^{\alpha_2+2}} \geq \frac{1}{2^3} \cdot \frac{1}{4^{2+\alpha_2}} \cdot \frac{1}{q_{n_k-1}^{(\alpha_1+1)(\alpha_2+2)}}. \tag{3.6}$$

- $n = n_k + 1$,

$$|J_{n_k+1}(x)| \geq \frac{1}{2^3} \cdot \frac{1}{q_{n_k+1}^2} \geq \frac{1}{2^7} \cdot \frac{1}{q_{n_k}^{2(1+\alpha_2)}}. \tag{3.7}$$

- *For each* $1 \leq \ell < \ell_{k+1}$,

$$|J_{n_k+1+\ell N}(x)| \geq \frac{1}{2^3} \cdot \bigg(\frac{1}{2^{2\ell}} \cdot \prod_{i=1}^{\ell} \frac{1}{q_N^2(w_i^{(k+1)})} \bigg) \cdot \frac{1}{q_{n_k+1}^2}$$
$$\geq \bigg(\prod_{i=1}^{\ell} \frac{1}{q_N^2(w_i^{(k+1)})} \bigg)^{1+\epsilon} \cdot \frac{1}{q_{n_k+1}^2}. \tag{3.8}$$

- *For* $n_k + 1 + (\ell - 1)N \leq n < n_k + 1 + \ell N$ *with* $1 \leq \ell \leq \ell_{k+1}$,

$$|J_n(x)| \geq c \cdot |J_{n_k+1+(\ell-1)N}(x)|, \tag{3.9}$$

where $c = c(M, N)$ *is an absolute constant.*

Proof. Applying equation (2.4) in Lemma 2.1 for ℓ_k times allows us to arrive the third inequality in equation (3.5), while the last inequality just follows from the choice of ℓ_k and ϵ in equation (3.3).

For the relation in (3.9), one notes that the partial quotients are all bounded by M except at the positions $n = n_k, n_k + 1$. The constant c can be taken as

$$\frac{1}{2^3} \cdot \bigg(\frac{1}{M+1} \bigg)^{2N}.$$ □

3.3. *Mass distribution.* We define a probability measure supported on the Cantor set E. Still express an element $x \in E$ as

$$x = [w_1^{(1)}, \ldots, w_{\ell_1}^{(1)}, a_{n_1}, a_{n_1+1}, w_1^{(2)}, \ldots, w_{\ell_2}^{(2)}, a_{n_2}, a_{n_2+1},$$
$$\ldots, w_1^{(k)}, \ldots, w_{\ell_k}^{(k)}, a_{n_k}, a_{n_k+1}, \ldots],$$

where

$$w_i^{(k)} \in \mathcal{U} \text{ for all } i, k \in \mathbb{N}, \text{ and } q_{n_t-1}^{\alpha_1} \leq a_{n_t} < 2q_{n_t-1}^{\alpha_1},\ q_{n_t}^{\alpha_2} \leq a_{n_t+1} < 2q_{n_t}^{\alpha_2} \text{ for all } t \geq 1.$$

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We define the measure along the basic cylinders $J_n(x)$ containing x as follows.

- Let $n \le n_1 + 1$:
  - for each $1 \le \ell \le \ell_1$, define
$$\mu(J_{Nl}(x)) = \prod_{i=1}^{\ell} \left(\frac{1}{q_N(w_i^{(1)})}\right)^{(\alpha_1+2)s}.$$
    Recall the definition of s (see equation (3.1)) and then once μ is a measure, it is a probability measure. Because of the arbitrariness of x, this defines the measure on all basic cylinders of order ℓN;
  - for each integer n with $(\ell-1)N < n < \ell N$ for some $1 \le \ell \le \ell_1$, define
$$\mu(J_n(x)) = \sum_{J_{\ell N} \subset J_n(x)} \mu(J_{\ell N}(x))$$
    where the summation is over all basic cylinders of order ℓN contained in $J_n(x)$. This is designed to ensure the consistency of a measure;
  - when $n = n_1$. Note that $n_1 = \ell_1 N + 1$, then define
$$\mu(J_{n_1}(x)) = \frac{1}{q_{n_1-1}^{\alpha_1}} \mu(J_{n_1-1}(x)) = \frac{1}{q_{n_1-1}^{\alpha_1}} \prod_{l=1}^{\ell_1} \frac{1}{q_N(w_l^{(1)})^{(\alpha_1+2)s}};$$
  - when $n = n_1 + 1$, define
$$\mu(J_{n_1+1}(x)) = \frac{1}{q_{n_1}^{\alpha_2}} \cdot \mu(J_{n_1}(x)) = \frac{1}{q_{n_1}^{\alpha_2}} \cdot \frac{1}{q_{n_1-1}^{\alpha_1}} \prod_{l=1}^{\ell_1} \frac{1}{q_N(w_l^{(1)})^{(\alpha_1+2)s}}.$$
- Let $n_{k-1} + 1 < n \le n_k + 1$. Assume the measure of all basic cylinders of order $n_{k-1} + 1$ has been defined:
  - for each $1 \le \ell \le \ell_k$, define
$$\mu(J_{n_{k-1}+1+N\ell}(x)) = \left(\prod_{i=1}^{\ell} \frac{1}{q_N(w_i^{(k)})^{(\alpha_1+2)s}}\right) \cdot \mu(J_{n_{k-1}+1}(x)); \tag{3.10}$$
  - for each integer n with $n_{k-1} + 1 + (\ell-1)N < n < n_{k-1} + 1 + \ell N$ for some $1 \le \ell \le \ell_k$, define
$$\mu(J_n(x)) = \sum_{J_{n_{k-1}+1+\ell N}(x) \subset J_n(x)} \mu(J_{n_{k-1}+1+\ell N}(x));$$
  - for each $n = n_k$ and $n = n_k + 1$, define
$$\mu(J_{n_k}(x)) = \frac{1}{q_{n_k-1}^{\alpha_1}} \cdot \mu(J_{n_k-1}(x)), \quad \mu(J_{n_k+1}(x)) = \frac{1}{q_{n_k}^{\alpha_2}} \cdot \mu(J_{n_k}(x)); \tag{3.11}$$
  - define the measure of the basic cylinders of other orders as the summation of the measure of its offsprings to ensure the consistency of a measure.

Look at equation (3.10) for the measure of a basic cylinder of order $n_k + 1 + \ell N$ and its predecessor of order $n_k + 1 + (\ell-1)N$: the former has one more term than the latter, that

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is the term

$$\left(\frac{1}{q_N(w_\ell^{(k+1)})}\right)^{(\alpha_1+2)s},$$

which is uniformly bounded. Thus there is an absolute constant $c > 0$, such that for each integer n:

- when $n_k + 1 + (\ell - 1)N \le n \le n_k + 1 + \ell N$,

$$\mu(J_n(x)) \ge c \cdot \mu(J_{n_k+1+(\ell-1)N}(x)); \tag{3.12}$$

- when $n \ne n_k - 1$ and $n \ne n_k$,

$$\mu(J_{n+1}(x)) \ge c \cdot \mu(J_n(x)). \tag{3.13}$$

3.4. *Hölder exponent of μ: for basic cylinders.* We compare the measure with the length of $J_n(x)$.

(1) When $n = n_k - 1$. Recall equations (3.5) and (3.10) on the length and measure of J_{n_k-1}. It follows that

$$\mu(J_{n_k-1}) \le \prod_{i=1}^{\ell_k} \frac{1}{q_N(w_i^{(k)})^{(\alpha_1+2)s}} \le |J_{n_k-1}(x)|^{s/(1+\epsilon)} \le \left(\frac{1}{q_{n_k-1}^{\alpha_1+2}}\right)^{s/(1+\epsilon)}.$$

(2) When $n = n_k$. Recall equations (3.11) and (3.6).

$$\mu(J_{n_k}(x)) = \frac{1}{q_{n_k-1}^{\alpha_1}} \cdot \mu(J_{n_k-1}(x)) \le \frac{1}{q_{n_k-1}^{\alpha_1}} \cdot \left(\frac{1}{q_{n_k-1}^{\alpha_1+2}}\right)^{s/(1+\epsilon)} := \left(\frac{1}{q_{n_k-1}^{(\alpha_1+1)(\alpha_2+2)}}\right)^t$$
$$\le c|J_{n_k}(x)|^t \le c \cdot \left(\frac{1}{q_{n_k}^{\alpha_2+2}}\right)^t,$$

where t is chosen as

$$t = \frac{\alpha_1 + (\alpha_1+2)s/(1+\epsilon)}{(\alpha_1+1)(\alpha_2+2)}.$$

(3) When $n = n_k + 1$. Recall equations (3.11) and (3.7). Note that $0 \le t \le 1$.

$$\mu(J_{n_k+1}(x)) = \frac{1}{q_{n_k}^{\alpha_2}} \cdot \mu(J_{n_k}(x)) \le \frac{1}{q_{n_k}^{\alpha_2}} \cdot c \cdot \left(\frac{1}{q_{n_k}^{\alpha_2+2}}\right)^t$$
$$\le c\left(\frac{1}{q_{n_k}^{2\alpha_2+2}}\right)^t \le c_2|J_{n_k+1}(x)|^t \le c_2\left(\frac{1}{q_{n_k+1}^2}\right)^t.$$

(4) When $n = n_k + 1 + \ell N$ for some $1 \le \ell \le \ell_k$. Recall equations (3.5) and (3.10).

$$\mu(J_{n_k+1+\ell N}) = \prod_{i=1}^{\ell} \frac{1}{q_N(w_i^{(k+1)})^{(\alpha_1+2)s}} \cdot \mu(J_{n_k+1}(x))$$
$$\le c_2 \cdot \prod_{i=1}^{\ell} \frac{1}{q_N(w_i^{(k+1)})^{2s}} \cdot \left(\frac{1}{q_{n_k+1}^2}\right)^t \quad \text{(by neglecting } \alpha_1\text{)}.$$

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Recall equation (3.8) for the length of $J_{n_k+1+\ell N}$. It follows that

$$\mu(J_{n_k+1+\ell N}(x)) \le c_2|J_{n_k+1+\ell N}(x)|^{\min\{s/(1+\epsilon),t\}}.$$

(5) Remaining cases. Then we are in the case that $n_k + 1 < n < n_{k+1} - 1$. Let $1 \le \ell \le \ell_{k+1}$ be the integer such that $n_k + 1 + (\ell - 1)N < n < n_k + 1 + \ell N$. Recall equation (3.9). Then

$$\mu(J_n(x)) \le \mu(J_{n_k+1+(\ell-1)N}(x)) \le c_2|J_{n_k+1+(\ell-1)N}(x)|^{\min\{s/(1+\epsilon),t\}}$$
$$\le c_2 \cdot c \cdot |J_n(x)|^{\min\{s/(1+\epsilon),t\}}.$$

In summary, we have shown that for some absolute constant c_3, for any $n \ge 1$ and $x \in E$,

$$\mu(J_n(x)) \le c_3 \cdot |J_n(x)|^{\min\{s/(1+\epsilon),t\}}. \tag{3.14}$$

3.5. *Hölder exponent of μ: for a general ball.* Write

$$s_o = \min\left\{\frac{s}{1+\epsilon}, t\right\}.$$

Recall Lemma 3.2 about the relation of the gap and the length of the basic cylinders:

$$G_n(x) \ge \frac{1}{M} \cdot |J_n(x)|.$$

We consider the measure of a general ball $B(x, r)$ with $x \in E$ and r small. Let $n \ge 1$ be the integer such that

$$G_{n+1}(x) \le r < G_n(x).$$

Then the ball $B(x, r)$ can only intersect one basic cylinder of order n, that is, the basic cylinder $J_n(x)$, and so all the basic cylinders of order $n + 1$ which have non-empty intersection with $B(x, r)$ are all contained in $J_n(x)$.

Let k be the integer such that

$$n_{k-1} + 1 \le n < n_k + 1.$$

(1) When $n_{k-1} + 1 \le n < n_k - 1$. By equations (3.13) and (3.14), it follows that

$$\mu(B(x, r)) \le \mu(J_n(x)) \le c \cdot \mu(J_{n+1}(x)) \le c \cdot c_3 \cdot |J_{n+1}(x)|^{s_o}$$
$$\le c \cdot c_3 \cdot M \cdot (G_{n+1}(x))^{s_o} \le c \cdot c_3 \cdot M \cdot r^{s_o}.$$

(2) When $n = n_k - 1$. The ball $B(x, r)$ can only intersect the basic cylinder $J_{n_k-1}(x)$ of order $n_k - 1$. Now we estimate how many basic cylinders of order n_k are contained in $J_{n_k-1}(x)$ and intersected with the ball $B(x, r)$.

We write a general basic cylinder of order n_k contained in $J_{n_k-1}(x)$ as

$$J_{n_k}(u, a) \quad \text{with } q_{n_k-1}^{\alpha_1} \le a < 2q_{n_k-1}^{\alpha_1}.$$

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It is clear that for each a, the basic cylinder $J_{n_k}(u, a)$ is contained in the cylinder $I_{n_k}(u, a)$ and the latter interval is of length $1/q_{n_k}(q_{n_k} + q_{n_k-1})$ with

$$\frac{1}{q_{n_k-1}(u)^{2\alpha_1+2}} \geq \frac{1}{q_{n_k}(q_{n_k} + q_{n_k-1})} \geq \frac{1}{2^5} \cdot \frac{1}{q_{n_k-1}(u)^{2\alpha_1+2}}.$$

- When
$$r < \frac{1}{2^5} \cdot \frac{1}{q_{n_k-1}(u)^{2\alpha_1+2}}.$$
Then the ball $B(x, r)$ can intersect at most three cylinders $I_{n_k}(u, a)$ and so three basic cylinders $J_{n_k}(u, a)$. Note that all those basic cylinders are of the same μ-measure, thus
$$\mu(B(x, r)) \leq 3\mu(J_{n_k}(x)) \leq 3 \cdot c_3 \cdot |J_{n_k}(x)|^{s_o} \leq 3 \cdot c_3 \cdot M \cdot G_{n+1}(x)^{s_o} \leq 3 \cdot c_3 \cdot M \cdot r^{s_o}.$$
- When
$$r \geq \frac{1}{2^5} \cdot \frac{1}{q_{n_k-1}(u)^{2\alpha_1+2}}.$$
The number of cylinders $I_{n_k}(u, a)$ for which the ball $B(x, r)$ can intersect is at most
$$2^6 \cdot r \cdot q_{n_k-1}(u)^{2\alpha_1+2} + 2 \leq 2^7 \cdot r \cdot q_{n_k-1}(u)^{2\alpha_1+2},$$
so at most this number of basic cylinders of order n_k can intersect $B(x, r)$. Thus,

$$\begin{aligned}
\mu(B(x, r)) &\leq \min \left\{ \mu(J_{n_k-1}(x)), 2^7 \cdot r \cdot q_{n_k-1}(u)^{2\alpha_1+2} \cdot \left(\frac{1}{q_{n_k-1}^{\alpha_1}} \cdot \mu(J_{n_k-1}(x)) \right) \right\} \\
&\leq c_3 \cdot |J_{n_k-1}|^{s_o} \cdot \min\{1, 2^7 \cdot r \cdot q_{n_k-1}(u)^{\alpha_1+2}\} \\
&\leq c_3 \cdot \left(\frac{1}{q_{n_k-1}(u)^{\alpha_1+2}} \right)^{s_o} \cdot 1^{1-s_o} \cdot (2^7 \cdot r \cdot q_{n_k-1}(u)^{\alpha_1+2})^{s_o} \\
&= c_4 \cdot r^{s_o}.
\end{aligned}$$

(3) When $n = n_k$. By changing $n_k - 1$ and α_1 in case (2) to n_k and α_2 respectively and then following the same argument as in case (2), we can arrive at the same conclusion.

We conclude by mass distribution principle (Proposition 2.4) that

$$\dim_H E \geq \min \left\{ \frac{s}{1+\epsilon}, \frac{\alpha_1 + (\alpha_1 + 2)s/(1+\epsilon)}{(\alpha_1 + 1)(\alpha_2 + 2)} \right\}. \tag{3.15}$$

Recall equation (3.2) on $s = s_N(M)$. Letting $N \to \infty$ as then $M \to \infty$, we arrive at

$$\dim_H E(\alpha_1, \alpha_2) \geq \min \left\{ \frac{2}{\alpha_1 + 2}, \frac{\alpha_1 + 2}{(\alpha_1 + 1)(\alpha_2 + 2)} \right\}.$$

This finishes the proof.

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4. *Simple facts for $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$*

4.1. *The condition for $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ non-empty.* Recall that

$$\mathcal{G}(t_1)\backslash\mathcal{K}(t_2) = \{x \in [0, 1) : a_n(x)a_{n+1}(x) \geq q_n(x)^{t_1}, \text{ i.m. } n \in \mathbb{N}; \\ \text{and } a_{n+1}(x) < q_n(x)^{t_2} \text{ for all } n \in \mathbb{N} \text{ large}\}.$$

It is clear that if t_1 is very large and t_2 is very small, one must have $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2) = \emptyset$. So there should be some boundary value between t_1 and t_2 ensuring the non-empty of $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$.

LEMMA 4.1. *When $t_1 > t_2 + t_2/(1 + t_2)$, the set $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ is empty.*

Proof. It is sufficient to show that under the restriction that $a_{n+1} < q_n^{t_2}$ for all n large, one ultimately has

$$a_n a_{n+1} < q_n^{t_1}.$$

It should be easy to see that $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ is non-empty when $t_1 \leq t_2$. So in the following, we ask $t_1 > t_2$. Thus,

$$a_n a_{n+1} < q_n^{t_1} \Longleftarrow a_n < q_n^{t_1-t_2} \\ \Longleftarrow a_n < a_n^{t_1-t_2} q_{n-1}^{t_1-t_2} \Longleftarrow a_n^{1-t_1+t_2} < q_{n-1}^{t_1-t_2}.$$

This is obviously true if $t_1 - t_2 \geq 1$, so assume that $t_1 - t_2 < 1$. Let us continue the above argument.

$$a_n a_{n+1} < q_n^{t_1} \Longleftarrow q_{n-1}^{t_2(1-t_1+t_2)} < q_{n-1}^{t_1-t_2} \\ \Longleftarrow t_2(1 - t_1 + t_2) < t_1 - t_2 \Longleftrightarrow t_1 > t_2 + \frac{t_2}{1+t_2}.$$

In conclusion, we have shown the desired claim. □

5. *Hausdorff dimension of $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ when $t_2 < t_1 < t_2 + t_2/(1 + t_2)$*

5.1. *Lower bound.* First we give some rough words for finding a suitable subset of $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$. Initially, we separate the restriction posed on the product $a_n a_{n+1}$. This leads us to consider the following set:

$$F := \{x : a_n \asymp q_{n-1}^{\alpha_1}, a_{n+1} \asymp q_n^{\alpha_2}, \text{ i.m. } n \in \mathbb{N}, \text{ and } 1 \leq a_n \leq M \text{ for all other } n \in \mathbb{N}\}.$$

We hope that F is a subset of $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ and at the same time, the dimension of F should be as large as possible.

- It is clear that the smaller α_1, α_2 will result in a larger dimension of F. So, we may choose α_1, α_2 satisfying

$$q_{n-1}^{\alpha_1} q_n^{\alpha_2} = q_n^{t_1}.$$

Combining with $q_n \asymp a_n q_{n-1}$, one has that

$$q_{n-1}^{\alpha_1} q_{n-1}^{(1+\alpha_1)\alpha_2} = q_{n-1}^{(1+\alpha_1)t_1} \Leftrightarrow \alpha_1 + (1 + \alpha_1)\alpha_2 = (1 + \alpha_1)t_1 \\ \Leftrightarrow \alpha_2 = t_1 - \frac{\alpha_1}{1+\alpha_1}. \tag{5.1}$$

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- However, we need that $\alpha_1 < t_2$ and $\alpha_2 < t_2$ which gives the range of α_1, α_2. More precisely,

$$
\begin{aligned}
\text{`}\alpha_1 < t_2, \alpha_2 < t_2\text{'} &\iff \text{`}\alpha_1 < t_2,\ \alpha_2 = t_1 - \frac{\alpha_1}{1+\alpha_1} < t_2\text{'} \\
&\iff \text{`}\frac{t_1 - t_2}{1 - t_1 + t_2} < \alpha_1 < t_2\text{'} \quad \text{(expressed in the range of } \alpha_1\text{)} \qquad (5.2) \\
&\iff t_1 - \frac{t_2}{1+t_2} < \alpha_2 < t_2 \quad \text{(expressed in the range of } \alpha_2\text{).} \qquad (5.3)
\end{aligned}
$$

Now we give a rigorous argument in defining a subset of $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$. Recall the set defined in equation (3.4) with a suitable choice of the constants c in $E(\alpha_1, \alpha_2)$:

$$
\begin{aligned}
E = \{x : q_{n_k-1}(x)^{\alpha_1} \le a_{n_k}(x) < 2q_{n_k-1}(x)^{\alpha_1},\ 2^{2t_1} q_{n_k}(x)^{\alpha_2} \\
\le a_{n_k+1}(x) < 2^{2t_1+1} q_{n_k}(x)^{\alpha_2} \\
\text{for all } k \ge 1;\ \text{and } a_n(x) \in \{1, \ldots, M\} \text{ for other } n \in \mathbb{N}\}. \qquad (5.4)
\end{aligned}
$$

PROPOSITION 5.1. *For any pair (α_1, α_2) satisfying equations (5.1) and (5.2), for any integer sequence $\{n_k\}_{k\ge1}$, the set E in equation (5.4) is a subset of $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ and thus*

$$
\dim_{\mathrm{H}} \mathcal{G}(t_1)\backslash\mathcal{K}(t_2) \ge \min\left\{\frac{2}{\alpha_1+2}, \frac{\alpha_1+2}{(\alpha_1+1)(\alpha_2+2)}\right\}.
$$

Proof. The fact that $a_n(x)q_{n-1}(x) \le q_n(x) \le 2a_n(x)q_{n-1}(x)$ will be used. Take a general element $x \in E$. We check that $x \in \mathcal{G}(t_1)$ but $x \notin \mathcal{K}(t_2)$.

- $x \in \mathcal{G}(t_1)$. This is done by checking that

$$
a_{n_k}(x)a_{n_k+1}(x) \ge q_{n_k}(x)^{t_1} \quad \text{for all } k \ge 1. \qquad (5.5)
$$

  More precisely, on one hand,

$$
\begin{aligned}
a_{n_k}(x)a_{n_k+1}(x) &\ge q_{n_k-1}^{\alpha_1} \cdot 2^{2t_1} \cdot q_{n_k}^{\alpha_2} \ge 2^{2t_1} \cdot q_{n_k-1}^{\alpha_1}(a_{n_k}q_{n_k-1})^{\alpha_2} \\
&\ge 2^{2t_1} \cdot q_{n_k-1}^{\alpha_1} \cdot q_{n_k-1}^{(\alpha_1+1)\alpha_2}.
\end{aligned}
$$

  On the other hand,

$$
q_{n_k}^{t_1} \le (2a_{n_k}q_{n_k-1})^{t_1} \le 2^{2t_1} \cdot q_{n_k-1}^{(\alpha_1+1)t_1}.
$$

  Then the inequality in equation (5.5) follows by recalling the first equivalence in equation (5.1).
- $x \notin \mathcal{K}(t_2)$. This is clear since $\alpha_1 < t_2, \alpha_2 < t_2$ by equation (5.2).

The dimensional result follows directly by recalling the dimension of E in equation (3.15). □

We claim that the second term is the minimal one under the condition in equation (5.1).

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LEMMA 5.2. *Under the condition in equation (5.1), one has*

$$\min\left\{\frac{2}{2+\alpha_1}, \frac{2+\alpha_1}{(1+\alpha_1)(2+\alpha_2)}\right\} = \frac{2+\alpha_1}{(1+\alpha_1)(2+\alpha_2)}.$$

Proof. At first, rewrite the relationship between α_1 and α_2:

$$\alpha_2 = t_1 - 1 + \frac{1}{1+\alpha_1}, \text{ so } \frac{1}{1+\alpha_1} = \alpha_2 - t_1 + 1.$$

Thus,

$$\begin{aligned}\frac{2+\alpha_1}{(1+\alpha_1)(2+\alpha_2)} &= \frac{1}{(1+\alpha_1)(2+\alpha_2)} + \frac{1}{2+\alpha_2} \\ &= \frac{\alpha_2 - t_1 + 1}{\alpha_2+2} + \frac{1}{2+\alpha_2} = 1 - \frac{t_1}{2+\alpha_2}.\end{aligned}$$

As a consequence,

$$\begin{aligned}&\frac{2}{2+\alpha_1} \geq \frac{2+\alpha_1}{(1+\alpha_1)(2+\alpha_2)} \iff \frac{2}{2+\alpha_1} \geq 1 - \frac{t_1}{2+\alpha_2} \\ \iff& \frac{t_1}{2+\alpha_2} \geq \frac{\alpha_1}{2+\alpha_1} \iff t_1\Big(1+\frac{2}{\alpha_1}\Big) \geq 2+\alpha_2 = t_1 + 1 + \frac{1}{\alpha_1+1} \\ \iff& \frac{2t_1}{\alpha_1} \geq 1 + \frac{1}{1+\alpha_1} \iff 2t_1 \geq \alpha_1 + \frac{\alpha_1}{\alpha_1+1}.\end{aligned}$$

Let

$$f(x) = x + \frac{x}{1+x} = x + 1 - \frac{1}{1+x}, \quad x \in [0, t_2].$$

Clearly f is increasing with respect to x and when $x = t_2$, it attains its maximal value

$$t_2 + \frac{t_2}{1+t_2}.$$

So, what we need is to show that

$$\begin{aligned}&2t_1 \geq t_2 + \frac{t_2}{1+t_2} \Longleftarrow 2t_2 \geq t_2 + \frac{t_2}{1+t_2} \\ \Longleftarrow\ & 2 \geq 1 + \frac{1}{1+t_2},\end{aligned}$$

which is clearly true. □

As a consequence,

$$\begin{aligned}\dim_{\mathrm{H}} \mathcal{G}(t_1)\backslash\mathcal{K}(t_2) &\geq \sup\left\{1 - \frac{t_1}{2+\alpha_2} : t_1 - \frac{t_2}{1+t_2} \leq \alpha_2 \leq t_2\right\} \\ &= 1 - \frac{t_1}{2+t_2}.\end{aligned}$$

In other words, the supremum is achieved at $\alpha_2 = t_2$.

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5.2. *Upper bound.* Recall that the lower bound of $\dim_{\mathrm H} \mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ given above is attained at

$$\alpha_2 = t_2, \quad \alpha_1 = \frac{t_1 - t_2}{1 + t_2 - t_1}.$$

LEMMA 5.3. *For any $x \in [0, 1)$,*

$$a_n(x)a_{n+1}(x) \geq q_n^{t_1}(x),\ a_{n+1}(x) < q_n^{t_2}(x) \Longrightarrow a_n(x) \geq q_{n-1}(x)^{(t_1-t_2)/(1+t_2-t_1)}.$$

Proof.

$$\begin{aligned} q_n^{t_1} \leq a_n a_{n+1} \leq a_n q_n^{t_2} &\Longrightarrow q_n^{t_1-t_2} \leq a_n \Longrightarrow a_n^{t_1-t_2} q_{n-1}^{t_1-t_2} \leq a_n \\ &\Longrightarrow q_{n-1}^{t_1-t_2} \leq a_n^{1-t_1+t_2} \Longrightarrow a_n \geq q_{n-1}^{(t_1-t_2)/(1+t_2-t_1)}. \end{aligned}$$

□

This lemma almost convinces us that the lower bound given above is the right dimension of $\dim_{\mathrm H} \mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$. Denote $\alpha_1 = (t_1 - t_2)/(1 + t_2 - t_1)$. Lemma 5.3 implies that

$$\mathcal{G}(t_1)\backslash\mathcal{K}(t_2) \subset \bigg\{x : a_n(x) \geq q_{n-1}(x)^{(t_1-t_2)/(1+t_2-t_1)}, a_{n+1}(x) \geq \frac{q_n(x)^{t_1}}{a_n(x)}, \text{i.m. } n \in \mathbb{N}\bigg\} := \mathcal{G}.$$

Fix $s > 1 - t_1/(2 + t_2)$. At first, it is easy to check that

$$\begin{aligned} s(1+t_1) > 1 &\Longleftrightarrow s > \frac{1}{1+t_1} \Longleftrightarrow 1 - \frac{t_1}{2+t_2} > \frac{1}{1+t_1} \\ &\Longleftrightarrow \frac{t_1}{1+t_1} > \frac{t_1}{2+t_2} \Longleftrightarrow 2 + t_2 > 1 + t_1 \\ &\Longleftrightarrow 1 + t_2 > t_1. \end{aligned}$$

The last inequality is clearly true since we are in the case that

$$t_1 \leq t_2 + \frac{t_2}{1+t_2}.$$

Now we search an upper bound of the dimension of $\mathcal{G}$. Still due to the limsup nature, there is a natural cover of $\mathcal{G}$. For any $a_1, \ldots, a_n \in \mathbb{N}$, define

$$J_n(a_1, \ldots, a_n) = \bigcup_{a_{n+1} \geq q_n^{t_1}/a_n} I_{n+1}(a_1, \ldots, a_n, a_{n+1}),$$

which is of length

$$|J_n(a_1, \ldots, a_n)| \asymp \frac{a_n}{q_n^{2+t_1}} \asymp \frac{1}{q_{n-1}^{2+t_1} a_n^{1+t_1}}.$$

It is clear that

$$\mathcal{G} = \bigcup_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_{n-1}\in\mathbb{N}} \bigcup_{a_n \geq q_{n-1}^{\alpha_1}} J_n(a_1, \ldots, a_n).$$

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Thus, the s-dimensional Hausdorff measure of $\mathcal{G}$ can be estimated as

$$\mathcal{H}^s(\mathcal{G}) \le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_{n-1}} \sum_{a_n \ge q_{n-1}^{\alpha_1}} \bigg(\frac{1}{q_{n-1}^{2+t_1} a_n^{1+t_1}}\bigg)^s$$
$$\ll \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_{n-1}} \bigg(\frac{1}{q_{n-1}^{2+t_1}}\bigg)^s \bigg(\frac{1}{q_{n-1}^{\alpha_1[(1+t_1)s-1]}}\bigg),$$

where we used the fact that $s(1+t_1) > 1$. The above series converges if

$$(2+t_1)s + \alpha_1[(1+t_1)s - 1] > 2 \iff (2+t_1)s + \alpha_1(1+t_1)s > \alpha_1 + 2$$
$$\iff s > \frac{\alpha_1 + 2}{2 + t_1 + \alpha_1(1+t_1)}.$$

Substituting the choice of α_1 into the last term gives that

$$\frac{\alpha_1+2}{2+t_1+\alpha_1(1+t_1)} = \frac{(t_1-t_2)/(1+t_2-t_1)+2}{1+(1+t_1)(1+\alpha_1)} = \frac{(1/(1+t_2-t_1))+1}{1+(1+t_1)\frac{1}{1+t_2-t_1}}$$
$$= \frac{2+t_2-t_1}{1+t_2-t_1+1+t_1} = \frac{2+t_2-t_1}{2+t_2}$$
$$= 1 - \frac{t_1}{2+t_2}.$$

This is what we choose about s. As a conclusion, we have shown that

$$\dim_{\mathrm{H}} \mathcal{G}(t_1)\backslash\mathcal{K}(t_2) \le \dim_{\mathrm{H}} \mathcal{G} \le 1 - \frac{t_1}{2+t_2}.$$

6. *Hausdorff dimension of $\mathcal{G}(t_1)\backslash\mathcal{K}(t_2)$ when $t_1 \le t_2$*

(1) When $t_1 = t_2$. In this case, for any t' with $t_2 + t_2/(1+t_2) > t' > t_1 = t_2$, we have that

$$\mathcal{G}(t')\backslash\mathcal{K}(t_2) \subset \mathcal{G}(t_1)\backslash\mathcal{K}(t_2).$$

Thus

$$\dim_{\mathrm{H}} \mathcal{G}(t_1)\backslash\mathcal{K}(t_2) \ge 1 - \frac{t'}{2+t_2},$$

then letting $t' \to t_1$ gives the lower bound. The upper bound is clear, since

$$\mathcal{G}(t_1)\backslash\mathcal{K}(t_2) \subset \mathcal{G}(t_1).$$

Thus we have

$$\dim_{\mathrm{H}} \mathcal{G}(t_1)\backslash\mathcal{K}(t_2) = \frac{2}{t_1+2}.$$

(2) When $t_1 < t_2$. Take $t_2' = t_1$, that is, we decrease t_2 to t_2'. Then

$$\mathcal{G}(t_1)\backslash\mathcal{K}(t_2') \subset \mathcal{G}(t_1)\backslash\mathcal{K}(t_2).$$

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Then we are in case (1). So,

$$\dim_{\rm H} \mathcal{G}(t_1)\backslash\mathcal{K}(t_2) \geq \frac{2}{t_1+2}.$$

The upper bound of the dimension is trival since it is always bounded by $\dim_{\rm H} \mathcal{G}(t_1)$.

7. *The two examples*

Assume that

$$t_1 = t_2 + \frac{t_2}{1+t_2}.$$

• Example 1.

$$E_1 = \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq q_n(x)^{t_1}, \text{ i.m. } n \in \mathbb{N}, \\ a_{n+1}(x) < q_n(x)^{t_2} \quad \text{for all } n \in \mathbb{N} \text{ large}\}.$$

We show that E_1 is an empty set. The proof is rather the same as that for case $t_1 > t_2 + t_2/(1+t_2)$. Let $x \in [0,1)$ and assume that for all $n \gg 1$, $a_{n+1}(x) < q_n(x)^{t_2}$. Then

$$\begin{aligned} a_n a_{n+1} < q_n^{t_1} &\Longleftarrow a_n(x)\cdot q_n(x)^{t_2} < q_n(x)^{t_1} \\ &\Longleftarrow a_n(x) < q_n(x)^{t_1-t_2} \Longleftarrow a_n(x) \leq (a_n(x)q_{n-1}(x))^{t_1-t_2} \\ &\Longleftarrow a_n(x)^{1-(t_1-t_2)} \leq q_{n-1}(x)^{t_1-t_2} \Longleftarrow q_{n-1}(x)^{t_2(1-t_1+t_2)} \leq (q_{n-1}(x))^{t_1-t_2} \\ &\Longleftarrow 1 \leq 1 \end{aligned}$$

by noticing that

$$t_2(1-t_1+t_2) = t_1 - t_2 \Leftrightarrow t_1 = t_2 + \frac{t_2}{1+t_2}.$$

• Example 2.

$$E_2 = \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq 4^{-t_1} q_n(x)^{t_1}, \text{ i.m. } n \in \mathbb{N}, \\ a_{n+1}(x) \leq 3q_n(x)^{t_2}, \text{ for all } n \in \mathbb{N} \text{ large}\}.$$

Choose $\alpha_2 = t_2$ and α_1 such that $\alpha_2 = t_1 - \alpha_1/(1+\alpha_1)$ (in fact, $\alpha_1 = t_2$ too). Then consider the set

$$F := \{x : q_{n-1}(x)^{\alpha_1} \leq a_n(x) < 2q_{n-1}(x)^{\alpha_1}, q_n(x)^{\alpha_2} \leq a_{n+1}(x) < 2q_n(x)^{\alpha_2}, \text{ i.m. } n \in \mathbb{N}; \\ \text{and } 1 \leq a_n(x) \leq M \text{ for all other } n \in \mathbb{N}\}.$$

We show that F is a subset of E_2. Let $x \in F$. At first,

$$q_n(x) \leq 2a_n(x)q_{n-1}(x) \leq 4q_{n-1}(x)^{1+\alpha_1} \Longrightarrow q_{n-1}(x) \geq (q_n(x)/4)^{1/(1+\alpha_1)}.$$

Therefore,

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- the first requirement in E_2:

$$a_n(x)a_{n+1}(x) \geq q_{n-1}(x)^{\alpha_1} q_n(x)^{\alpha_2} \geq \left(\frac{q_n(x)}{4}\right)^{\alpha_1/(1+\alpha_1)} \cdot q_n(x)^{\alpha_2}$$
$$\geq \left(\frac{q_n(x)}{4}\right)^{\alpha_2+\alpha_1/(1+\alpha_1)} = 4^{-t_1} q_n(x)^{t_1}.$$

- The second requirement in E_2: the relation between t_1 and t_2 and the choice of α_1, α_2 yield that $\alpha_1 = \alpha_2 = t_2$. So it is clear

$$a_{n+1}(x) < 2q_n(x)^{\alpha_2} \leq 3q_n(x)^{t_2}, \quad a_n(x) < 2q_{n-1}(x)^{\alpha_1} \leq 3q_{n-1}(x)^{t_2}.$$

This means that F is a subset of E, so we have that

$$\dim_{\mathrm{H}} E \geq 1 - \frac{t_1}{2+t_2}.$$

The upper bound of the dimension of E_2 is clear by the result for the case $t_1 < t_2 + t_2/(1+t_2)$, since E_2 is enlarged if we decrease the value of t_1.

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