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THE MEAN-VALUE PROPERTY AND (α, β)-HARMONICITY

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Abstract

In this article we consider whether, for integrable functions on the unit ball of $\mathbb{C}^n$, the mean-value property implies (α, β)-harmonicity. We find that the answer is affirmative when $0 < n + \alpha + \beta \le \rho_0$, but is negative when $n + \alpha + \beta > \rho_0$. Here ρ_0 is a constant between 11.025 and 11.069.

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1. Introduction

Let B_n be the open unit ball in $\mathbb{C}^n$, and let S and $\bar{B}_n$ be its boundary and closure. The group of all one-to-one holomorphic maps of B_n onto B_n will be denoted by $\mathrm{Aut}(B_n)$. This group is generated by the unitary transformations and the involutions φ_a of the form

$$\varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z, a \rangle} \quad \forall z \in B_n$$

where P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one-dimensional subspace generated by a and $Q_a z := z - P_a z$, for all $a \in B_n$.

The invariant Laplacian or Bergman Laplacian on B_n is defined, for all $f \in C^2(B_n)$, by

$$\widetilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0) = 4(1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - z_i \bar{z}_j) D_i \bar{D}_j f,$$

where Δ is the ordinary Laplacian and $D_j = \partial/\partial z_j$. It commutes with every $\psi \in \mathrm{Aut}(B_n)$ in the sense that

$$(\widetilde{\Delta} f) \circ \psi = \widetilde{\Delta}(f \circ \psi).$$

The $\mathcal{M}$-harmonic functions in B_n are those for which $\widetilde{\Delta} f = 0$.

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Let ν denote Lebesgue measure on $\mathbb{C}^n$ normalized so that $\nu(B_n) = 1$. If f is $\mathcal{M}$-harmonic, then its mean value over spheres of center 0 and radius r at most 1 is $f(0)$ (see [17, p. 51]). If f is also in $L^1(B_n, \nu)$, then

$$\int_{B_n} f \circ \psi \, d\nu = f(\psi(0)) \quad \forall \psi \in \mathrm{Aut}(B_n). \tag{1.1}$$

We can also describe this property by saying that f satisfies the invariant mean-value property.

In 1993 Ahern *et al.* (see [3]) proved a surprising result.

Theorem 1. *The invariant mean-value property (1.1) characterizes $\mathcal{M}$-harmonic functions in $L^1(B_n, \nu)$ if and only if $n \le 11$.*

This paper is motivated by an attempt to understand the reason for the curious dependence on the dimension in Theorem 1. To this end we generalize Theorem 1 to (α, β)-harmonic functions.

Generalized $\mathcal{M}$-harmonic functions, or (α, β)-harmonic functions, are the functions annihilated by the differential operator $\Delta_{\alpha,\beta}$, given by

$$\Delta_{\alpha,\beta} = (1 - |z|^2)\Big\{\sum_{i,j=1}^{n} (\delta_{ij} - z_i \bar{z}_j) D_i \bar{D}_j + \alpha R + \beta \bar{R} - \alpha\beta\Big\},$$

where $\alpha, \beta \in \mathbb{C}$ and R is the radial derivative, given by $R = \sum_j z_j D_j$. If $\alpha = \beta = 0$, then the operator $4\Delta_{0,0}$ is just the invariant Laplacian $\widetilde{\Delta}$. The operator $\Delta_{\alpha,\alpha}$ is the Laplacian for the Bergman space with weight $(1 - |z|^2)^\alpha$. The more general operators $\Delta_{\alpha,\beta}$ were introduced by Geller [13]. They appear in a natural way when we consider certain derivatives of $\mathcal{M}$-harmonic functions. It was proved in [2] that $\Delta_{\alpha,\beta} u = 0$ implies that $\Delta_{\alpha,\beta-1}(Ru - \beta u) = 0$. The operators $\Delta_{\alpha,\beta}$ also appear when computing the Laplace–Beltrami operator on forms.

We now give the definition of the mean-value property to which the title of our paper refers. To make this notion meaningful, as well as for simplicity, we will from now on assume that α and β are two real numbers satisfying the conditions that

$$n + \alpha + \beta > 0, \quad n + \alpha > 0, \quad n + \beta > 0.$$

For notational simplicity, we write

$$\rho := n + \alpha + \beta.$$

It was shown in [13, Theorem 1.1] that if $\Delta_{\alpha,\beta} f = 0$ in B_n and $0 < r < 1$, then

$${}_2F_1(-\alpha, -\beta; n; r^2) f(0) = \int_S f(r\zeta) \, d\sigma(\zeta), \tag{1.2}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function (see Section 2 for the definition) and σ denotes the normalized surface-area measure on the unit sphere S.

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We multiply (1.2) by $2nr^{2n-1}$ and integrate over $[0, 1)$ to get

$$C_0 \int_{B_n} f(z)\,d\nu(z) = f(0) \tag{1.3}$$

and

$$C_0 = \left\{2n \int_0^1 r^{2n-1}\,{}_2F_1(-\alpha, -\beta; n; r^2)\,dr\right\}^{-1} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

Here the last equality follows by (2.1) in Section 2. It is proved in [2, Lemma 1.2] that

$$\Delta_{\alpha,\beta}[(f \circ \varphi_a)h_a^{(\alpha,\beta)}] = [(\Delta_{\alpha,\beta} f) \circ \varphi_a]h_a^{(\alpha,\beta)} \tag{1.4}$$

for each $a \in B_n$. Here

$$h_a^{(\alpha,\beta)}(z) = (1 - \langle a, z\rangle)^{\alpha}(1 - \langle z, a\rangle)^{\beta} \quad \forall z \in B_n.$$

This implies that $(f \circ \varphi_a)h_a^{(\alpha,\beta)}$ is also (α, β)-harmonic. Thus applying (1.3) to $(f \circ \varphi_a)h_a^{(\alpha,\beta)}$ shows that

$$f(a) = C_0 \int_{B_n} (f \circ \varphi_a)h_a^{(\alpha,\beta)}\,d\nu \quad \forall a \in B_n. \tag{1.5}$$

This is the mean-value property to which the title of our paper refers. This property is invariant in the sense that $(f \circ \varphi_a)h_a^{(\alpha,\beta)}$ satisfies this property for each $a \in B_n$ whenever f satisfies the property.

The question again arises as to whether this invariant mean-value property characterizes the (α, β)-harmonic functions in $L^1(B_n, \nu)$. In our main result, ρ_0 is a constant between 11.025 and 11.069, which is made precise in Definition 4.8.

Theorem 1.1. *Suppose that $n + \alpha + \beta > 0$, $n + \alpha > 0$ and $n + \beta > 0$. For integrable functions in B_n the mean-value property (1.5) implies that (α, β)-harmonicity if and only if $n + \alpha + \beta \le \rho_0$.*

The proof of the above theorem follows the main lines of [3]. However our study of Berezin-type transforms is interesting in its own right. These results are included in Section 3. In Section 2, we review some of the standard facts on Möbius transformations and the Gauss hypergeometric function and establish a few elementary lemmas. Section 4 is devoted to the study of the eigenspaces of $\Delta_{\alpha,\beta}$. Finally we complete the proof of our main result in Section 5.

2. Preliminaries

We begin by summarizing some properties of the mapping φ_a which will be used later.

Lemma 2.1. *For all $a \in B_n$, the mapping φ_a has the following properties.*

(i) $\varphi_a(0) = a$ *and* $\varphi_a(a) = 0$.

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(ii) *The mapping* φ_a *is an involution; that is,* $\varphi_a \circ \varphi_a = \mathrm{id}$, *where* id *is the identity mapping.*

(iii) *For all* $z, w \in \bar{B}_n$,

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}.$$

(iv) *The (real) Jacobian of* φ_a *is given by*

$$(J_R\, \varphi_a)(z) = \frac{(1 - |a|^2)^{n+1}}{|1 - \langle z, a \rangle|^{2(n+1)}}.$$

A number of special functions will appear in this paper. We use the classical notation ${}_2F_1(a, b; c; z)$ to denote the hypergeometric series

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

where $c \neq 0, -1, -2, \ldots$, $(a)_0 = 1$, and for all positive integers k,

$$(a)_k = a(a+1) \cdots (a+k-1).$$

We list the following formulas for easy reference (see [4, Ch. 2]):

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2.1}$$

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c; z), \tag{2.2}$$

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}\, {}_2F_1(a, b; \lambda; tz)\, dt, \tag{2.3}$$

where $\mathrm{Re}\, c > \mathrm{Re}\, \lambda > 0$ and $z \notin [1, +\infty)$.

Lemma 2.2. *Suppose that* $\mathrm{Re}\, c > 0$, $\mathrm{Re}\, \delta > 0$ *and* $\mathrm{Re}(c - a - b + \delta) > 0$. *Then*

$$\int_0^1 t^{c-1}(1-t)^{\delta-1}\, {}_2F_1(a, b; c; t)\, dt = \frac{\Gamma(c)\Gamma(\delta)\Gamma(c-a-b+\delta)}{\Gamma(c-a+\delta)\Gamma(c-b+\delta)}. \tag{2.4}$$

Proof. Note that, under our hypotheses, both sides of (2.3) are continuous functions of z at 1. The lemma then follows by letting z tend to 1 and applying (2.1). $\square$

Lemma 2.3. *For all* $z \in B_n$, $s, t \in \mathbb{R}$ *and* $c > -1$,

$$\int_S \frac{d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^s (1 - \langle \zeta, z \rangle)^t} = {}_2F_1(s, t; n; |z|^2) \tag{2.5}$$

and

$$\int_{B_n} \frac{(1 - |w|^2)^c\, dv(w)}{(1 - \langle z, w \rangle)^s (1 - \langle w, z \rangle)^t} = \frac{\Gamma(n+1)\Gamma(1+c)}{\Gamma(n+1+c)}\, {}_2F_1(s, t; n+1+c; |z|^2). \tag{2.6}$$

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Proof. It is well known that, if $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and $\gamma \in \mathbb{R}$, then

$$(1-\lambda)^{-\gamma} = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} \lambda^k.$$

We also recall that

$$\int_S \langle z, \zeta \rangle^k \overline{\langle z, \zeta \rangle}^\ell \, d\sigma(\zeta) = \delta_{k\ell} \frac{k!}{(n)_k} |z|^{2k}$$

for all nonnegative integers k and ℓ. Thus

$$\begin{aligned}
\int_S \frac{d\sigma(\zeta)}{(1-\langle z, \zeta \rangle)^s (1-\langle \zeta, z \rangle)^t} &= \int_S \bigg\{ \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \langle z, \zeta \rangle^k \bigg\} \bigg\{ \sum_{\ell=0}^{\infty} \frac{(t)_\ell}{\ell!} \langle \zeta, z \rangle^\ell \bigg\} \, d\sigma(\zeta) \\
&= \sum_{k,\ell=0}^{\infty} \frac{(s)_k}{k!} \frac{(t)_\ell}{\ell!} \int_S \langle z, \zeta \rangle^k \overline{\langle z, \zeta \rangle}^\ell \, d\sigma(\zeta) \\
&= \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \frac{(t)_k}{k!} \frac{k!}{(n)_k} |z|^{2k} \\
&= {}_2F_1(s, t; n; |z|^2).
\end{aligned}$$

We may now use polar coordinates ($w = r\zeta$) and (2.5) to obtain

$$\int_{B_n} \frac{(1-|w|^2)^c \, dv(w)}{(1-\langle z, w \rangle)^s (1-\langle w, z \rangle)^t} = 2n \int_0^1 r^{2n-1} (1-r^2)^c \, {}_2F_1(s, t; n; r^2|z|^2) \, dr.$$

Now (2.6) follows after an application of (2.3). □

Lemma 2.4. *For all $k \in \mathbb{Z}^+$,*

$$\frac{\Gamma(n+\alpha+1+k)\Gamma(n+\beta+1+k)}{\Gamma^2(n+\frac{1}{2}(\alpha+\beta)+1+k)} \leq \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma^2(n+\frac{1}{2}(\alpha+\beta)+1)}. \tag{2.7}$$

Proof. The logarithmic derivative of the gamma function, ψ, which is defined by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is called the psi or digamma function in the literature. Its second derivative satisfies

$$\psi''(x) = -2 \sum_{j=0}^{\infty} \frac{1}{(x+j)^3} < 0.$$

Hence ψ is concave on $[0, \infty)$. Thus

$$\psi(n+\alpha+1+x) + \psi(n+\beta+1+x) - 2\psi\Big(n + \frac{\alpha+\beta}{2} + 1 + x\Big) \leq 0.$$

This implies that the function

$$x \longmapsto \frac{\Gamma(n+\alpha+1+x)\Gamma(n+\beta+1+x)}{\Gamma^2(n+\frac{1}{2}(\alpha+\beta)+1+x)}$$

is decreasing on $[0, \infty)$, and our result follows. $\square$

3. Berezin-type transforms $\mathscr{B}_k$

For all nonnegative integers k, define the measure ν_k on B_n by

$$d\nu_k(w) = (1-|w|^2)^k \, d\nu(w).$$

For all $f \in L^1(\nu_k)$, the Berezin-type transform $\mathscr{B}_k f$ of f is defined by

$$\mathscr{B}_k f(z) = C_k \int_{B_n} f(\varphi_z(w)) h_z^{(\alpha,\beta)}(w) \, d\nu_k(w) \quad \forall z \in B_n, \tag{3.1}$$

where

$$C_k = \frac{\Gamma(n+\alpha+k+1)\Gamma(n+\beta+k+1)}{\Gamma(n+1)\Gamma(k+1)\Gamma(n+\alpha+\beta+k+1)}.$$

Hence we can rephrase the definition of the mean-value property in the form $\mathscr{B}_0 f = f$. It will be helpful to recast our mean-value property using the integral operator $\mathscr{B}_0$.

If we replace w by $\varphi_z(w)$ in (3.1), we obtain a second formula for $\mathscr{B}_k f$, namely

$$\mathscr{B}_k f(z) = C_k \int_{B_n} \frac{(1-|z|^2)^{n+\alpha+\beta+k+1}(1-|w|^2)^k}{(1-\langle z, w\rangle)^{n+\alpha+k+1}(1-\langle w, z\rangle)^{n+\beta+k+1}} f(w) \, d\nu(w). \tag{3.2}$$

Lemma 3.1. *Suppose that $a, z, w \in B_n$. Then*

$$h_{\varphi_z(a)}^{(\alpha,\beta)}(w) h_a^{(\alpha,\beta)}(z) = h_a^{(\alpha,\beta)}(\varphi_z(w)) h_z^{(\alpha,\beta)}(w). \tag{3.3}$$

Proof. It suffices to show that

$$(1-\langle \varphi_z(a), w\rangle)(1-\langle a, z\rangle) = (1-\langle a, \varphi_z(w)\rangle)(1-\langle z, w\rangle),$$

which follows easily from Lemma 2.1. $\square$

Now we prove the 'Möbius invariance' of the Berezin-type transforms.

Proposition 3.2. *Let $k \geq 0$. If $f \in L^1(\nu_k)$ and $a \in B_n$, then*

$$\mathscr{B}_k[(f \circ \varphi_a) h_a^{(\alpha,\beta)}] = [(\mathscr{B}_k f) \circ \varphi_a] h_a^{(\alpha,\beta)}. \tag{3.4}$$

Proof. Note first that, if $f \in L^1(\nu_k)$, then $(f \circ \varphi_a) h_a^{(\alpha,\beta)} \in L^1(\nu_k)$. For all $z \in B_n$, the map $\varphi_{\varphi_a(z)} \circ \varphi_a \circ \varphi_z$ is an automorphism of B_n that fixes 0. Hence it is a unitary transformation (see, for example [17, Theorem 2.2.5]), which we denote by U.

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Since $\varphi_{\varphi_a(z)}$ and φ_z are involutions,

$$\varphi_a \circ \varphi_z = \varphi_{\varphi_a(z)} \circ U \tag{3.5}$$

and

$$U \circ \varphi_z(a) = \varphi_a(z). \tag{3.6}$$

Thus

$$\begin{aligned}
\mathscr{B}_k[(f \circ \varphi_a)h_a^{(\alpha,\beta)}](z) &= C_k \int_{B_n} f(\varphi_a(\varphi_z(w)))h_a^{(\alpha,\beta)}(\varphi_z(w))h_z^{(\alpha,\beta)}(w)\, d\nu_k(w) \\
&= C_k \int_{B_n} f(\varphi_{\varphi_a(z)}(Uw))h_{\varphi_z(a)}^{(\alpha,\beta)}(w)h_a^{(\alpha,\beta)}(z)\, d\nu_k(w) \\
&= C_k\Big\{\int_{B_n} f(\varphi_{\varphi_a(z)}(w))h_{U\varphi_z(a)}^{(\alpha,\beta)}(w)\, d\nu_k(w)\Big\}h_a^{(\alpha,\beta)}(z) \\
&= (\mathscr{B}_k f)(\varphi_a(z))h_a^{(\alpha,\beta)}(z),
\end{aligned}$$

where, in the second inequality, we use (3.5) and (3.3); in the third equality, we use the rotation invariance of ν_k and the inner product $\langle\cdot,\cdot\rangle$; and in the fourth inequality we use (3.6) and (3.1). □

PROPOSITION 3.3. *If $f \in L^\infty(B_n, \nu)$, then for every nonnegative integer k,*

$$\|\mathscr{B}_k f\|_\infty \le \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma^2(n+\frac{1}{2}(\alpha+\beta)+1)}\|f\|_\infty. \tag{3.7}$$

PROOF. It follows immediately from (3.2) that

$$\|\mathscr{B}_k f\|_\infty \le C_k \sup_{z\in B_n}\Big\{(1-|z|^2)^{\rho+k+1}\int_{B_n}\frac{(1-|w|^2)^k}{|1-\langle z,w\rangle|^{\rho+n+2k+2}}\, d\nu(w)\Big\}\|f\|_\infty.$$

By applying (2.6) and (2.2), we may write the expression in braces in the form

$$\frac{\Gamma(n+1)\,\Gamma(k+1)}{\Gamma(n+k+1)}\,{}_2F_1\Big(-\frac{\alpha+\beta}{2}, -\frac{\alpha+\beta}{2}; n+k+1; |z|^2\Big).$$

Note that the above hypergeometric function is increasing on the interval $[0, 1)$ since its Taylor coefficients are all positive. Thus this hypergeometric function is bounded above by

$${}_2F_1\Big(-\frac{1}{2}(\alpha+\beta), -\frac{1}{2}(\alpha+\beta); n+k+1; 1\Big) = \frac{\Gamma(n+1+k)\Gamma(n+\alpha+\beta+1+k)}{\Gamma^2(n+\frac{1}{2}(\alpha+\beta)+1+k)}.$$

Thus

$$\|\mathscr{B}_k f\|_\infty \le \frac{\Gamma(n+\alpha+1+k)\Gamma(n+\beta+1+k)}{\Gamma^2(n+\frac{1}{2}(\alpha+\beta)+1+k)}\|f\|_\infty.$$

Now (3.7) follows from the above equality and (2.7). □

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Proposition 3.4. *Let k be a strictly positive integer. Then $\mathscr{B}_k$ is a bounded linear operator on $L^1(\nu)$, and*

$$\|\mathscr{B}_k\| \le (\rho+2)\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma^2(n+\frac{1}{2}(\alpha+\beta)+1)}.$$

Proof. Let $f \in L^1(\nu)$. It follows immediately from (3.2) that

$$\|\mathscr{B}_k f\|_1 \le C_k \sup_{w\in B_n}\Big\{(1-|w|^2)^k \int_{B_n} \frac{(1-|z|^2)^{\rho+1+k}}{|1-\langle z,w\rangle|^{\rho+n+2k+2}}\, d\nu(z)\Big\}\|f\|_1.$$

After the change of variables $z \mapsto \varphi_w(z)$, the integral becomes

$$\int_{B_n}\Big(\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}\Big)^{\rho+1+k}\Big(\frac{|1-\langle z,w\rangle|}{1-|w|^2}\Big)^{\rho+n+2+2k}\Big(\frac{1-|w|^2}{|1-\langle z,w\rangle|^2}\Big)^{n+1} d\nu(z)$$
$$= (1-|w|^2)^{-k}\int_{B_n}\frac{(1-|z|^2)^{\rho+k+1}}{|1-\langle z,w\rangle|^{\rho+n+2}}\, d\nu(z).$$

Hence

$$\begin{aligned}\|\mathscr{B}_k\| &\le C_k \sup_{w\in B_n}\Big\{\int_{B_n}\frac{(1-|z|^2)^{\rho+k+1}}{|1-\langle z,w\rangle|^{\rho+n+2}}\, d\nu(z)\Big\}\\ &\le C_k\frac{\Gamma(n+1)\Gamma(\rho+k+2)}{\Gamma(\rho+n+k+2)}\\ &\qquad\times \sup_{w\in B_n} {}_2F_1\Big(\frac{1}{2}(\rho+n+2), \frac{1}{2}(\rho+n+2); \rho+n+k+2; |w|^2\Big).\end{aligned}$$

Applying a similar argument to that in the proof of Proposition 3.3, we obtain

$$\begin{aligned}\|\mathscr{B}_k\| &\le \Big(1+\frac{\rho+1}{k}\Big)\frac{\Gamma(n+\alpha+1+k)\Gamma(n+\beta+1+k)}{\Gamma^2(\frac{1}{2}(n+\rho)+1+k)}\\ &\le (\rho+2)\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma^2(\frac{1}{2}(n+\rho)+1)},\end{aligned}$$

where the last inequality follows by (2.7). □

Note that $\mathscr{B}_0$ does not carry $L^1(\nu)$ into $L^1(\nu)$ when $\rho > 0$ because

$$\int_{B_n}\frac{(1-|z|^2)^{\rho+1}}{|1-\langle z,w\rangle|^{\rho+n+2}}\, d\nu(z)$$

tends to ∞ as $|w|$ tends to 1. However we do have the following result, whose proof is similar to that of Proposition 3.4 and is omitted.

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Proposition 3.5. *The operator $\mathscr{B}_0$ is a bounded linear operator from $L^1(\nu)$ to $L^1(\nu_1)$. Moreover,*

$$\|\mathscr{B}_0 f\|_{L^1(\nu_1)} \leq (\rho+2)(\rho+1)\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma^2(n+\frac{1}{2}(\alpha+\beta)+2)}\|f\|_{L^1(\nu)}$$

for all $f \in L^1(\nu)$.

Proposition 3.6. *If $f \in L^1(\nu)$, then*

$$\lim_{k\to\infty} \|\mathscr{B}_k f - f\|_1 = 0.$$

Proof. We first assume that $f \in C(\bar{B}_n)$. We proceed to show that $\mathscr{B}_k f \to f$ pointwise as $k \to \infty$. Note that it is enough to prove that $\mathscr{B}_k f(0) \to f(0)$ as $k \to \infty$. The general result then follows by Proposition 3.2: we simply apply this limit to $(f \circ \varphi_z)\, h_z^{(\alpha,\beta)}$.

Note that

$$\mathscr{B}_k f(0) - f(0) = C_k \int_{B_n} [f(w) - f(0)]\, d\nu_k(w) + (\widetilde{C}_k - 1) f(0),$$

where

$$\widetilde{C}_k = C_k \int_{B_n} d\nu_k = \frac{\Gamma(n+\alpha+1+k)\Gamma(n+\beta+1+k)}{\Gamma(n+1+k)\Gamma(n+\alpha+\beta+1+k)}.$$

It is an immediate consequence of Stirling's formula that $\widetilde{C}_k \to 1$ as $k \to \infty$.

To show that

$$C_k \int_{B_n} [f(w) - f(0)]\, d\nu_k(w) \to 0 \tag{3.8}$$

as $k \to \infty$, we split B_n into a (sufficiently) small ball δB_n and a spherical shell $B_n \setminus \delta B_n$. We correspondingly decompose the integral into two parts. Then (3.8) follows by estimating the integral over δB_n using the continuity of f and estimating the integral over $B_n \setminus \delta B_n$ using the boundedness of f and the inequality

$$C_k \int_{B_n \setminus \delta B_n} d\nu_k(y) \leq \frac{\Gamma(n+\alpha+k+1)\Gamma(n+\beta+k+1)}{\Gamma(n+1)\Gamma(k+1)\Gamma(n+\alpha+\beta+k+1)}(1-\delta^2)^k.$$

Finally, by Proposition 3.3 and Lebesgue's dominated convergence theorem, we find that

$$\lim_{k\to\infty} \int_{B_n} |\mathscr{B}_k f - f|\, d\nu = 0 \quad \forall f \in C(\bar{B}_n).$$

The general case follows by Proposition 3.4 and the density of $C(\bar{B}_n)$ in $L^1(\nu)$. □

Proposition 3.7. *Suppose that k and ℓ are nonnegative integers, and $k + \ell > 0$. Then $\mathscr{B}_k \mathscr{B}_\ell = \mathscr{B}_\ell \mathscr{B}_k$ on $L^1(\nu)$.*

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PROOF. By Proposition 3.4, the expression $\mathscr{B}_k\mathscr{B}_\ell f$ makes sense for all $f \in L^1(\nu)$ when $k \geq 0$ and $\ell \geq 1$. Proposition 3.5 implies that the expression $\mathscr{B}_k\mathscr{B}_0 f$ also makes sense for all $f \in L^1(\nu)$ when $k \geq 1$. Hence $\mathscr{B}_k\mathscr{B}_\ell f$ is well-defined for all $f \in L^1(\nu)$ when $k + \ell > 0$.

Let $f \in L^1(\nu)$. To prove that $\mathscr{B}_k\mathscr{B}_\ell f = \mathscr{B}_\ell\mathscr{B}_k f$ it suffices to show that

$$\mathscr{B}_k\mathscr{B}_\ell f(0) = \mathscr{B}_\ell\mathscr{B}_k f(0),$$

by Proposition 3.2. For notational simplicity, we write

$$\begin{aligned} &c := \alpha + \beta + n + k + \ell + 1,\\ &s' := n + \alpha + \ell + 1, \quad t' := n + \beta + \ell + 1,\\ &s := n + \alpha + k + 1, \quad t := n + \beta + k + 1. \end{aligned}$$

Note that $c + n + 1 - s' = t$ and $c + n + 1 - t' = s$. In addition, a simple calculation shows that

$$\mathscr{B}_k\mathscr{B}_\ell f(0) = C_k C_\ell \int_{B_n} f(w)(1 - |w|^2)^\ell \left(\int_{B_n} \frac{(1 - |z|^2)^c \, d\nu(z)}{(1 - \langle z, w\rangle)^{s'}(1 - \langle w, z\rangle)^{t'}}\right) d\nu(w). \quad (3.9)$$

The change of variables $z \mapsto \varphi_w(z)$ in the inner integral yields

$$\begin{aligned} &\int_{B_n} \frac{(1 - |z|^2)^c}{(1 - \langle z, w\rangle)^{s'}(1 - \langle w, z\rangle)^{t'}} \, d\nu(z)\\ &\quad = \int_{B_n} \left(\frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w\rangle|^2}\right)^c \left(\frac{1 - \langle z, w\rangle}{1 - |w|^2}\right)^{s'} \left(\frac{1 - \langle w, z\rangle}{1 - |w|^2}\right)^{t'} \left(\frac{1 - |w|^2}{|1 - \langle z, w\rangle|^2}\right)^{n+1} d\nu(z)\\ &\quad = \int_{B_n} \frac{(1 - |z|^2)^c (1 - |w|^2)^{c+n+1-s'-t'}}{(1 - \langle z, w\rangle)^{c+n+1-s'}(1 - \langle w, z\rangle)^{c+n+1-t'}} \, d\nu(z)\\ &\quad = (1 - |w|^2)^{k-\ell} \int_{B_n} \frac{(1 - |z|^2)^c}{(1 - \langle z, w\rangle)^t (1 - \langle w, z\rangle)^s} \, d\nu(z)\\ &\quad = \frac{\Gamma(n + 1)\Gamma(c + 1)}{\Gamma(n + 1 + c)} (1 - |w|^2)^{k-\ell} \, {}_2F_1(t, s; n + 1 + c; |w|^2), \end{aligned}$$

where the last equality follows by (2.6). Inserting this into (3.9), we deduce that

$$\begin{aligned} \mathscr{B}_k\mathscr{B}_\ell f(0) = C_k C_\ell &\frac{\Gamma(n + 1)\Gamma(c + 1)}{\Gamma(n + 1 + c)}\\ &\times \int_{B_n} f(w)(1 - |w|^2)^k \, {}_2F_1(t, s; n + 1 + c; |w|^2) \, d\nu(w). \end{aligned} \quad (3.10)$$

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On the other hand, interchanging the indices k and ℓ in (3.9) shows that

$$\begin{aligned}\mathscr{B}_\ell\mathscr{B}_k f(0) &= C_k C_\ell \int_{B_n} f(w)(1-|w|^2)^k \\ &\quad\times \left\{\int_{B_n} \frac{(1-|z|^2)^c}{(1-\langle z,w\rangle)^s(1-\langle w,z\rangle)^t}\,d\nu(z)\right\} d\nu(w) \\ &= C_k C_\ell \frac{\Gamma(n+1)\,\Gamma(c+1)}{\Gamma(n+1+c)} \\ &\quad\times \int_{B_n} f(w)(1-|w|^2)^k\,{}_2F_1(s,t;n+1+c;|w|^2)\,d\nu(w).\end{aligned} \tag{3.11}$$

A comparison of (3.10) and (3.11) now establishes our result. □

The following proposition gives the relationship between the Berezin-type transforms $\mathscr{B}_k$ and the differential operators $\Delta_{\alpha,\beta}$.

Proposition 3.8. *For all nonnegative integers k and $f \in L^1(\nu)$,*

$$\Delta_{\alpha,\beta}\mathscr{B}_k f = (\rho+k+1)(k+1)(\mathscr{B}_k f - \mathscr{B}_{k+1} f).$$

Proof. By (1.4) and (3.4), it suffices to establish that

$$\Delta_{\alpha,\beta}\mathscr{B}_k f(0) = (\rho+k+1)(k+1)\{\mathscr{B}_k f(0) - \mathscr{B}_{k+1} f(0)\} \quad \forall f \in L^1(\nu).$$

This may be shown by differentiating under the integral sign and regrouping terms. □

In other words, for all strictly positive integers k, the operator identity

$$\mathscr{B}_k = \left(1 - \frac{\Delta_{\alpha,\beta}}{k(\rho+k)}\right)\mathscr{B}_{k-1}$$

holds. The next corollary is an immediate consequence of this identity.

Corollary 3.9. *Suppose that k is a positive integer and set*

$$G_{k,\rho}(\lambda) = \prod_{j=1}^{k}\left(1 - \frac{\lambda}{j(\rho+j)}\right) \quad \forall \lambda \in \mathbb{C}.$$

Then $\mathscr{B}_k = G_{k,\rho}(\Delta_{\alpha,\beta})\mathscr{B}_0$ on $L^1(\nu)$.

Let

$$G_\rho(\lambda) := \prod_{j=1}^{\infty}\left(1 - \frac{\lambda}{j(\rho+j)}\right) \quad \forall \lambda \in \mathbb{C}. \tag{3.12}$$

It is clear that G_ρ is an entire function and that $G_{k,\rho}$ converges to G_ρ, uniformly on compact subsets of $\mathbb{C}$ as $k \to \infty$. It should not be surprising that the function G_ρ plays an important role in the solution of our problem.

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4. The eigenspaces of $\Delta_{\alpha,\beta}$

Let α and β be fixed. For all $\lambda \in \mathbb{C}$, we define the function space

$$X_\lambda := \{f \in C^2(B_n) \mid \Delta_{\alpha,\beta} f = \lambda f\}.$$

In particular, when $\lambda = 0$, we see that X_0 is the space of all (α, β)-harmonic functions.

Proposition 4.1. *Let $\lambda, \gamma \in \mathbb{C}$ be related by*

$$\lambda = -\gamma(\rho - \gamma). \tag{4.1}$$

Then the radial functions in X_λ are the constant multiples of

$$(1 - |z|^2)^\gamma \, {}_2F_1(\gamma - \alpha, \gamma - \beta; n; |z|^2).$$

Remark. From now on, we will always assume that λ and γ are related as in (4.1).

Proof. We will only sketch of the proof, as it is similar to that of [1, Theorem 2.1].

Recall the following 'radial-tangential' expression (see [2]):

$$\Delta_{\alpha,\beta} = (1 - |z|^2)\Big\{\frac{1}{|z|^2}\Big((1 - |z|^2)R\bar{R} - \mathcal{L}_0 + \frac{n-1}{2}(R + \bar{R})\Big) + \alpha R + \beta\bar{R} - \alpha\beta\Big\},$$

where

$$L_{ij} = \bar{z}_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial z_i} \quad \text{and} \quad \mathcal{L}_0 = -\frac{1}{2}\sum_{i<j}(\bar{L}_{ij}L_{ij} + L_{ij}\bar{L}_{ij}).$$

For radial functions, the operator $\Delta_{\alpha,\beta}$ has the form

$$\Delta_{\alpha,\beta} f = \frac{(1 - r^2)}{4}\Big\{(1 - r^2)\frac{d^2}{dr^2} + \Big[\frac{2n-1}{r} + (2\alpha + 2\beta - 1)r\Big]\frac{d}{dr} - 4\alpha\beta\Big\}f,$$

where $r = |z|$.

Let $f \in C^2(B_n)$ be radial. Then $f(z) = (1 - r^2)^\gamma g(r^2)$ for some function g defined on $[0, 1)$. This converts $\Delta_{\alpha,\beta} f = \lambda f$ to the form

$$x(1 - x)g''(x) + \{n - (2\gamma - \alpha - \beta + 1)x\}g'(x) - (\gamma - \alpha)(\gamma - \beta)g(x) = 0$$

for all $x \in (0, 1)$. But this is just a hypergeometric equation, and its only solutions that are smooth at 0 are multiples of

$${}_2F_1(\gamma - \alpha, \gamma - \beta; n; x).$$

Hence f is a multiple of

$$(1 - |z|^2)^\gamma \, {}_2F_1(\gamma - \alpha, \gamma - \beta; n; |z|^2),$$

and our result is established. □

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Corollary 4.2. *If $\Delta_{\alpha,\beta} f = \lambda f$ in B_n and $0 < r < 1$, then*

$$\int_S f(\varphi_z(r\zeta)) h_z^{(\alpha,\beta)}(r\zeta)\, d\sigma(\zeta) = (1 - r^2)^\gamma\, {}_2F_1(\gamma - \alpha, \gamma - \beta; n; r^2) f(z). \tag{4.2}$$

Proof. If $f \in X_\lambda$, then so is its radialization $f^\sharp$, given by

$$f^\sharp = \int_{\mathscr{U}(n)} (f \circ U)\, dU,$$

where $\mathscr{U}(n)$ denotes the group of all unitary transformations on $\mathbb{C}^n$ and dU denotes the Haar measure element on $\mathscr{U}$. It follows by Proposition 4.1 that

$$f^\sharp(z) = C(1 - |z|^2)^\gamma\, {}_2F_1(\gamma - \alpha, \gamma - \beta; n; |z|^2) \tag{4.3}$$

for some constant C. Letting $z = 0$ in (4.3) we get $C = f(0)$ and hence

$$\int_S f(r\zeta)\, d\sigma(\zeta) = (1 - r^2)^\gamma\, {}_2F_1(\gamma - \alpha, \gamma - \beta; n; r^2) f(0) \quad \forall r \in (0, 1).$$

This is (4.2) with f in place of $(f \circ \varphi_z) h_z^{(\alpha,\beta)}$. The general case of (4.2) follows from (1.4), that is, the Möbius-invariance of X_λ. $\square$

Corollary 4.3. *Suppose that $-1 < \operatorname{Re} \gamma < \rho + 1$ and $f \in X_\lambda \cap L^1(\nu)$. Then*

$$\mathscr{B}_0 f(z) = \frac{\Gamma(\gamma + 1)\Gamma(\rho + 1 - \gamma)}{\Gamma(\rho + 1)} f(z).$$

Proof. We integrate using polar coordinates and apply (4.2) and (2.4) to obtain

$$\begin{aligned}
\mathscr{B}_0 f(z) &= C_0 \int_{B_n} f(\varphi_z(w)) h_z^{(\alpha,\beta)}(w)\, d\nu(w) \\
&= 2nC_0 \int_0^1 r^{2n-1} \Big\{ \int_S f(\varphi_z(r\zeta)) h_z^{(\alpha,\beta)}(r\zeta)\, d\sigma(\zeta) \Big\}\, dr \\
&= \Big\{ 2nC_0 \int_0^1 r^{2n-1} (1 - r^2)^\gamma\, {}_2F_1(\gamma - \alpha, \gamma - \beta; n; r^2)\, dr \Big\} f(z) \\
&= nC_0 \frac{\Gamma(n)\Gamma(\gamma + 1)\Gamma(n + \alpha + \beta + 1 - \gamma)}{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)} f(z),
\end{aligned}$$

which is precisely the assertion of this corollary. $\square$

For all $s \in \mathbb{C}$, we set

$$\Phi_s(\eta) := \frac{\Gamma(\eta + 1)\Gamma(s + 1 - \eta)}{\Gamma(s + 1)} \quad \forall \eta \in \mathbb{C}.$$

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LEMMA 4.4. *Let λ and γ be related as in (4.1). Then*

$$G_\rho(\lambda) = \frac{1}{\Phi_\rho(\gamma)}.$$

PROOF. The jth factor in the product (3.12)

$$G_\rho(\lambda) = \prod_{j=1}^{\infty}\Big(1 + \frac{\gamma(\rho-\gamma)}{j(\rho+j)}\Big)$$

is equal to

$$\Big\{\Big(1+\frac{\gamma}{j}\Big)e^{-\gamma/j}\Big\}\Big\{\Big(1+\frac{\rho-\gamma}{j}\Big)e^{(\gamma-\rho)/j}\Big\}\Big\{\Big(1+\frac{\rho}{j}\Big)^{-1}e^{\rho/j}\Big\}.$$

Recalling the well-known identity

$$\frac{1}{\Gamma(y+1)} = e^{\theta y}\prod_{j=1}^{\infty}\Big(1+\frac{y}{j}\Big)e^{-y/j},$$

where θ is the Euler–Mascheroni constant, given by

$$\theta = \lim_{j\to\infty}\Big(\sum_{i=1}^{j}\frac{1}{i} - \log j\Big),$$

the conclusion now follows easily. □

We define the region

$$\Sigma_\rho := \{\gamma \in \mathbb{C} \mid -1 < \operatorname{Re}\gamma < \rho + 1\}.$$

Then $\gamma \mapsto \lambda$ is a two-to-one map from the region Σ_ρ onto a region Ω_ρ. Arguing as in [3, Section 3.1], it is easy to deduce that

$$\Omega_\rho \subseteq \Big\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda + \Big(\frac{\operatorname{Im}\lambda}{\rho+2}\Big)^2 < \rho + 1\Big\} \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < \rho + 1\}. \tag{4.4}$$

The relationship (4.1) between λ and γ enables us to work with the region Σ_ρ, which is more manageable than Ω_ρ. Changes of variable of this kind have already been used extensively (see, for example, [3, Section 3.1]).

COROLLARY 4.5. *Suppose that $\lambda \in \Omega_\rho$ and $f \in X_\lambda \cap L^1(\nu)$. Then*

$$\mathscr{B}_0 f = f/G_\rho(\lambda).$$

PROPOSITION 4.6. *We have $X_\lambda \cap L^1(\nu) \neq \{0\}$ if and only if $\lambda \in \Omega_\rho$.*

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PROOF. Take $f \in X_\lambda \cap L^1(\nu) \setminus \{0\}$ and $a \in B_n \setminus \{0\}$. Then g, defined by

$$g := (f \circ \varphi_a) h_a^{(\alpha,\beta)},$$

also lies in $X_\lambda \cap L^1(\nu)$ (by 'Möbius invariance') and hence so does $g^\sharp$, its radialization. Moreover, $g^\sharp \neq 0$ since

$$g^\sharp(0) = g(0) = f(a) \neq 0,$$

by the choice of a. Thus $g^\sharp$ is a nonzero multiple of the function g_0, given by

$$g_0(z) := (1 - |z|^2)^\gamma \, {}_2F_1(\gamma - \alpha, \gamma - \beta; n; |z|^2).$$

Now it suffices to show that $g_0 \in L^1(\nu)$ if and only if $\lambda \in \Omega_\rho$. Integration in polar coordinates yields

$$\int_{B_n} |g_0| \, d\nu = n \int_0^1 |(1 - r^2)^\gamma \, {}_2F_1(\gamma - \alpha, \gamma - \beta; n; r^2)| r^{2(n-1)} \, dr^2.$$

By [12, Vol. I, p. 76, (9)],

$${}_2F_1(a, b; c; z) \sim c_1 z^{-a} + c_2 z^{-b}$$

for large z (with a logarithmic factor if $a - b$ is an integer). Therefore $g_0 \in L^1(\nu)$ exactly when

$$-1 < \operatorname{Re} \gamma < \rho + 1,$$

that is, exactly when $\lambda \in \Omega_\rho$. □

PROPOSITION 4.7. *If $\lambda \in \Omega_\rho$, then $G_\rho(\lambda) \neq 0$ and $G'_\rho(\lambda) \neq 0$.*

PROOF. From (3.12), the zeroes of G_ρ occur at $\rho + 1, 2(\rho + 2), 3(\rho + 3), \ldots$, and none of these lies in Ω_ρ, by (4.4). Logarithmic differentiation of (3.12) yields

$$G'_\rho(\lambda) = G_\rho(\lambda) \sum_{j=1}^{\infty} \frac{1}{\lambda - j(\rho + j)}.$$

Each of these summands has a negative real part when

$$\operatorname{Re} \lambda < \rho + 1$$

and hence has a negative real part for all $\lambda \in \Omega_\rho$. □

For all $\gamma \in \Sigma_\rho$ such that $\lambda \in \Omega_\rho$, Corollary 4.5 shows that $X_\lambda \cap L^1(\nu)$ is a subspace of the eigenspace of $\mathscr{B}_0$ with eigenvalue $\Phi_\rho(\gamma)$. Since our main concern is the equation $\mathscr{B}_0 f = f$, we need to investigate the set of points in Σ_ρ at which $\Phi_\rho(\gamma) = 1$.

DEFINITION 4.8. Let $S \subseteq \mathbb{R}$ be the collection of all positive real numbers s for which there exists a positive real number t such that $\Phi_s(-1-it) \in \mathbb{R}$ and $\Phi_s(-1-it) > 1$. We define

$$\rho_0 := \inf S.$$

Then $11.025 < \rho_0 < 11.069$ by [20, p. 34].

PROPOSITION 4.9 (See [20, Theorem 4.5]). *There are no zeroes of the function $\Phi_\rho - 1$ in $\Sigma_\rho \backslash \{0, \rho\}$ if and only if $\rho \le \rho_0$.*

Further, if $\rho > \rho_0$, then the equation $\Phi_\rho(\gamma) = 1$ has solutions in $\Sigma_\rho \backslash \{0, \rho\}$. The number of these solutions is finite for each ρ, but tends to ∞ as $\rho \to \infty$.

We define the set E_ρ by

$$E_\rho := \{\lambda \in \Omega_\rho \mid G_\rho(\lambda) = 1\}.$$

COROLLARY 4.10. *We have $E_\rho = \{0\}$ if and only if $\rho \le \rho_0$.*

5. Proof of Theorem 1.1

The rest of the proof of our main result, Theorem 1.1, mimics the argument given in [3], and we only sketch it.

By Proposition 3.5, the subspace

$$M := \{f \in L^1(\nu) \mid \mathscr{B}_0 f = f\}$$

is closed in $L^1(\nu)$. We define $\Delta^M_{\alpha,\beta}$ to be the restriction of $\Delta_{\alpha,\beta}$ to M. Propositions 3.8 and 3.4 imply that $\Delta^M_{\alpha,\beta}$ is bounded from M to $L^1(\nu)$. Moreover, for all $f \in M$,

$$\mathscr{B}_0 \Delta^M_{\alpha,\beta} f = (\rho + 1)(\mathscr{B}_0 f - \mathscr{B}_1 \mathscr{B}_0 f) = \Delta^M_{\alpha,\beta} f,$$

by Propositions 3.8 and 3.7. Thus $\Delta^M_{\alpha,\beta}$ carries M into M.

Since

$$G_{k,\rho}(\lambda) \to G_\rho(\lambda)$$

uniformly on compact subsets of $\mathbb{C}$ as $k \to \infty$, it follows that

$$G_{k,\rho}(\Delta^M_{\alpha,\beta}) \to G_\rho(\Delta^M_{\alpha,\beta})$$

in the topology induced by the operator norm. Corollary 3.9 and Proposition 3.6 now imply that $f = G_\rho(\Delta^M_{\alpha,\beta}) f$ for all $f \in M$. That is,

$$G_\rho(\Delta^M_{\alpha,\beta}) = \mathrm{id}_M, \tag{5.1}$$

where id_M denotes the identity map on the subspace M.

By Corollary 4.5 and Proposition 4.6 the set E_ρ is exactly the point spectrum of $\Delta^M_{\alpha,\beta}$. In addition, E_ρ is finite for all positive ρ by Propositions 4.9 and 4.4.

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Now let Q be the monic polynomial which has a simple zero at each point of E_ρ and no other zeros in $\mathbb{C}$. As $G'_\rho(\lambda) \neq 0$ in Ω_ρ, there exists an entire function H such that $HQ = G_\rho - 1$ and $H(\lambda) \neq 0$ for all $\lambda \in E_\rho$. Since H has no zero on the point spectrum of $\Delta^M_{\alpha,\beta}$, the spectral mapping theorem implies that 0 is not in the point spectrum of $H(\Delta^M_{\alpha,\beta})$. Therefore $H(\Delta^M_{\alpha,\beta})$ is a one-to-one operator. Now (5.1) shows that

$$H(\Delta^M_{\alpha,\beta})Q(\Delta^M_{\alpha,\beta}) = 0,$$

and it follows that $Q(\Delta^M_{\alpha,\beta}) = 0$.

Corollary 4.5 tells us that

$$M \cap X_\lambda = X_\lambda \cap L^1(\nu)$$

for all $\lambda \in E_\rho$. We now apply [3, Lemma 4.1] to the Banach space M and the operator $\Delta^M_{\alpha,\beta}$ to deduce that

$$M = \bigoplus_{\lambda \in E_\rho} X_\lambda \cap L^1(\nu).$$

The theorem now follows from this decomposition and Corollary 4.10. □

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