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VECTOR-VALUED THEOREM FOR THE UNCENTRED MAXIMAL OPERATOR ON BESSEL–KINGMAN HYPERGROUPS

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Abstract. In this paper we introduce a vector-valued uncentred maximal operator in the setting of one-dimensional Bessel–Kingman hypergoups, and prove a maximal theorem for it.

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1. Introduction. A hypergroup is a pair $(K, *)$, where K is a locally compact space and $*$ is a binary operation (usually called generalized convolution), which is defined on the measure space on K and satisfies certain properties. The reader is referred to a monograph by Bloom and Heyer [**5**] for a precise definition and a thorough description of hypergroups.

An important class of hypergroups is the Chébli–Trimèche hypergroups, which are one-dimensional hypergroups on $\mathbb{R}_+$ with a convolution structure related to the second-order differential operator

$$L_A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where A is a continuous function on $\mathbb{R}_+$, twice continuously differentiable on $]0; +\infty[$ and satisfies the following properties (see [**15**, p. 12]):

(1) $A(0) = 0$ and for every $x > 0$, $A(x) > 0$,
(2) A is increasing and unbounded,
(3) $\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x)$ on a neighbourhood of 0, where $\alpha > -\frac{1}{2}$, and B is an odd and smooth function on $\mathbb{R}$,
(4) $\frac{A'}{A}$ is a decreasing and smooth function on $]0; +\infty[$ and
(5) $\rho = \frac{1}{2}\lim_{x\to+\infty}\left(\frac{A'(x)}{A(x)}\right) \geqslant 0$ exists.

Harmonic analysis of these hypergroups has been developed recently by several authors (see, for instance, [**3**, **4**, **6–9**, **13**]). In particular, a theory of scalar maximal functions has been established (see [**6**, **9**, **13**]). The main aim of this paper is to prove some vector-valued analogues which could be useful for a thorough study of both singular integrals and the Littlewood–Paley theory in this setting.

Therefore, we introduce a vector-valued uncentred maximal operator associated with Bessel–Kingman hypergroups which corresponds to the special case where the function A is defined for every $x \in \mathbb{R}_+$ by $A(x) = x^{2\alpha+1}$ (with $\alpha > -\frac{1}{2}$), and we prove a maximal theorem for it. We restrict ourselves to this case because Haar measure satisfies doubling condition enjoyed by Euclidean spaces or homogeneous spaces; in other words, we do not consider a hypergroup of exponential growth (like Jacobi hypergroup) for which a complete vector-valued maximal theorem seems to be out of reach for the moment. To become more precise, we first define the scalar uncentred maximal operator M by

$$Mf(x) = \sup_{\varepsilon>0,\, z\in I(x,\varepsilon)} \frac{1}{\mathcal{A}(]0,\varepsilon[)} \int_0^{\varepsilon} T_z|f|(y)A(y)\,\mathrm{d}y, \quad x \in \mathbb{R}_+,$$

where we denote by $I(x,\varepsilon)$ the open interval $]\max\{0; x-\varepsilon\}, x+\varepsilon[$, by $\mathcal{A}(]0,\varepsilon[)$ the Haar measure of the interval $]0,\varepsilon[$, with $\mathcal{A}$ the Haar measure on the Bessel–Kingman hypergroup and by T_x (for $x \in \mathbb{R}_+$) the generalized translation by x (see Section 2 for more details). We then define the vector-valued uncentred maximal operator by

$$\overline{M}_r f(\cdot) = \left(\sum_{n=0}^{+\infty} (Mf_n(\cdot))^r\right)^{\frac{1}{r}}, \quad 1 < r < +\infty,$$

where $f = (f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions on $\mathbb{R}_+$. In order to state the main result of this paper, let us introduce some notations.

For $1 < r < +\infty$, we use the following notation:

$$|f(\cdot)|_r = \left(\sum_{n=0}^{+\infty} |f_n(\cdot)|^r\right)^{\frac{1}{r}},$$

and we write $|f(\cdot)|_r \in L^p_A$ (where we denote by L^p_A the space $L^p(\mathbb{R}_+; A(x)\,\mathrm{d}x)$) if

$$\left(\int_{\mathbb{R}_+} \left(\sum_{n=0}^{+\infty} |f_n(x)|^r\right)^{\frac{p}{r}} A(x)\,\mathrm{d}x\right)^{\frac{1}{p}} < +\infty.$$

We also use the notation $\|\cdot\|_{A,p}$ instead of $\|\cdot\|_{L^p_A}$. With these notations in mind, we can now state the following theorem that we will prove.

THEOREM 1.1. *Let $f = (f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}_+$ and A be the function defined on $\mathbb{R}_+$ by $A(x) = x^{2\alpha+1}$, with $\alpha > -\frac{1}{2}$. Let $1 < r < +\infty$.*

(1) *If $|f(\cdot)|_r \in L^1_A$, then for every $\lambda > 0$ we have*

$$\mathcal{A}(\overline{E}_\lambda) \leqslant \frac{C}{\lambda} \int_{\mathbb{R}_+} |f(x)|_r A(x)\,\mathrm{d}x,$$

where $\overline{E}_\lambda = \{x \in \mathbb{R}_+ : \overline{M}_r f(x) > \lambda\}$ and $C = C(\alpha, r)$ is a positive constant independent of f and λ.

(2) *If $|f(\cdot)|_r \in L^p_A$, with $1 < p < +\infty$, then $\overline{M}_r f \in L^p_A$ and*

$$\|\overline{M}_r f\|_{A,p} \leqslant C\|f\|_{A,p},$$

where $C = C(\alpha, r, p)$ is a positive constant independent of f.

The proof for the classical vector-valued maximal operator (associated with the Hardy–Littlewood maximal operator on $\mathbb{R}^d$) is due to Fefferman and Stein [**11**]. Their proof is mainly based on three tools: a Calderón–Zygmund decomposition, a maximal theorem and a weighted inequality for the Hardy–Littlewood maximal operator. However, we cannot apply this method in our setting because of the generalized translation operator which prevents from using classical techniques of real analysis. Thus, our aim is to construct a more convenient operator $\mathcal{M}$ which controls M in the sense that for every $x \in \mathbb{R}_+$ $Mf(x) \leqslant C\mathcal{M}f(x)$ (with C a positive constant independent of x and f) and to prove for $\mathcal{M}$ a maximal theorem and a decisive weighted inequality. Recently, similar techniques have been used in the setting of Dunkl's analysis (see [**10**]).

The paper is organized as follows. In the next section, we recall some definitions and properties which are related to Bessel–Kingman hypergroups and will be relevant for the sequel. Section 3 is devoted to the proof of our main result.

Throughout this paper, C denotes a positive constant, which depends only on fixed parameters, and whose value may vary from line to line.

2. Preliminaries. This section is concerned with the preliminaries and background. We consider the Bessel–Kingman hypergroup $(\mathbb{R}_+, *_A)$, where the function A is given for $x \in \mathbb{R}_+$ by $A(x) = x^{2\alpha+1}$ with $\alpha > -\frac{1}{2}$. The convolution structure is related to the second-order differential operator

$$L = L_A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\alpha+1}{x}\frac{\mathrm{d}}{\mathrm{d}x}.$$

Let us clarify our statement. The solutions φ_λ, $\lambda \in \mathbb{C}$ of the differential equation

$$L\varphi_\lambda(x) = \lambda^2\varphi_\lambda(x), \quad \varphi_\lambda(0) = 1, \quad \varphi_\lambda'(0) = 0 \tag{2.1}$$

are multiplicative (and these solutions give all multiplicative functions on the hypergroup) in the sense that $\varphi_\lambda(x)\varphi_\lambda(y) = \int_{\mathbb{R}_+} \varphi_\lambda(z)\,\mathrm{d}(\varepsilon_x *_A \varepsilon_y)(z)$, where ε_t is the unit point mass at $t \in \mathbb{R}_+$. Solutions of (2.1) are $\varphi_\lambda(\cdot) = j_\alpha(\lambda\cdot)$, where we denote by j_α, for $\alpha > -\frac{1}{2}$, the normalized Bessel function of the first kind and of order α, that is

$$j_\alpha(x) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(x)}{x^\alpha},$$

with J_α, the usual Bessel function of the first kind and of order α, given as

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!\,\Gamma(n+\alpha+1)}.$$

Then the well-known product formula for $x > 0$ and $y > 0$ (see [**14**, p. 367] or [**2**, p. 217]),

$$j_\alpha(x)j_\alpha(y) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi j_\alpha\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)\sin^{2\alpha}\theta\,\mathrm{d}\theta$$

implies for $x > 0$ and $y > 0$ the following one

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^{+\infty} \varphi_\lambda(z)K^\alpha_{x,y}(z)A(z)\,\mathrm{d}z,$$

with $K^{\alpha}_{x,y}$ being the positive function given by

$$K^{\alpha}_{x,y}(z) = \frac{\Gamma(\alpha+1)2^{2\alpha-3}\Big(\big((x+y)^2-z^2\big)\big(z^2-(x-y)^2\big)\Big)^{\alpha-\frac{1}{2}}}{\Gamma\big(\alpha+\frac{1}{2}\big)\Gamma\big(\frac{1}{2}\big)(xyz)^{2\alpha}}\chi_{[|x-y|,x+y]}(z),$$

where χ_X is the characteristic function of the set X. The convolution on the measure space on $\mathbb{R}_+$ is then defined by $\mathrm{d}(\varepsilon_x *_A \varepsilon_y)(z) = K^{\alpha}_{x,y}(z)A(z)\,\mathrm{d}z$ and we have the following support property $\mathrm{supp}(\varepsilon_x *_A \varepsilon_y) = [|x-y|, x+y]$. It is well known (see [**15**, especially Proposition 2.3, Corollary 2.4 and Theorem 4.5]) that $(\mathbb{R}_+, *_A)$ is commutative with neutral element 0 and the identity mapping as involution. Haar measure $\mathcal{A}$ on $(\mathbb{R}_+, *_A)$ is absolutely continuous with respect to the Lebesgue measure and can be chosen to have Lebesgue density A. We denote by $\mathcal{A}(]a,b[)$ the Haar measure of the interval $]a,b[$ for any $0 \leqslant a < b$, that is $\mathcal{A}(]a,b[) = \int_a^b A(x)\,\mathrm{d}x$.
The convolution of two functions f and g is defined by

$$f *_A g(x) = \int_{\mathbb{R}+} T_x f(y) g(y) A(y)\,\mathrm{d}y, \quad x \in \mathbb{R}_+,$$

where T_x is the generalized (left) translation given by

$$T_x f(y) = \int_{\mathbb{R}_+} f(z)\,\mathrm{d}(\varepsilon_x *_A \varepsilon_y)(z) = \int_{\mathbb{R}_+} f(z) K^{\alpha}_{x,y}(z) A(z)\,\mathrm{d}z, \quad y \in \mathbb{R}_+.$$

The convolution is associative and commutative, and since T_x is for every $x \in \mathbb{R}_+$ a bounded operator on L^p_A (for $1 \leqslant p \leqslant +\infty$), the convolution satisfies usual Young's inequalities (see in particular [**1**]). We conclude this section with a sharp inequality which is due to Bloom and Xu [**6**, Proposition 4.6 and Lemma 5.1].

PROPOSITION 2.1. *There exists a positive constant C such that for every $x, y \in \mathbb{R}_+$ and for every $\varepsilon > 0$ we have*

$$\big|T_x(\chi_{]0,\varepsilon[})(y)\big| \leqslant C\,\frac{\mathcal{A}(]0,\varepsilon[)}{\mathcal{A}\big(I(x,\varepsilon)\big)},$$

where we denote by $I(x,\varepsilon)$ the following set

$$I(x,\varepsilon) = \big]\max\{0; x-\varepsilon\}, x+\varepsilon\big[.$$

3. Proof of the main result. This section is devoted to the proof of Theorem 1.1. As we have already claimed, we shall construct a more convenient operator $\mathcal{M}$, which controls M point-wise and for which we can apply standard techniques. For the construction, the idea is to use the inequality of Proposition 2.1 to bypass some difficulties related to the translation operator. The following proposition gives us this new operator $\mathcal{M}$.

PROPOSITION 3.1. *There exists a positive constant C such that for every locally integrable (with respect to $\mathcal{A}$) function f and every $x \in \mathbb{R}_+$ we have*

$$Mf(x) \leqslant C\mathcal{M}f(x),$$

where the operator $\mathcal{M}$ is given by

$$\mathcal{M}f(x) = \sup_{\varepsilon>0,\, z\in I(x,\varepsilon)} \frac{1}{\mathcal{A}\big(I(z,\varepsilon)\big)} \int_{I(z,\varepsilon)} |f(y)|A(y)\,dy.$$

Proof. Let $\varepsilon > 0$, $x \in \mathbb{R}_+$ and $z \in I(x,\varepsilon)$. The commutativity of $*_A$ implies that

$$\int_0^{\varepsilon} T_z|f|(y)A(y)\,dy = \int_{\mathbb{R}_+} |f(y)|T_z(\chi_{]0,\varepsilon[})(y)A(y)\,dy.$$

Using the support property of the generalized translation, it follows at once that

$$\int_0^{\varepsilon} T_z|f|(y)A(y)\,dy = \int_{I(z,\varepsilon)} |f(y)|T_z(\chi_{]0,\varepsilon[})(y)A(y)\,dy.$$

According to Proposition 2.1, we get the existence of a positive constant C such that

$$\int_0^{\varepsilon} T_z|f|(y)A(y)\,dy \leqslant C\frac{\mathcal{A}(]0,\varepsilon[)}{\mathcal{A}\big(I(z,\varepsilon)\big)} \int_{I(z,\varepsilon)} |f(y)|A(y)\,dy.$$

Since this inequality is valid for every $\varepsilon > 0$ and $z \in I(x,\varepsilon)$, we deduce that

$$Mf(x) \leqslant C\mathcal{M}f(x),$$

which is precisely what we wanted to prove. □

As a trivial consequence of the above proposition, we have for $1 < r < +\infty$

$$\overline{M}_r f(\cdot) = \left(\sum_{n=0}^{+\infty}\Big(Mf_n(\cdot)\Big)^r\right)^{\frac{1}{r}} \leqslant C\left(\sum_{n=0}^{+\infty}\Big(\mathcal{M}f_n(\cdot)\Big)^r\right)^{\frac{1}{r}} = C\overline{\mathcal{M}}_r f(\cdot).$$

Then we are left with the task of establishing the following result in order to prove Theorem 1.1.

THEOREM 3.1. *Let $f = (f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}_+$. Let $1 < r < +\infty$.*

(1) *If $|f(\cdot)|_r \in L^1_A$, then for every $\lambda > 0$ we have*

$$\mathcal{A}\big(\overline{\mathcal{E}}_\lambda\big) \leqslant \frac{C}{\lambda}\int_{\mathbb{R}_+} |f(x)|_r A(x)\,dx,$$

where $\overline{\mathcal{E}}_\lambda = \{x \in \mathbb{R}_+ : \overline{\mathcal{M}}_r f(x) > \lambda\}$, and $C = C(\alpha, r)$ is a positive constant independent of f and λ.

(2) *If $|f(\cdot)|_r \in L^p_A$, with $1 < p < +\infty$, then $\overline{\mathcal{M}}_r f \in L^p_A$ and*

$$\|\overline{\mathcal{M}}_r f\|_{A,p} \leqslant C\|f\|_{A,p},$$

where $C = C(\alpha, r, p)$ is a positive constant independent of f.

Following the proof given in **[11]**, we claim that this theorem is proved if we establish a maximal theorem and a weighted inequality for $\mathcal{M}$. Indeed, the case $p = r$ is nothing more than a scalar case (that is, we only use a maximal theorem); the case

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$p = 1$ is based on a Calderón–Zygmund decomposition; the case $1 < p < r$ is easily deduced from the two previous cases by the Marcinkiewicz interpolation theorem; the case $r < p < +\infty$ is based on a weighted inequality. Let us begin with the following maximal theorem.

THEOREM 3.2. *Let f be a measurable function defined on $\mathbb{R}_+$.*

(1) *If $f \in L^1_A$, then for every $\lambda > 0$ we have*

$$\mathcal{A}(\mathcal{E}_\lambda) \leqslant \frac{C}{\lambda} \int_{\mathbb{R}_+} |f(x)| A(x)\, \mathrm{d}x,$$

where $\mathcal{E}_\lambda = \{x \in \mathbb{R}_+ : \mathcal{M}f(x) > \lambda\}$ and $C = C(\alpha, r)$ is a positive constant independent of f and λ.

(2) *If $f \in L^p_A$, with $1 < p \leqslant +\infty$, then $\mathcal{M}f \in L^p_A$ and*

$$\|\mathcal{M}f\|_{A,p} \leqslant C\|f\|_{A,p},$$

where $C = C(\alpha, r, p)$ is a positive constant independent of f.

For the first inequality of the previous theorem, we need the following covering lemma of Vitali type (the proof can be found in [**12**, p. 9], see also [**6**, Lemma 4.21]).

LEMMA 3.1. *Let E be a measurable (with respect to $\mathcal{A}$) subset of $\mathbb{R}_+$. Suppose that we have $E \subset \cup_{j\in J} I_j$ with $I_j = I(z_j, r_j)$ bounded for every $j \in J$ (where $z_j \in \mathbb{R}_+$ and $r_j > 0$). Then, from this family, we can choose a sequence (which may be finite) of disjoint sets $I_1, \ldots, I_n, \ldots$, such that*

$$\mathcal{A}(E) \leqslant C \sum_n \mathcal{A}(I_n),$$

where C is a positive constant, which depends only on α.

REMARK 3.1. In the standard proof of this lemma (which uses the doubling property of $\mathcal{A}$), we note that $E \subset \bigcup_n I_n^5$, where for every integer $k \geqslant 1$, $I^k(x, \varepsilon) = I(x, k\varepsilon)$.

Thanks to this lemma, we can now turn to the proof of Theorem 3.2.

Proof. Let us begin with the first inequality.
Let $f \in L^1_A$, $\lambda > 0$ and $x \in \mathcal{E}^\times_\lambda = \mathcal{E}_\lambda \setminus \{0\}$. By definition, there exist $\varepsilon_x > 0$ and $z_x \in I(x, \varepsilon_x)$ such that

$$\lambda \mathcal{A}\big(I(z_x, \varepsilon_x)\big) < \int_{I(z_x,\varepsilon_x)} |f(y)| A(y)\, \mathrm{d}y. \tag{3.1}$$

Since we have $x \in I(z_x, \varepsilon_x)$ (since $x \in]0, +\infty[$ and $z_x \in I(x, \varepsilon_x)$), we assert that $\mathcal{E}^\times_\lambda \subset \bigcup_{x\in\mathcal{E}^\times_\lambda} I(z_x, \varepsilon_x)$. Thanks to the previous lemma, we can then select a disjoint collection of intervals denoted by $I_1 = I(z_1, \varepsilon_1), \ldots, I_n = I(z_n, \varepsilon_n), \ldots$, with each I_n satisfying (3.1) and such that $\mathcal{A}(\mathcal{E}_\lambda) = \mathcal{A}(\mathcal{E}^\times_\lambda) \leqslant C \sum_n \mathcal{A}(I_n)$, with C being a positive constant depends only on α. It follows that

$$\mathcal{A}(\mathcal{E}_\lambda) \leqslant \frac{C}{\lambda} \sum_n \int_{I_n} |f(y)| A(y)\, \mathrm{d}y \leqslant \frac{C}{\lambda} \int_{\cup_n I_n} |f(y)| A(y)\, \mathrm{d}y \leqslant \frac{C}{\lambda} \|f\|_{A,1},$$

where we have used inequality (3.1) in the first step, the disjoint property of the intervals I_n in the second step and where we have enlarged the domain of the integral in the last step. The first inequality of Theorem 3.2 is then proved. There is nothing to do for the second one. Indeed, by the Marcinkiewicz interpolation theorem (see **[12]**), it is a simple consequence of the trivial fact that $\mathcal{M}$ is bounded on L^∞_A together with the first inequality. The whole theorem is then proved. □

We now state a weighted inequality for the operator $\mathcal{M}$.

THEOREM 3.3. *Let W be a positive and locally integrable (with respect to $\mathcal{A}$) function defined on $\mathbb{R}_+$. For $1 < r < +\infty$, there exists a positive constant C, which depends only on α and r and such that for every $f \in L^r(\mathbb{R}_+; \mathcal{M}W(x)A(x)\,dx)$*

$$\int_{\mathbb{R}_+} (\mathcal{M}f(y))^r W(y)A(y)\,dy \leqslant C \int_{\mathbb{R}_+} |f(y)|^r \mathcal{M}W(y)A(y)\,dy.$$

Proof. By the Marcinkiewicz interpolation theorem and since the operator $\mathcal{M}$ is obviously bounded on L^∞_A, this theorem is a consequence of the the following inequality

$$\mathcal{A}^W(\mathcal{E}_\lambda) \leqslant \frac{C}{\lambda} \int_{\mathbb{R}_+} |f(y)|\mathcal{M}W(y)A(y)\,dy, \quad \lambda > 0, \tag{3.2}$$

where $\mathcal{A}^W(X) = \int_X W(y)A(y)\,dy$ and C is a positive constant, which depends only on α. Thus, we now turn to the proof of (3.2).
Let E be any compact subset of $\mathcal{E}^\times_\lambda$. By a reprise of the argument given in the proof of Theorem 3.2, we have the existence of a disjoint collection of intervals denoted by $I_1 = I(z_1, \varepsilon_1), \ldots, I_n = I(z_n, \varepsilon_n), \ldots$ so that $\mathcal{E}^\times_\lambda \subset \bigcup_n I^5_n$ (invoking Remark 3.1), with each I_n satisfying

$$\lambda\mathcal{A}(I_n) < \int_{I_n} |f(y)|A(y)\,dy. \tag{3.3}$$

Since E is a compact subset of $\mathcal{E}^\times_\lambda$, we can then select a finite and disjoint subcollection $(I_{n_k})_{1\leqslant k\leqslant m}$ from the sequence $(I_n)_n$ such that $E \subset \bigcup_{1\leqslant k\leqslant m} I^5_{n_k}$.
Let t be an element of I_{n_k}. Then $z_{n_k} \in I(t, 5\varepsilon_{n_k})$ and we can write

$$\int_{I(z_{n_k},5\varepsilon_{n_k})} W(y)A(y)\,dy \leqslant \mathcal{A}\big(I\big(z_{n_k}, 5\varepsilon_{n_k}\big)\big)\mathcal{M}W(t) \leqslant C\mathcal{A}\big(I\big(z_{n_k}, \varepsilon_{n_k}\big)\big)\mathcal{M}W(t),$$

where we have used the definition of the operator $\mathcal{M}$ for the first inequality and the doubling property of the measure $\mathcal{A}$ for the second one. We obtain by multiplying both sides by $|f(t)|A(t)$ and by integrating over I_{n_k}

$$\left(\int_{I_{n_k}} |f(t)|A(t)\,dt\right)\left(\int_{I(z_{n_k},5\varepsilon_{n_k})} W(y)A(y)\,dy\right) \leqslant C\mathcal{A}\big(I\big(z_{n_k}, \varepsilon_{n_k}\big)\big) \int_{I_{n_k}} |f(t)|\mathcal{M}W(t)A(t)\,dt.$$

On account of (3.3), we are readily led to

$$\left(\int_{I_{n_k}^5} W(y)\mathcal{A}(y)\,\mathrm{d}y\right) \leqslant \frac{C}{\lambda}\int_{I_{n_k}} |f(t)|\mathcal{M}W(t)A(t)\,\mathrm{d}t. \tag{3.4}$$

Since we have

$$\mathcal{A}^W(E) \leqslant \mathcal{A}^W\left(\bigcup_{1\leqslant k\leqslant m} I_{n_k}^5\right) \leqslant \sum_{1\leqslant k\leqslant m}\left(\int_{I_{n_k}^5} W(y)A(y)\,\mathrm{d}y\right),$$

we can deduce from (3.4) that

$$\mathcal{A}^W(E) \leqslant \frac{C}{\lambda}\sum_{1\leqslant k\leqslant m}\int_{I_{n_k}} |f(t)|\mathcal{M}W(t)A(t)\,\mathrm{d}t.$$

We obtain by using the disjoint property of $(I_{n_k})_{1\leqslant k\leqslant m}$ and then by enlarging the domain of the integral

$$\mathcal{A}^W(E) \leqslant \frac{C}{\lambda}\int_{\mathbb{R}_+} |f(t)|\mathcal{M}W(t)A(t)\,\mathrm{d}t.$$

It follows at once that

$$\mathcal{A}^W(\mathcal{E}_\lambda^\times) \leqslant \frac{C}{\lambda}\int_{\mathbb{R}_+} |f(t)|\mathcal{M}W(t)A(t)\,\mathrm{d}t,$$

from which we deduce inequality (3.2). Then the theorem is proved. □

REMARK 3.2. Since we readily have $\mathrm{M}f(x) \leqslant Mf(x)$, where M is the centred maximal operator introduced by Bloom and Xu [**6**] and given by

$$\mathrm{M}f(x) = \sup_{\varepsilon>0}\frac{1}{\mathcal{A}(]0,\varepsilon[)}\int_0^\varepsilon T_x|f|(y)A(y)\,\mathrm{d}y, \quad x\in\mathbb{R}_+,$$

it follows that Theorem 1.1 is also true if we replace $\overline{M}_r$ by the operator $\overline{\mathrm{M}}_r$ given by

$$\overline{\mathrm{M}}_r f(\cdot) = \left(\sum_{n=0}^{+\infty}\left(\mathrm{M}f_n(\cdot)\right)^r\right)^{\frac{1}{r}}, \quad f=(f_n)_{n\in\mathbb{N}}.$$

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