TRACE FORMULAS FOR POWERS OF A STURM-LIOUVILLE OPERATOR

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1. Introduction. Let H_0 be the mth power (m a positive integer) of the self-adjoint operator defined in the Hilbert space $L^2(0, \pi)$ by the differential operator $-(d^2/dx^2)$ and the boundary conditions $u(0) = u(\pi) = 0$. The eigenvalues of H_0 are $\mu_n = n^{2m}$ and the corresponding eigenfunctions are $\phi_n = (2/\pi)^{1/2} \sin nx$, $n = 1, 2, \ldots$.

Let p be a $(2m - 2)$-times continuously differentiable real valued function defined over the interval $[0, \pi]$ satisfying the conditions $p^{(j)}(0) = p^{(j)}(\pi) = 0$ for j odd and less than $2m - 4$. (This condition is vacuous in the cases $m = 1, 2$.) Let H_1 be the mth power of the operator defined in $L^2(0, \pi)$ by the differential operator $-d^2/dx^2 + p(x)$ and the boundary conditions $u(0) = u(\pi) = 0$. Then H_1 and H_0 are self-adjoint operators with a common domain. We define V as $H_1 - H_0$.

Let λ_n be the eigenvalues of H_1 arranged in increasing order. Let $\mu_n^{(1)}$, $\mu_n^{(2)}, \ldots$ be the coefficients in the perturbation series

$$\eta_n(\epsilon) = \mu_n + \epsilon\mu_n^{(1)} + \epsilon^2\mu_n^{(2)} + \ldots$$

for the eigenvalue $\eta_n(\epsilon)$ of $H_0 + \epsilon V$ corresponding to μ_n.

In (**2**, Theorem 7) it is stated that if all odd order derivatives of p vanish at 0 and π,

$$\sum_{n=1}^{\infty} \{\lambda_n - \mu_n - \mu_n^{(1)} - \ldots - \mu_n^{(s)}\} = 0 \tag{1}$$

for all s sufficiently large. The only cases considered in the proof are $m = 1$, $s = 1$ or 2, and $m = 2$, $s = 2$. In Section 3 of this paper we present a simple proof which is valid for each m and for all $s \geqslant 2m$. The conditions on p are those given above in the definition of H_1. The method offers the prospect of wider application, and generalizations are under study. In Section 4 other methods are used for the cases $m = 1$, $s = 1$ and $m = 2$, $s = 2$. Additional conditions on p are necessary in this section.

Dikii (**2**) uses equations very similar to (1) to obtain approximate values for the first few eigenvalues of H_1 for $p(x) \equiv \cos 2x$ and $m = 1$. Also (1) may be regarded as an extension of the result (true in some cases) that the average value of the perturbation terms is zero.

Received December 26, 1962. This research was supported by a National Science Foundation Grant, No. G17824 and by the Mathematics Research Center, U.S. Army, Madison, Wis., under contract No. DA-11-022-ORD-2059.

2. A contour integral formulation. In this section we shall be concerned with operators in a Hilbert space $\mathfrak{H}$. The uniform norm of a bounded operator H will be denoted by $\|H\|$. For H in the Schmidt class **S** we denote the Schmidt norm of H by $\|H\|_2$. For H in the trace class **T** we denote the trace norm of H by $\|H\|_1$ and the trace of H by $S\{H\}$. For H in **S** we have $\|H\| \leqslant \|H\|_2$ and for H in **T** we have $\|H\| \leqslant \|H\|_2 \leqslant \|H\|_1$. If H is in **S** [or in **T**] and A is any bounded operator, then AH and HA are in **S** [or in **T**] and both $\|AH\|_2$ and $\|HA\|_2$ are less than or equal to $\|A\| \cdot \|H\|_2$ [or both $\|AH\|_1$ and $\|HA\|_1$ are less than or equal to $\|A\| \cdot \|H\|_1$]. The product of two operators A, B in **S** is in **T** and $\|AB\|_1 \leqslant \|A\|_2 \|B\|_2$.

From the identity

$$A(y)B(y) - A(z)B(z) = A(y)\{B(y) - B(z)\} + \{A(y) - A(z)\}B(z)$$

we see that the product of two operators continuous in the Schmidt norm is continuous in the trace norm. Also, if A is continuous in the Schmidt [or trace] norm and B is continuous in the uniform norm, then the product is continuous in the Schmidt [or trace] norm.

For further information we refer the reader to Schatten **(4)**.

The resolvent set of an operator will be denoted by $\Lambda(H)$ and its domain by $\mathbf{D}(H)$.

Note that if H_0 and V are operators with $\mathbf{D}(H_0) \subset \mathbf{D}(V)$ such that H_0 and $H_0 + \epsilon_0 V (\epsilon_0 \neq 0)$ are self-adjoint and $VR_0(\omega)$ is bounded for some ω in $\Lambda(H_0)$ then $H_\epsilon = H_0 + \epsilon V$ is self-adjoint for all ϵ sufficiently small. It is easily seen that H_ϵ is symmetric for all ϵ. From the resolvent equation

$$R_0(\omega) - R_0(z) = (\omega - z)R_0(\omega)R_0(z)$$

it follows that $VR_0(z)$ is bounded for all $z \in \Lambda(H_0)$. Taking $z = i$ and $z = -i$ in the equation

$$H_0 + \epsilon V - zI = (I + \epsilon VR_0(z))(H_0 - zI),$$

one sees that H_ϵ is self-adjoint for all ϵ sufficiently small.

The following lemma is a modification of a result of Kato **(3)**.

LEMMA 1. *Let H_0 and V be operators such that $\mathbf{D}(H_0) \subset \mathbf{D}(V)$ and $H_\epsilon = H_0 + \epsilon V$ is self-adjoint for $\epsilon = 0$ and $\epsilon = \epsilon_0 \neq 0$. Suppose $R_0(\omega) = (H_0 - \omega I)^{-1}$ and $VR_0(\omega)$ are in* **S** *for some ω in $\Lambda(H_0)$. Let μ be an isolated simple eigenvalue of H_0, let Γ be a closed contour in $\Lambda(H_0)$ which surrounds μ but no other point of the spectrum of H_0. Then for sufficiently small ϵ, Γ is in $\Lambda(H_\epsilon)$, surrounds precisely one simple eigenvalue $\eta(\epsilon)$ of H_ϵ and*

$$\eta(\epsilon) = \mu + \epsilon\mu^{(1)} + \epsilon^2\mu^{(2)} + \ldots$$

where

https://doi.org/10.4153/CJM-1964-043-4 Published online by Cambridge University Press

(2) $$\mu^{(j)} = S\left\{-(2\pi i)^{-1}\int_\Gamma zR_0(z)[-VR_0(z)]^j dz\right\}$$
$$= S\left\{-(2\pi i)^{-1}\int_\Gamma (z-\mu)R_0(z)[-VR_0(z)]^j dz\right\}.$$

Proof. From the resolvent equation it follows that

$$\|R_0(\omega) - R_0(z)\|_2 \leqslant |\omega - z|\, \|R_0(\omega)\|_2\, \|R_0(z)\|,$$

and

$$\|VR_0(\omega) - VR_0(z)\| \leqslant |\omega - z|\, \|VR_0(\omega)\|_2\, \|R_0(z)\|$$

for any z in $\Lambda(H_0)$. Therefore $R_0(z)$ and $VR_0(z)$ are in **S** and are continuous in the Schmidt norm (as well as in the uniform norm). For any positive integer t, $R_0(z)\ [VR_0(z)]^t$ will be in **T** and continuous in the trace norm. It follows that the series

$$\sum_{t=1}^{\infty} \epsilon^t R_0(z)[-VR_0(z)]^t$$

converges in the trace norm to $R_\epsilon(z) - R_0(z)$, uniformly for z on Γ and for $|\epsilon| \leqslant \epsilon_0$, where $0 < \epsilon_0 < \min_{z \text{ on } \Gamma} \|VR_0(z)\|_2{}^{-1}$. (We consider the difference $R_\epsilon - R_0$ because neither R_ϵ nor R_0 is necessarily in **T**.) Since the same is true in the uniform norm, we have that Γ is in $\Lambda(H_\epsilon)$ for $|\epsilon| \leqslant \epsilon_0$, and

$$(-2\pi i)^{-1}\int_\Gamma R_\epsilon(z)dz$$
$$= -(2\pi i)^{-1}\int_\Gamma R_0(z)dz + \sum_{t=1}^{\infty}\epsilon^t\left\{(-2\pi i)^{-1}\int_\Gamma R_0(z)[-VR_0(z)]^t dz\right\}.$$

It follows that

$$E_\epsilon = E_0 + \sum_{t=1}^{\infty}\epsilon^t\left\{(-2\pi i)^{-1}\int_\Gamma R_0(z)[-VR_0(z)]^t dz\right\},$$

where E_ϵ is the projection corresponding to that part of the spectrum of H_ϵ which is enclosed by Γ, and E_0 is the similar projection for H_0. Hence by Kato (**3**, Corollary to Lemma 1.2), $\dim E_\epsilon = \dim E_0 = 1$. Thus for $|\epsilon| \leqslant \epsilon_0$, H_ϵ has one simple eigenvalue $\eta(\epsilon)$ within Γ.

From

$$R_\epsilon(z) - R_0(z) = \sum_{t=1}^{\infty}\epsilon^t R_0(z)[-VR_0(z)]^t,$$

the series converging in the trace norm, uniformly for z on Γ, it follows that

(3) $$(-2\pi i)^{-1}\int_\Gamma zR_\epsilon(z)dz + (2\pi i)^{-1}\int_\Gamma zR_0(z)dz$$
$$= \sum_{t=1}^{\infty}(-2\pi i)^{-1}\int_\Gamma \epsilon^t zR_0(z)[-VR_0(z)]^t dz.$$

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The first term on the left is the operator $H_\epsilon E_\epsilon$. Thus this term is in **T** and its trace is $\eta(\epsilon)$. Similarly the trace of the second term is $-\mu$. Taking the trace of both sides of (3) and using the fact that the series converges in the trace norm we obtain the first equality of (2). If we multiply by $z - \mu$ instead of z, we obtain the second.

LEMMA 2. *Let H_0, V, μ, Γ be as in the previous lemma except that μ is to be of multiplicity w. Then for sufficiently small ϵ, Γ contains precisely w eigenvalues (counting multiplicity) $\eta_n(\epsilon)$, $n = 1, \ldots, w$, of H_ϵ. Each $\eta_n(\epsilon)$ is an analytic function of ϵ:*

$$\eta_n(\epsilon) = \mu + \epsilon\mu_n^{(1)} + \epsilon^2\mu_n^{(2)} + \ldots,$$

where

$$\sum_{n=1}^{w} \mu_n^{(j)} = S\left\{-(2\pi i)^{-1}\int_\Gamma zR_0(z)[-VR_0(z)]^j dz\right\}. \tag{4}$$

The proof of (4) is essentially the same as that of (2). Kato **(3)** gives a similar lemma.

THEOREM 1. *Let H_0 and V be operators such that $\mathbf{D}(H_0) \subset \mathbf{D}(V)$ and $R_0(\omega)$ and $VR_0(\omega)$ are in **S** for some ω in $\Lambda(H_0)$. Suppose $H_\epsilon = H_0 + \epsilon V$ is self-adjoint for $\epsilon = 0$ and $\epsilon = 1$. Let Ω be a closed contour in $\Lambda(H_0) \cap \Lambda(H_1)$. Suppose that Ω surrounds k eigenvalues $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_k$ of H_0 and k eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k$ of H_1 and no other spectral points of either operator. Then for sufficiently small ϵ, the operator H_ϵ has k eigenvalues*

$$\eta_n(\epsilon) = \mu_n + \epsilon\mu_n^{(1)} + \epsilon^2\mu_n^{(2)} + \ldots$$

inside Γ and for $s \geqslant 0$,

$$\sum_{n=1}^{k} \{\lambda_n - \mu_n - \mu_n^{(2)} - \ldots - \mu_n^{(s)}\} = -(2\pi i)^{-1}\int_\Omega S\{zR_1(z)[-VR_0(z)]^{s+1}\}dz.$$

Proof. Note first that we have no assurance that the conclusions of Lemmas 1 and 2 hold for $\epsilon = 1$. The operator H_ϵ is used only to define the perturbation coefficients.

The equation

$$R_1(z) = R_0(z) + R_0(z)[-VR_0(z)] + \ldots + R_0(z)[-VR_0(z)]^s + R_1(z)[-VR_0(z)]^{s+1} \tag{5}$$

holds for all z in $\Lambda(H_0) \cap \Lambda(H_1)$ and therefore on all points of Ω. It follows that

$$-(2\pi i)^{-1}\int_\Omega zR_1(z)dz$$
$$= -(2\pi i)^{-1}\int_\Omega zR_0(z)dz - (2\pi i)^{-1}\int_\Omega z\{R_0(z)[-VR_0(z)]\}dz \ldots$$
$$-(2\pi i)^{-1}\int_\Omega z\{R_0(z)[-VR_0(z)]^s\}dz - (2\pi i)^{-1}\int_\Omega z\{R_1(z)[-VR_0(z)]^{s+1}\}dz.$$

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The term on the left is the operator H_1 reduced by the projection on the subspace corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$. Hence the trace of this term is the sum $\lambda_1 + \lambda_2 + \ldots + \lambda_k$. Similarly, the trace of the first term on the right is $\mu_1 + \mu_2 + \ldots + \mu_k$. Each remaining term on the right except the last can be replaced by a sum of contour integrals over contours $\Gamma_1, \ldots, \Gamma_r$ each surrounding an isolated eigenvalue of H_0. By (4) the trace of the sum of all such terms is

$$\sum_{n=1}^{k} \{\mu_n + \mu_n^{(1)} + \mu_n^{(2)} + \ldots + \mu_n^{(s)}\}.$$

We obtain

$$\sum_{n=1}^{k} \{\lambda_n - \mu_n - \mu_n^{(1)} - \ldots - \mu_n^{(s)}\}$$

$$= -(2\pi i)^{-1} S\left[\int_{\Omega} z\{R_1(z)[-VR_0(z)]^{s+1}\}dz\right].$$

The integrand in the last term is continuous in the trace norm. Thus we can interchange the two operations, integration and taking the trace, to obtain the desired result.

3. Applications to powers of a Sturm-Liouville operator. Let H_0 and H_1 be the operators defined in the Introduction. In this case $VR_0(z)$ is an integral operator with kernel:

$$H(x, y; z)$$
$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{a_0(x)n^{2m-2} \sin nx + a_1(x)\, n^{2m-3} \cos nx + \ldots + a_{2m-2}(x) \sin nx}{n^{2m} - z} \sin ny,$$

where $a_0(x), a_1(x), \ldots, a_{2m-2}(x)$ are polynomials in p and its derivatives up to order $2m - 2$. If z is in $\Lambda(H_0)$, then $VR_0(z)$ is in $\mathbf{S}$; indeed, we may estimate $||VR_0(z)||_2$ as follows:

$$||VR_0(z)||_2^2 = \int_0^{\pi} \int_0^{\pi} |H(x, y; z)|^2 dy dx$$
$$\leqslant A \sum_{n=1}^{\infty} \frac{n^{4m-4}}{|n^{2m} - z|^2},$$

where A is a constant.

LEMMA 3. *If z lies on a circle Γ_k with centre at the origin and radius $\rho_k = (k + \frac{1}{2})^m$, where k is a positive integer, then*

$$\sum_{n=1}^{\infty} \frac{n^{4m-4}}{|n^{2m} - z|^2} = O(k^{-2}) \quad \textit{and} \quad ||VR_0(z)||_2 = O(k^{-1}) \quad \textit{as} \quad k \to \infty.$$

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Proof. Since $|n^{2m} - z| \geqslant |n^{2m} - \rho_k|$,

$$\sum_{n=1}^{\infty} \frac{n^{4m-4}}{|n^{2m} - z|^2} \leqslant \sum_{n=1}^{\infty} \frac{n^{4m-4}}{|n^{2m} - \rho_k|^2}$$

$$= \sum_{n=1}^{k-1} \frac{n^{4m-4}}{|n^{2m} - (k+\frac{1}{2})^{2m}|^2} + \sum_{n=k+2}^{\infty} \frac{n^{4m-4}}{|n^{2m} - (k+\frac{1}{2})^{2m}|^2}$$

$$+ \frac{k^{4m-4}}{|k^{2m} - (k+\frac{1}{2})^{2m}|^2} + \frac{(k+1)^{4m-4}}{|(k+1)^{2m} - (k+\frac{1}{2})^{2m}|^2}.$$

Clearly each of the last two terms is $O(k^{-2})$. The first sum on the right is dominated by the integral

$$\int_0^k \frac{x^{4m-4}}{|x^{2m} - (k+\frac{1}{2})^{2m}|^2} dx.$$

The substitution $x = (k + \frac{1}{2})v$ and the inequality

$$\frac{v^{4m-4}}{(v^{2m} - 1)^2} \leqslant \frac{C}{(v-1)^2},$$

where C is a constant, shows that this integral is $O(k^{-2})$. Similarly

$$\sum_{n=k+2}^{\infty} \frac{n^{4m-4}}{|n^{2m} - (k+\frac{1}{2})^{2m}|^2} \leqslant \int_{k+1}^{\infty} \frac{x^{4m-4}}{|x^{2m} - (k+\frac{1}{2})^{2m}|^2} dx$$

$$= (k+\tfrac{1}{2})^{-3} \int_{(k+1)/(k+\frac{1}{2})}^{\infty} \frac{v^{4m-4}}{(v^{2m}-1)^2} dv$$

$$= (k+\tfrac{1}{2})^{-3} \int_{(k+1)/(k+\frac{1}{2})}^{2} \frac{v^{4m-4}}{(v^{2m}-1)^2} + O(k^{-3}) = O(k^{-2}).$$

This completes the proof of the lemma.

LEMMA 4. *If z lies on the circle Γ_k of Lemma 3 and $|I(z)| \geqslant k^{2m-\alpha}$ for some $\alpha > 0$, then $\|VR_0(z)\| = O(k^{\alpha-2})$ as $k \to \infty$.*

Proof. We can express $VR_0(z)$ as a sum of operators of the form

$$a(x) \sum_{n=1}^{\infty} \frac{n^{2m-q}}{n^{2m} - z} (\cdot\,, \phi_n)\psi_n(x),$$

where q is an integer such that $2 \leqslant q \leqslant 2m$, $\psi_n(x)$ is either $(2/\pi)^{1/2} \sin nx$ or $(2/\pi)^{1/2} \cos nx$, and $a(x)$ is a bounded, continuous function. To prove the assertion of the lemma we must show that

$$\left\{\sup_n \frac{n^{4m-2q}}{|n^{2m} - z|^2}\right\}^{1/2} = O(k^{\alpha-2})$$

for all z satisfying the conditions. It is clearly sufficient to show this for the case $q = 2$ and $z = z_k = \sigma_k + i\tau_k$, where $\tau_k = k^{2m-\alpha}$ and $\sigma_k = (\rho_k^2 - \tau_k^2)^{1/2}$.

Let $f(x) = x^{4m-4}[(x^{2m} - \sigma_k)^2 + \tau_k^2]^{-1}$. Then by elementary methods

$$\left\{\max_{0 \leqslant x < \infty} f(x)\right\}^{1/2} = \{\tfrac{1}{2}[2 - m + (m^2 + 4(m-1)(\tau_k/\sigma_k)^2)^{1/2}]\}^{1-(1/m)}$$
$$\times \{\tfrac{1}{4}[-m + \{m^2 + 4(m-1)(\tau_k/\sigma_k)^2\}^{1/2}]^2 + (\tau_k/\sigma_k)^2\}^{-1/2} \times \sigma_k^{-1/m}.$$

Since $\tau_k/\sigma_k = k^{-\alpha}[1 + O(k^{-1}) + O(k^{-2\alpha})]^{-\frac{1}{2}}$, we obtain

$$\left\{\max_{0 \leqslant x < \infty} f(x)\right\}^{1/2} \leqslant Mk^{\alpha-2}$$

for all k sufficiently large, where M is a constant. This completes the proof of the lemma.

For the case $m = 1$ it is well known that $\lambda_n = n^2 + O(1)$; see, for example, (**1**, Chapter VI). Then for any m, $\lambda_n = n^{2m} + O(n^{2m-2})$ and consequently for z on Γ_k,

$$||R_1(z)|| \leqslant \max\{|\lambda_k - \rho_k|^{-1}, |\lambda_{k+1} - \rho_k|^{-1}\}.$$

Therefore, $||R_1(z)|| = O(k^{1-2m})$ as $k \to \infty$.

THEOREM 2. *Let H_0 be the* m*th power (m a positive integer) of the operator defined in $L^2(0, \pi)$ by the differential operator $-d^2/dx^2$ and the boundary conditions $u(0) = u(\pi) = 0$. Let p be a real valued, $(2m-2)$-times continuously differentiable function defined on the interval $[0, \pi]$ such that $p^{(j)}(0) = p^{(j)}(\pi) = 0$ for j odd and less than $2m-4$. Let H_1 be the* m*th power of the operator defined by the same boundary conditions and $-d^2/dx^2 + p(x)$. Let $\mu_n = n^{2m}$ be the eigenvalues of H_0, and let λ_n be the eigenvalues of H_1 arranged in increasing order. Let $\mu_n^{(1)}$, $\mu_n^{(2)}, \ldots$ be the coefficients in the perturbation series*

$$\lambda_n(\epsilon) = \mu_n + \epsilon\mu_n^{(1)} + \epsilon^2\mu_n^{(2)} + \ldots$$

for the eigenvalue $\lambda_n(\epsilon)$ of $H_0 + \epsilon V$ corresponding to μ_n. Then, if $s \geqslant 2m$

$$\sum_{n=1}^{\infty} (\lambda_n - \mu_n - \mu_n^{(1)} - \ldots - \mu_n^{(s)}) = 0.$$

Proof. By Theorem 1, for k sufficiently large,

$$\sum_{n=1}^{k} (\lambda_n - \mu_n - \mu_n^{(1)} - \ldots - \mu_n^{(s)}) = -(2\pi i)^{-1} \int_{\Gamma_k} S\{zR_1(z)[-VR_0(z)]^{s+1}\}dz.$$

We shall show that the integral tends to zero as $k \to \infty$.

Since $s \geqslant 2m$, there exists a positive α such that $2m - s < \alpha < 2(s-m)/s$ or $2m - \alpha - s < 0$ and $2m + (\alpha - 2)s < 0$. For z on Γ_k such that

$$|\mathbf{I}(z)| \leqslant k^{2m-\alpha},$$

we use the estimate

$$\begin{aligned} |S\{zR_1(z)[-VR_0(z)]^{s+1}\}| &\leqslant |z|\ ||R_1(z)||\ ||VR_0(z)||_2^{s+1} \\ &= O(k^{2m})\cdot O(k^{1-2m})\cdot O(k^{-(s+1)}) \\ &= O(k^{-s}). \end{aligned}$$

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Since the length of this part of the contour is $O(k^{2m-\alpha})$, its contribution to the integral is $O(k^{2m-\alpha-s})$, which tends to zero as $k \to \infty$.

For z on Γ_k such that $|\mathbf{I}(z)| > k^{2m-\alpha}$ we use the estimate

$$\begin{aligned}|S\{zR_1(z)[-VR_0(z)]^{s+1}\}| &\leqslant |z|\,||R_1(z)||\,||VR_0(z)||^{s-1}\,||VR_0(z)||_2^2 \\ &= O(k^{2m})O(k^{-(2m-\alpha)})O(k^{(s-1)(\alpha-2)})O(k^{-2}) \\ &= O(k^{(\alpha-2)s}).\end{aligned}$$

Since the length of this part of the contour is $O(k^{2m})$, its contribution is $O(k^{2m+(\alpha-2)s})$, which also tends to zero as $k \to \infty$. This completes the proof of the theorem.

Remark 1. It is a trivial consequence that

$$\sum_{n=1}^{\infty} \mu_n^{(s)} = 0 \qquad \text{for } s > 2m.$$

Remark 2. By similar methods with a circle centred at μ_k as our contour and with the second of equations (2), we may show that for all k sufficiently large

$$|\lambda_k - \mu_k - \mu_k^{(1)} - \ldots - \mu_k^{(s)}| \leqslant C_1(C_2/k)^{s+1}k^{2m},$$

where C_1 and C_2 are constants independent of s and k. This expression shows that for all k sufficiently large,

$$\lambda_k = \sum_{j=0}^{\infty} \mu_k^{(j)}.$$

Further, if $s \geqslant 2m - 1$,

$$\lim_{k\to\infty} |\lambda_k - \mu_k - \mu_k^{(1)} - \ldots - \mu_k^{(s)}| = 0;$$

and if $s \geqslant 2m$,

$$\sum_{k=1}^{\infty} (\lambda_k - \mu_k - \mu_k^{(1)} - \ldots - \mu_k^{(s)})$$

converges absolutely.

4. Other methods. In this section we shall show by other methods that the conclusion of Theorem 2 can hold for smaller values of s provided p satisfies additional restrictions. The section is intended only to illustrate the methods involved.

The conditions and notation of the previous section are retained without further comment.

LEMMA 5. *If $s + 1 > 4m/3$, then*

$$\lim_{\tau\to\infty} \tau^2 S\{R_1(i\tau) - R_0(i\tau) - R_0(i\tau)[-VR_0(i\tau)] - \ldots - R_0(i\tau)[-VR_0(i\tau)]^s\} = 0. \tag{6}$$

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Proof. By the estimate given in the previous section,

$$||VR_0(i\tau)||_2^2 \leqslant A \sum_{n=1}^{\infty} \frac{n^{4m-4}}{n^{4m} + \tau^2}.$$

Using methods similar to those used for Lemma 3, we can show this sum is $O(\tau^{\epsilon-3/2m})$ where ϵ is an arbitrary positive number. From equation (5) it follows that the trace of the expression within braces in (6) is of order

$$O(\tau^{-(3/4m)(s+1)-1+\epsilon(s+1)}).$$

This is sufficient to establish (6).

LEMMA 6. *Each term within the braces in* (6) *is in the trace class. In particular,*

$$S\{R_1(i\tau)\} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - i\tau},$$

$$S\{R_0(i\tau)\} = \sum_{n=1}^{\infty} \frac{1}{\mu_n - i\tau},$$

$$S\{R_0(i\tau)VR_0(i\tau)\} = \sum_{n=1}^{\infty} \frac{\mu_n^{(1)}}{(\mu_n - i\tau)^2},$$

$$S\{R_0(i\tau)[VR_0(i\tau)]^2\} = -\sum_{n=1}^{\infty} \frac{\mu_n^{(2)}}{(\mu_n - i\tau)^2} - \sum_{n=1}^{\infty} \frac{(V\phi_n, \phi_n)^2}{(\mu_n - i\tau)^3}.$$

Proof. It is easily seen that each of the terms is in the trace class. The first two equations are obvious and the third follows from the relation

$$\mu_n^{(1)} = (V\phi_n, \phi_n).$$

If $\eta_n(\epsilon)$ is the eigenvalue and $\chi_n(\epsilon)$ is the eigenfunction of the operator $H_0 + \epsilon V$ corresponding to the unperturbed eigenvalue μ_n, then for all sufficiently small ϵ,

$$\eta_n(\epsilon) = \mu_n + \epsilon\mu_n^{(1)} + \epsilon^2\mu_n^{(2)} + \ldots,$$

and

$$\chi_n(\epsilon) = \phi_n + \epsilon\phi_n^{(1)} + \epsilon^2\phi_n^{(2)} + \ldots.$$

If we replace $R_\epsilon(z)$, $\eta_n(\epsilon)$, and $\chi_n(\epsilon)$ in the equation

$$(\eta_n(\epsilon) - z)R_\epsilon(z)\chi_n(\epsilon) = \chi_n(\epsilon)$$

by their series expansions in powers of ϵ and identify coefficients of like powers of ϵ, we obtain a set of equations, one of which is

$$\begin{aligned}(\mu_n - z)R_0(z)[-VR_0(z)]^2\phi_n &+ \mu_n^{(1)}R_0(z)[-VR_0(z)]\phi_n \\ &+\mu_n^{(2)}R_0(z)\phi_n + (\mu_n - z)R_0(z)[-VR_0(z)]\phi_n^{(1)} \\ &+\mu_n^{(1)}R_0(z)\phi_n^{(1)} + (\mu_n - z)R_0(z)\phi_n^{(2)} = \phi_n^{(2)}.\end{aligned}$$

Taking the inner product of both sides with ϕ_n and using the relation

$$\phi_n^{(1)} = \sum_{r=1}^{\infty}{}' \frac{(V\phi_n, \phi_r)}{\mu_n - \mu_r}\phi_r,$$

we obtain

$S\{R_0(z)[VR_0(z)]^2\}$

$$= \sum_{n=1}^{\infty} \frac{\mu_n^{(2)}}{(\mu_n - z)^2} + \sum_{n=1}^{\infty} \frac{(V\phi_n, \phi_n)^2}{(\mu_n - z)^3} + \sum_{n=1}^{\infty} \sum_{r=1}^{\infty}{}' \frac{(V\phi_n, \phi_r)(\phi_r, V\phi_n)}{(\mu_r - z)(\mu_n - z)(\mu_r - \mu_n)},$$

where the prime indicates the omission of the term corresponding to $r = n$. The last sum is zero by virtue of the anti-symmetry of the summand in n and r. This completes the proof of the lemma.

THEOREM 3. *If p is in C^2, then*

$$\sum_{n=1}^{\infty} (\lambda_n - \mu_n - \mu_n^{(1)}) = 0$$

when $m = 1$. If p is in C^4 and $p'(0) = p'(\pi) = 0$, then

$$\sum_{n=1}^{\infty} (\lambda_n - \mu_n^{(1)} - \mu_n^{(2)}) = 0$$

when $m = 2$.

Proof. For $m = 1$, (6) is valid for $s \geqslant 1$. From (6) for $s = 1$ and Lemma 6, we obtain

$$\lim_{\tau\to\infty} \tau^2 \left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n - i\tau} - \sum_{n=1}^{\infty} \frac{1}{\mu_n - i\tau} + \sum_{n=1}^{\infty} \frac{\mu_n^{(1)}}{(\mu_n - i\tau)^2} \right] = 0,$$

which may be transformed into

$$(7) \qquad \lim_{\tau\to\infty} \tau^2 \left[\sum_{n=1}^{\infty} \frac{\mu_n + \mu_n^{(1)} - \lambda_n}{(\mu_n - i\tau)^2} + \sum_{n=1}^{\infty} \frac{(\mu_n - \lambda_n)^2}{(\lambda_n - i\tau)(\mu_n - i\tau)^2} \right] = 0.$$

The condition p in C^2 is sufficient to establish the estimates

$$\mu_n^{(1)} = \frac{1}{\pi} \int_0^{\pi} p(x)dx + O(n^{-2})$$

and

$$\lambda_n = \mu_n + \frac{1}{\pi} \int_0^{\pi} p(x)dx + O(n^{-2}).$$

Therefore,

$$\sum_{n=1}^{\infty} |\lambda_n - \mu_n - \mu_n^{(1)}| < \infty,$$

$$\sum_{n=1}^{\infty} \tau^2 \frac{\mu_n + \mu_n^{(1)} - \lambda_n}{(\mu_n - i\tau)^2}$$

converges uniformly for $\tau > 0$, and

$$\lim_{\tau\to\infty} \sum_{n=1}^{\infty} \tau^2 \frac{\mu_n + \mu_n^{(1)} - \lambda_n}{(\mu_n - i\tau)^2} = \sum_{n=1}^{\infty} (\lambda_n - \mu_n - \mu_n^{(1)}).$$

Similarly, since

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$$\sum_{n=1}^{\infty} \frac{(\mu_n - \lambda_n)^2}{\mu_n} < \infty,$$

the contribution of the second term in (7) to the limit is zero. This establishes the first conclusion of the theorem.

For $m = 2$, (6) is valid for $s = 2$ and yields

$$(8) \quad \lim_{\tau\to\infty} \tau^2 \left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n - i\tau} - \sum_{n=1}^{\infty} \frac{1}{\mu_n - i\tau} + \sum_{n=1}^{\infty} \frac{\mu_n^{(1)}}{(\mu_n - i\tau)^2} + \sum_{n=1}^{\infty} \frac{\mu_n^{(2)}}{(\mu_n - i\tau)^2} + \sum_{n=1}^{\infty} \frac{(V\phi_n, \phi_n)^2}{(\mu_n - i\tau)^3} \right] = 0.$$

If we impose the additional condition that

$$\int_0^{\pi} p(x)dx = 0,$$

then it can be shown that p in C^4 and $p'(0) = p'(\pi) = 0$ implies $(V\phi_n, \phi_n) = O(1)$,

$$\lambda_n = \mu_n + \frac{1}{2\pi} \int_0^{\pi} p^2(x)dx + O(n^{-2}),$$

and that $\lambda_n - \mu_n - \mu_n^{(1)} - \mu_n^{(2)} = O(n^{-2})$. The first estimate allows us to conclude that

$$\lim_{\tau\to\infty} \tau^2 \sum_{n=1}^{\infty} \frac{(V\phi_n, \phi_n)^2}{(\mu_n - i\tau)^3} = 0$$

so that (8) may be rewritten as

$$(9) \quad \lim_{\tau\to\infty} \tau^2 \left[\sum_{n=1}^{\infty} \frac{\mu_n + \mu_n^{(1)} + \mu_n^{(2)} - \lambda_n}{(\mu_n - i\tau)^2} + \sum_{n=1}^{\infty} \frac{(\mu_n - \lambda_n)^2}{(\lambda_n - i\tau)(\mu_n - i\tau)^2} \right] = 0.$$

The remaining two estimates applied to (9) yield the desired conclusion

$$\sum_{n=1}^{\infty} (\lambda_n - \mu_n - \mu_n^{(1)} - \mu_n^{(2)}) = 0.$$

An auxiliary argument allows us to dispense with the condition

$$\int_0^{\pi} p(x)dx = 0.$$

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https://doi.org/10.4153/CJM-1964-043-4 Published online by Cambridge University Press