The summation of a slowly convergent series

By S. Paterson.

The series

$$S_{2r}(x) \equiv \sum_{n=1}^{\infty} n^{2r} e^{-n^2 x} \tag{1}$$

in which r is zero or an integer is rapidly convergent if x is large but may be very slowly convergent if x is small. The object of this note is to derive an alternative series for $S_{2r}(x)$ which is rapidly convergent for small values of x.

If we let

$$\Phi_{-r}(z) \equiv \frac{2}{\sqrt{\pi}} \int_z^{\infty} \int_z^{\infty} \ldots \int_z^{\infty} e^{-z} \ (dz)^{r+1}, \ r \geqq -1,$$

then it can readily be established[1] by induction that

$$2r\,\Phi_{-r}(z) = -2z\,\Phi_{-(r-1)}(z) + \Phi_{-(r-2)}(z) \tag{2}$$

from which it follows that

$$(2\sqrt{x})^{r-2}\,\Phi_{-(r-2)}(n\pi/\sqrt{x}) = \frac{d}{dx}\left\{(2\sqrt{x})^{r}\,\Phi_{-r}(n\pi/\sqrt{x})\right\}. \tag{3}$$

Now as a special case of Poisson's identity we have

$$S_0(x) \equiv \sum_{n=1}^{\infty} e^{-n^2 x} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\pi/x)} + \sqrt{(\pi/x)} \sum_{n=1}^{\infty} e^{-n^2\pi^2/x}$$

which may be written in the form

$$S_0(x) = \tfrac{1}{2}(\sqrt{\pi/x} - 1) + \tfrac{1}{2}\,\pi x^{-\frac{1}{2}} \sum_{n=1}^{\infty} \Phi_1(n\pi/\sqrt{x}). \tag{4}$$

Integrating this equation with respect to x from x to ∞ leads to the relation

$$S_{-2}(x) = \frac{\pi^2}{6} - \sqrt{\pi x} + \tfrac{1}{2}x - 2\pi\sqrt{x} \sum_{n=1}^{\infty} \Phi_{-1}(n\pi/\sqrt{x}) \tag{5}$$

which in turn leads to

$$S_{-4}(x) = \frac{\pi^4}{90} - \frac{\pi^2 x}{6} + \frac{2(\pi x^3)^{\frac{1}{2}}}{3} - \frac{x^2}{4} + \pi(2\sqrt{x})^3 \sum_{n=1}^{\infty} \Phi_{-3}(n\pi/\sqrt{x}). \tag{6}$$

---

[1] Hartree, D. R., *Mem. Proc. Manchester Lit. Phil. Soc.*, **80**, 85, (1936).

In the general case we obtain the expression

$$S_{-2r}(x) = 2^{2r-1}\pi^{2r}\left\{\frac{B_r}{2r!\,0!} - \frac{B_{r-1}\,x}{4\pi^2(2r-2)!\,1!}\right.$$

$$\left. + \ldots\ldots + (-1)^{r-1}\frac{B_1 x^{r-1}}{(4\pi^2)^{r-1}\,2!\,(r-1)!}\right\} + (-1)^r\sqrt{\pi}\,\frac{2^{2r-1} r!\,x^{r-}}{(2r)!}$$

$$+ (-1)^{r-1}\frac{x^r}{2(r!)} + (-1)^{r+1}\pi(2\sqrt{x})^{2r-1}\sum_{n=1}^{\infty}\Phi_{-(2r-1)}(n\pi/\sqrt{x}) \qquad (7)$$

in which B_r denotes the r-th Bernoulli number.

The magnitude of the contribution $\pi(2\sqrt{x})^{2r-1}\ \Sigma\ \Phi_{-(2r-1)}$ to the final result can be estimated by expanding $\Phi_{-(2r-1)}$ in descending powers of $\pi x^{-\frac{1}{2}}$; we then find that $\pi(2\sqrt{x})^{2r-1}\Sigma\Phi_{-(2r-1)}$ is of the order of $(x/\pi)^{2r-\frac{1}{2}}e^{-\pi^2/x}$ which is always negligible if $0 < x \leqq 1$.

On the other hand if we differentiate (4) repeatedly and make use of equation (3) in the form

$$(4x)^{-r-\frac{1}{2}}\Phi_{2r+1}(n\pi/\sqrt{x}) = \frac{d^r}{dx^r}\left\{(4x)^{-\frac{1}{2}}\Phi_1(n\pi/\sqrt{x})\right\}$$

we find that

$$S_{2r} = \frac{(2r)!\,\sqrt{\pi}}{r!\,(2\sqrt{x})^{2r+1}} + \frac{(-1)^r\,2\sqrt{\pi}}{(2\sqrt{x})^{2r+1}}\sum_{n=1}^{\infty}\left[\frac{d^{2r}e^{-z^2}}{dz^{2r}}\right]_{z=n\pi/\sqrt{x}}$$

$$= \frac{\sqrt{\pi}}{2^r x^{r+\frac{1}{2}}}\left\{\frac{(2r)!}{r!\,2^{r+1}} + (-1)^r\sum_{n=1}^{\infty}e^{-n^2\pi^2/2x}D_{2r}(n\pi\sqrt{2/x})\right\} \qquad (8)$$

where D denotes Weber's parabolic cylinder function.

Since the first term in the asymptotic expansion of $D_{2r}(z)$ is $e^{-\frac{1}{4}z^2}\cdot z^{2r}$ the ratio of the series in (8) to the other term is of order $2(r!)\,(4\pi^2/x)^r\,e^{-\pi^2 x}/(2r)!$, and this is of the order 10^{-4}, or smaller, if

$$r \leqq \pi^2(1/x - 1)/\log(\pi^2/x).$$

Thus we conclude that

$$S_{2r} = \frac{\sqrt{\pi}\,(2r)!}{r!\,(4x)^{r+\frac{1}{2}}},\ r \geqq 1 \qquad (9)$$

to better than 1 in 10,000 provided that

$x \leqq$	0·7	0·62	0·52	0·45	0·38	0·32	0·28	0·25	0·22	0·20
if $r =$	1	2	3	4	5	6	7	8	9	10 .

Since we have neglected the factor $2^{2r+1}(r!)/(2r)!$ which decreases as r increases, the range is actually somewhat wider for the larger values of r.

By writing y for $\pi/\sqrt{x}$ in the above relations, we obtain a corresponding set of expressions for $\sum_{n=1}^{\infty} \Phi_{2r+1}(ny)$ for $y \geqq 10$. For example from (8) we have

$$\sum_{n=1}^{\infty} \Phi_{2r+1}(ny) \doteq (-1)^{r+1}(2r)!/\pi^{\frac{1}{2}} r!$$

which is independent of y ($\geqq 10$), and for $r = 1$ gives

$$\sum_{n=1}^{\infty} e^{-\frac{1}{4}n^2 z^2} D_2(nz) \doteq \tfrac{1}{2}, \text{ if } z > 14.$$

GAIRVIEW, GLENCAIRN ST.,
STEVENSTON, AYRSHIRE.

A Generalisation of Dirichlet's Multiple Integral

By I. J. GOOD.

The purpose of this note is to generalise the Dirichlet-Liouville formula which expresses a certain type of multiple integral in terms of a single integral.[1] In our formula the multiple integral will involve several arbitrary functions instead of only one, and it will be expressed as a product of single integrals.

Let n be a positive integer. Let $f_1(t), f_2(t), \ldots, f_n(t)$ be Lebesgue measurable functions when $0 \leqq t \leqq 1$. A finite sequence of n real numbers $m_1, m_2, \ldots, m_n$ is given. We write $m_{n+1} = 0$ and

$$M_r = m_1 + m_2 + \ldots + m_r$$

$$X_r = x_1 + x_2 + \ldots + x_r$$

[1] See, for example, G. F. Meyer, *Vorlesungen über die Theorie der bestimmten Integrale* (Leipzig, 1871), 566 *et seq.*; or E. T. Whittaker and G. N. Watson, *Modern Analysis* (4th edn., Cambridge, 1935), section 12.5; or H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge, 1946), section 15.08; or L. J. Mordell, "Dirichlet's integrals," *Edin. Math. Notes*, No. 34 (1944), 15-17.

https://doi.org/10.1017/S0950184300002950 Published online by Cambridge University Press