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Some results on the derived series of finite p-groups

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It is well-known that in a finite p-group G, the condition $|G'/G''| \leqslant p^2$ implies that G' is abelian. More generally, if $|G^{(k)}/G^{(k+1)}| \leqslant p^{2^k}$ for some $k \geqslant 1$, then $G^{(k)}$ is abelian. The main objective of this dissertation is to report on finite p-groups G, such that $|G^{(k)}/G^{(k+1)}| = p^{2^k+1}$ and $G^{(k+1)} \neq 1$ for some $k \geqslant 1$.

First we study this condition for $k = 1$; that is we assume that $|G'/G''| = p^3$ and $G'' \neq 1$. It is a result of Hall that for odd primes $|G''| = p$, and so $|G'| = p^4$. We improve Hall's result by showing that $\gamma_3(G)$ is a maximal subgroup of G' and $G'' = \gamma_5(G)$. Moreover G' is the direct product of an extraspecial group of order p^3 and a cyclic group of order p. If p is even then the order of G'' can be arbitrarily large. In this case Blackburn showed that G'' is abelian with at most two generators. The set of such 2-groups can be divided into two classes according to whether $\gamma_3(G)$ is a maximal subgroup of G' or not, and we show that neither class is empty. Furthermore, we refine Blackburn's results by obtaining a more detailed description of the quotients $\gamma_i(G)/\gamma_{i+1}(G)$ for $i \geqslant 2$.

It is an undecided question whether for $k \geqslant 2$ and $p \geqslant 3$ there exists a finite p-group G, such that $|G^{(k)}/G^{(k+1)}| = p^{2^k+1}$ and $G^{(k+1)} \neq 1$. We find that if such G exists, then $[G^{(k)}, G]$ must be a maximal subgroup in $G^{(k)}$. For $p \geqslant 5$ we also show that $G^{(k)}$ has nilpotency class at most 3, and its order is bounded by a function of the form $p^{f(k)}$, where f is independent of p. These results are based on Bokut's similar results on Lie algebras.

Our discussion leads to a new lower bound for the order of a p-group with a given soluble length. In particular, we prove that for $p \neq 3$ the condition $G^{(k)} \neq 1$ implies $|G| \geqslant p^{2^k+3k-8}$. We describe some examples for groups and Lie algebras of high soluble length and low order. Among others, we present constructions published elsewhere by the author, M.F. Newman and S. Evans-Riley for p-groups with soluble length k and order p^{2^k-2} [1]. These groups are currently the smallest known.

Most of our theorems are proved using Lie methods, and some interesting results about Lie algebras are also obtained.

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