ON THE STRUCTURE OF HALF-GROUPS

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1. Introduction. Furstenberg **(1)** and the author **(4)**, among others, have exhibited postulate systems for groups in terms of the operation $x - y = x + (-y)$. Furstenberg also investigated a system obtained by removing one of his postulates which defines what he called a half-group. A structure theorem for half-groups was given in **(1)**. In the present paper, we prove another structure theorem for half-groups. A more restricted entity, called a pseudo-group, is introduced, and its structure, together with that of a half-group satisfying the left and right cancellation laws, is studied. Finally, some topological questions concerning their structure are also considered.

2. Pseudo-groups. Let G be a set of elements over which is defined a binary operation $x - y$ which satisfies, for any $x, y, z \in G$, the following conditions:

(i) $x - y \in G$,

(ii) There is an $e \in G$ such that $x - y = e$ if, and only if, $x = y$,

(iii) $(x - z) - (y - z) = x - y$.

It was shown in **(4)** that, in terms of the operation $x + y = x - (e - y)$, G is a group. It was also shown that if G satisfies (i), (ii), and

(iv) $(x - z) - (x - y) = y - z$,

then G is a commutative group.

We define a half-group to be a set with an operation satisfying (i) and (iii). A half-group which also has the property

(v) There is an $e \in G$ such that $x - x = e$

will be called a pseudo-group. Before proceeding to a discussion of half-groups, we shall consider briefly the structure of pseudo-groups.

We define a set S to be an extension of a group G if $G \subset S$ and if there exists a single-valued function f defined on S such that $f(x) \in G$ for all $x \in S$ and $f(u) = u$ for all $u \in G$. The set S becomes a pseudo-group under the operation

$$x \circ y = f(x) - f(y), \qquad x, y \in S. \tag{1}$$

Indeed, $x \circ y \in G \subset S$, and $x \circ x = f(x) - f(x) = e$. Finally,

$$\begin{aligned}(x \circ z) \circ (y \circ z) &= f(f(x) - f(z)) - f(f(y) - f(z)) \\ &= (f(x) - f(z)) - (f(y) - f(z)) = f(x) - f(y) = x \circ y,\end{aligned}$$

and (i), (iii), and (v) are satisfied.

THEOREM 1. *Every pseudo-group S contains a group G such that S is an extension of G and the operation in S satisfies* (1).

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Proof. Let $x - y$ be the operation in S, and set $f(x) = x - e$. We will show that S is an extension of the group $G = f(S)$. For any $x, y \in S$, we have

$$(x - y) - e = (x - y) - (y - y) = x - y.$$

Setting $y = e$, we obtain $f(f(x)) = f(x)$, whence $f(u) = u$ for all $u \in G$. It also follows that if $u, v \in G$, then $u - v = f(u - v) \in G$, and G satisfies (i). Since (iii) and (v) hold in S, they hold also in G. To verify the remaining half of (ii), we observe that if $u, v \in G$ and $u - v = e$, then, since $f(u) = u$ and $f(v) = v$,

$$\begin{aligned} u = u - e &= (u - v) - (e - v) = e - (e - v) \\ &= (v - v) - (e - v) = v - e = v. \end{aligned}$$

Hence, G is a group. Finally, it follows from (iii) that, for all $x, y \in S$, $x - y = f(x) - f(y)$, and the proof is complete.

In any pseudo-group S, we may introduce the sum $x + y = x - (e - y)$ for all $x, y \in S$. Then

$$x + y = f(x) - f(e - y) = f(x) - (e - f(y)) = f(x) + f(y).$$

Hence, this operation is associative. A pseudo-group is said to be commutative if $x + y = y + x$.

THEOREM 2. *If S is a set with an operation satisfying* (i) *and* (iv), *then S is a commutative pseudo-group.*

Proof. For any $x, y, z \in S$, we have

$$\begin{aligned} (x - z) - (y - z) &= ((z - z) - (z - x)) - ((z - z) - (z - y)) \\ &= (z - y) - (z - x) = x - y, \end{aligned}$$

which proves (iii). Next, $x - x = (x - y) - (x - y) = y - y = e$. Since the group $G = S - e$ also satisfies (iv), G is commutative. From the remark preceding Theorem 2, it follows that $x + y = y + x$ for all $x, y \in S$.

In terms of subtraction, the left and right cancellation laws take the following forms: for all $x, y, z \in S$,

(vi) $x - y = x - z$ implies $y = z$,

(vii) $x - z = y - z$ implies $x = y$.

The next theorem shows, incidentally, that (vi) implies (vii) in a half-group. That (vii) does not imply (vi) can be seen by considering what we shall call the simple half-group, a set in which $x - y$ is defined to be x. The structure of half-groups satisfying (vii) will be taken up in the next section.

THEOREM 3. *A half-group in which the left cancellation law holds is a group.*

Proof. Suppose that S satisfies (i), (iii), and (vi). For any $u, x, y, z \in S$, we have

$$\begin{aligned} (u - x) - z &= ((u - x) - (x - x)) - (z - (x - x)) \\ &= (u - x) - (z - (x - x)). \end{aligned}$$

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Hence, $z - (x - x) = z = z - (y - y)$, and $x - x = y - y = e$. If $x - y = e = x - x$, then $x = y$ and (ii) is satisfied. Therefore, G is a group.

3. Half-groups. Following Furstenberg **(1)**, we introduce the following definitions. We call an element a of a half-group H an idempotent if $a - a = a$. A subset $K \subset H$ is called invariant in H if $x - (x - K) \subset K$ for all $x \in H$. If K is a sub-half-group of H which contains all idempotents in H, then the relation $x \equiv y$ defined by $x - y \in K$ is an equivalence relation. In fact, $x \equiv x$ since $x - x$ is an idempotent. If $x \equiv z$ and $y \equiv z$, then $x - y = (x - z) - (y - z) \in K$, and $x \equiv y$. The resulting quotient space will be denoted by H/K. A weaker form of the following theorem was proved in **(1**, Theorem 4**)**.

THEOREM 4. *In the half-group H, let K be an invariant sub-half-group containing all idempotents of H. Then H/K is a group.*

Proof. We refer to **(1)** for the proof that if $u, v, x, y \in H$ and $u \equiv x$, $v \equiv y$, then $u - v \equiv x - y$. Thus, if x^* and y^* are the equivalence classes containing x and y, respectively, then $(x - y)^* = x^* - y^*$. Clearly H/K satisfies (i) and (iii). If e^* is the equivalence class containing K, then $x^* - x^* = e^*$, since $x - x$ is an idempotent. Finally, if $x^* - y^* = e^*$, then $x^* = y^*$, and (ii) is satisfied. Hence, H/K is a group.

Let G be a group and $\{S_\alpha\}_{\alpha \in A}$ a family of disjoint sets, indexed by the set A, with the following properties:

(a) For each $\alpha \in A$, there exists a single-valued function f_α mapping S_α onto G,

(b) For each $\alpha, \beta \in A$, there exists a one-to-one function $\phi_{\alpha\beta}$ mapping G into S_α such that $f_\alpha(\phi_{\alpha\beta}(g)) = g$ for each $g \in G$.

We now define an operation $x \circ y$ in $H = \bigcup S_\alpha$ as follows: if $x \in S_\alpha$ and $y \in S_\beta$, then

$$x \circ y = \phi_{\alpha\beta}(f_\alpha(x) - f_\beta(y)). \tag{2}$$

We note that $x \circ y \in S_\alpha$. Under this operation, H becomes a half-group. Since (i) is clearly satisfied, we have only to verify (iii). If $z \in S_\gamma$, we have

$$\begin{aligned}(x \circ z) \circ (y \circ z) &= \phi_{\alpha\beta}\{f_\alpha(\phi_{\alpha\gamma}(f_\alpha(x) - f_\gamma(z))) - f_\beta(\phi_{\beta\gamma}(f_\beta(y) - f_\gamma(z)))\} \\ &= \phi_{\alpha\beta}((f_\alpha(x) - f_\gamma(z)) - (f_\beta(y) - f_\gamma(z))) \\ &= \phi_{\alpha\beta}(f_\alpha(x) - f_\beta(y)) = x \circ y.\end{aligned}$$

Each S_α becomes a pseudo-group in which $\phi_{\alpha\alpha} f_\alpha$ plays the role of f in the representation of pseudo-groups given in § 2. If e is the identity of G, then $\phi_{\alpha\alpha}(e) = e_\alpha$ is the identity in S_α, since $x \circ x = e_\alpha$ for all $x \in S_\alpha$. The idempotents in H are precisely the elements e_α for all $\alpha \in A$. It follows from (2) that $\phi_{\alpha\beta}(f_\alpha(x)) = x \circ e_\beta$ for all $x \in S_\alpha$. In particular, $S_\alpha \circ e_\alpha$ is isomorphic to G.

THEOREM 5. *Every half-group H can be represented as the union of disjoint pseudo-groups in which the operation is defined by* (2).

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Proof. Let $E = \{e_\alpha\}_{\alpha \in A}$ be the set of idempotents of H indexed by the set A, and, for each $\alpha \in A$, let $S_\alpha = \{x \in H: x - x = e_\alpha\}$. If $x \in S_\alpha$ and $y \in H$, then clearly $x - y \in S_\alpha$, and S_α satisfies (i). Since (iii) holds in H, it holds also in S_α. Finally, S_α satisfies (v) with $e = e_\alpha$, so that S_α is a pseudo-group. In addition, $\alpha \neq \beta$ implies $S_\alpha \cap S_\beta = \emptyset$, and $H = \bigcup S_\alpha$.

We turn now to the construction of G and the functions appearing in (2). For any $x, y \in H$ and $a \in E$, we have

$$\begin{aligned}(x - a) - y &= ((x - a) - (a - a)) - (y - (a - a)) \\ &= (x - a) - (y - a) = x - y.\end{aligned}$$

Similarly, if $b \in E$, then $x - (y - b) = x - y$. Combining these relations, we obtain

$$(3) \qquad (x - a) - (y - b) = x - y.$$

(For the sake of compactness, we shall also let (3) stand for the two preceding relations.) If we set $F = E - E = \{u - v: u, v \in E\}$, then F is a sub-half-group by virtue of (3). Repeated application of (3) also shows that $x - (x - z) = x - x \in F$ for all $x \in H$ and $z \in F$. Hence, by Theorem 4, $H/F = G$ is a group. The corresponding equivalence relation will be written $x \equiv y$, and the quotient mapping from H onto G will be denoted by h.

For each $\alpha \in A$, let $G_\alpha = S_\alpha - e_\alpha$. We now define functions $\psi_\alpha: H \to G_\alpha$ as follows:

$$\psi_\alpha(x) = e_\alpha - (e_\alpha - x).$$

For any $x \in H$, we have $e_\alpha - x \in S_\alpha$. From Theorem 1 it follows that the difference of any two elements in S_α lies in $S_\alpha - e_\alpha = G_\alpha$, whence $\psi_\alpha(x) \in G_\alpha$. In addition,

$$h(\psi_\alpha(x)) = h(e_\alpha) - (h(e_\alpha) - h(x)) = e - (e - h(x)) = h(x),$$

where e is the identity of G. Hence, $x \equiv \psi_\alpha(x)$ and

$$(4) \qquad x - \psi_\alpha(x) = e_\gamma - e_\delta.$$

If $x \in S_\beta$, then $x - \psi_\alpha(x) \in S_\beta$ and $e_\gamma - e_\delta \in S_\gamma$, whence $\gamma = \beta$. (Further calculation shows that $\delta = \alpha$, but we shall not need this fact.) Thus every equivalence class intersects each G_α. Furthermore, no two elements of G_α are equivalent, that is, $G_\alpha \cap F = \{e_\alpha\}$. In fact, if $y \in G_\alpha$ and $y = e_\alpha - e_\beta$, then $y = y - e_\alpha = (e_\alpha - e_\beta) - e_\alpha = e_\alpha$. Hence, the restriction of h to G_α is an isomorphism h_α between G_α and G.

The functions f_α are now defined to be the restrictions of h to S_α, and the $\phi_{\alpha\beta}$ are defined, for each $\alpha, \beta \in A$, and each $g \in G$, by

$$\phi_{\alpha\beta}(g) = h_\alpha^{-1}(g) - e_\beta.$$

Since $x - e_\beta \equiv x$ for all $x \in S_\alpha$ because of (3), it follows that $f_\alpha(\phi_{\alpha\beta}(g)) = g$ for all $g \in G$, and conditions (a) and (b) are fulfilled. Moreover, if $x \in S_\alpha$, then $h_\alpha{}^{-1}(f_\alpha(x)) = x - e_\alpha$, for each member is equivalent to x and lies in

G_α, and such elements in G_α must be identical. Similarly, if $y \in S_\beta$, then $h_\alpha^{-1}(f_\beta(y)) = \psi_\alpha(y)$, for each member lies in G_α and is equivalent to y. Finally, if $x \in S_\alpha$ and $y \in S_\beta$, then, using (3) and (4),

$$\begin{aligned} x - y &= (x - \psi_\alpha(y)) - (y - \psi_\alpha(y)) \\ &= ((x - e_\alpha) - \psi_\alpha(y)) - (e_\beta - e_\delta) \\ &= (h_\alpha^{-1}(f_\alpha(x)) - h_\alpha^{-1}(f_\beta(y))) - e_\beta \\ &= h_\alpha^{-1}(f_\alpha(x) - f_\beta(y)) - e_\beta = \phi_{\alpha\beta}(f_\alpha(x) - f_\beta(y)). \end{aligned}$$

This completes the proof.

COROLLARY. *Let $K \supset E$ be an invariant sub-half-group of H. Then there exists an invariant subgroup N of G such that $K/F \cong N$ and $H/K \cong G/N$.*

Proof. We adopt the notation of the proof of Theorem 5. Let $h(K) = N$. Then N is clearly an invariant subgroup of G. Since the restriction of h to K is the canonical mapping from K onto K/F, N and K/F are isomorphic. Now let $L = h^{-1}(N)$ and k be the canonical mapping from H onto H/K. Clearly $x - y \in K$ implies $x - y \in L$ for all $x, y \in H$. Conversely, if $x - y \in L$, then there exists a $z \in K$ such that $x - y \equiv z \pmod F$. Since $K \supset F$, $x - y \equiv z \pmod K$ and

$$k(x) - k(y) = k(x - y) = k(z) = \bar{e},$$

where $\bar{e}$ is the identity in H/K. Hence, $k(x) = k(y)$ and $x - y \in K$.

Let θ be the canonical mapping from G onto G/N. For any $x, y \in H$, the relation $\theta(h(x)) = \theta(h(y))$ is evidently equivalent to $h(x) - h(y) \in N$ and, hence, to $x - y \in L$. Thus, the correspondence between H/K and G/Ndefined by $x^* \leftrightarrow \theta(h(x))$, where x^* is the equivalence class (mod K) containing x, is one-to-one and onto. Finally,

$$x^* - y^* = (x - y)^* \leftrightarrow \theta(h(x - y)) = \theta(h(x)) - \theta(h(y)),$$

and the correspondence is a group isomorphism.

Let $\{G_\alpha\}_{\alpha \in A}$ be a family of disjoint groups, indexed by the set A, satisfying the following condition:

(c) For each $\alpha, \beta \in A$, there exists a unique isomorphism $\theta_{\alpha\beta}: G_\beta \to G_\alpha$. In addition, for all $\alpha, \beta, \gamma \in A$, $\theta_{\alpha\beta}.\theta_{\beta\gamma} = \theta_{\alpha\gamma}$.

It follows from (c) that $\theta_{\alpha\alpha}(x) = x$ for all $x \in G_\alpha$. We now define an operation $x \circ y$ in $H = \bigcup G_\alpha$ as follows: if $x \in G_\alpha$ and $y \in G_\beta$, then

$$x \circ y = x - \theta_{\alpha\beta}(y). \tag{5}$$

Thus $x \circ y \in G_\alpha$. If $z \in G_\gamma$, then

$$\begin{aligned} (x \circ z) \circ (y \circ z) &= (x - \theta_{\alpha\gamma}(z)) - \theta_{\alpha\beta}(y - \theta_{\beta\gamma}(z)) \\ &= (x - \theta_{\alpha\gamma}(z)) - (\theta_{\alpha\beta}(y) - \theta_{\alpha\gamma}(z)) = x \circ y. \end{aligned}$$

Hence, H is a half-group. Furthermore, if $x \circ z = y \circ z$, then $\alpha = \beta$ and $x = y$, whence H satisfies (vii).

We can define a second operation $x \oplus y$ in H as follows: $x \oplus y = x \circ (e_\gamma \circ y)$, where e_γ is the identity in G_γ. We have

$$(6)\quad x \oplus y = x - \theta_{\alpha\gamma}(e_\gamma - \theta_{\gamma\beta}(y)) = x - (e_\alpha - \theta_{\alpha\beta}(y)) = x + \theta_{\alpha\beta}(y).$$

Thus $x \oplus y$ is independent of γ, and it is easily verified that this operation is associative. In addition, each e_γ is a right identity of H.

THEOREM 6. *If H is a half-group in which the right cancellation law holds, then H can be represented as the union of disjoint, isomorphic groups in which the operation is defined by* (5). *Moreover, H is a semigroup under the operation defined by* (6).

Proof. We adopt the notation of the proof of Theorem 5. It follows from (3) that, for all $x \in H$ and $\alpha \in A$, $x - e_\alpha = (x - e_\alpha) - e_\alpha$, whence, by (vii), $x = x - e_\alpha$. Thus $S_\alpha = G_\alpha$, and f_α is one-to-one. From (b) we infer that $\phi_{\alpha\beta} = f_\alpha^{-1}$ for all $\beta \in A$. If $x \in G_\alpha$ and $y \in G_\beta$, then

$$x - y = f_\alpha^{-1}(f_\alpha(x) - f_\beta(y)) = x - f_\alpha^{-1}(f_\beta(y)).$$

If we set $f_\alpha^{-1}f_\beta = \theta_{\alpha\beta}$, then the conditions of (c) are fulfilled and (5) is verified. The rest of the theorem follows from the discussion preceding it.

COROLLARY. *If H is a half-group satisfying* (vii) *in which the operation $x \oplus y$ is commutative, then H is a* (*commutative*) *group.*

Proof. If $x \in G_\alpha$ and $y \in G_\beta$, then $x \oplus y \in G_\alpha$ and $y \oplus x \in G_\beta$, whence $\alpha = \beta$ and $x \oplus y = x + y$.

4. Topological pseudo-groups.

A topological pseudo-group S is a pseudo-group with a topology J in which $x - y$ is continuous jointly in x and y. Thus $G = S - e$ is a topological group in its relative topology. This definition, however, places very little restriction on the topology in the complement of G, and it will sometimes be necessary to impose the additional condition that the topology does not separate x and $x - e$:

(viii) For every $U \in J$ and every $x \in S$, $x \in U$ if, and only if, $x - e \in U$.

Let S be an extension of a topological group G with topology $J(G)$, and let f be defined as in § 2. We introduce a topology $J(S)$ into S as follows:

$$(7)\qquad J(S) = \{f^{-1}(U) : U \in J(G)\}.$$

With this topology and the operation defined in (1), S becomes a topological pseudo-group. To show that subtraction is continuous, let $x, y \in S$ and $x \circ y \in W$, where $W \in J(S)$. Then there exist sets $U_0, V_0 \in J(G)$ such that $f(x) \in U_0$, $f(y) \in V_0$, and $U_0 - V_0 \subset f(W)$. Hence, $x \in U = f^{-1}(U_0)$, $y \in V = f^{-1}(V_0)$, and $U \circ V \subset W$. Since $f(x) = x \circ e$, (viii) is also satisfied.

THEOREM 7. *Every topological pseudo-group S which satisfies* (viii) *can be represented as an extension of a topological group in which the operation is defined by* (1) *and the topology by* (7).

Proof. Let $G = S - e$ and $f(x) = x - e$ for $x \in S$. Then G is a topological group in its relative topology, and f is continuous on S. From (viii) it follows that if $U \in J(S)$, then $f^{-1}(U \cap G) = U$, which verifies (7). The rest follows from Theorem 1.

In each of the following corollaries, S denotes a topological pseudo-group satisfying (viii). We define S to be T_i $(i = 1, 2)$ if $S - e$ is T_i in its relative topology. A topological space is T_1 if each of its points is closed, and T_2 if every two distinct points have disjoint neighbourhoods. A space is first countable if every point has a countable base of neighbourhoods.

COROLLARY 1. *If $C \subset S$ is closed, and $x \in S$ is such that $x - e \notin C - e$, then there exists a continuous ϕ: $S \to [0, 1]$ such that $\phi(x) = 0$ and $\phi(y) = 1$ for $y \in C$.*

Proof. Since $f(x) \notin f(C)$ and $f(C)$ is closed in G, there exists (**3**, p. 188) a continuous ϕ_0: $G \to [0, 1]$ such that $\phi_0(f(x)) = 0$ and $\phi_0 = (f(y)) = 1$ for $y \in C$. Hence, $\phi = \phi_0 f$ has the desired properties.

COROLLARY 2. *If S is first countable, then S is pseudo-metrizable with a right-invariant pseudo-metric.*

Proof. Since G is first countable in its relative topology, it follows from (**3**, p. 210) that G is pseudo-metrizable with a right-invariant pseudo-metric $r(x, y)$. Then $s(x, y) = r(f(x), f(y))$ is the desired pseudo-metrix for S. Since $f(x - z) = f(x) - f(z)$, s is also right-invariant.

COROLLARY 3. *If S is T_1 and locally compact, then there exists a right-invariant Haar measure μ in S.*

Proof. Since G is T_1 and locally compact, there exists a right-invariant Haar measure μ_0 on the Borel sets of G (cf. **2**, p. 254). Each Borel set $B \subset S$ is evidently of the form $f^{-1}(B_0)$ where B_0 is a Borel set relative to G. Hence, $\mu(B) = \mu_0(f(B))$, and the right-invariance follows as in the proof of Corollary 2.

To show the necessity for some additional assumption such as (viii) in the preceding corollaries, we give the following example, based on one in (**1**), in which S is a topological pseudo-group which does not satisfy (viii). In addition, S is compact, connected, first countable, and T_1, but not T_2, and hence, not pseudo-metrizable. Let S be the interval $[0, 2]$, and let the operation in S be defined as subtraction (mod 1). Then S is evidently a commutative pseudo-group, and $G = [0, 1)$. The neighbourhoods of the points in $(0, 2]$ are those of the relative Euclidean topology, while the neighbourhoods of 0 are of the form $[0, \rho) \cup (\sigma, 1)$, where $\rho, \sigma \in (0, 1)$. Evidently S is a topological pseudo-group, and the other properties of S are easily verified. In particular, 0 and 1 have no disjoint neighbourhoods.

5. Topological half-groups. A topological half-group H is defined as a half-group with a topology in which $x - y$ is jointly continuous in x and y.

For the theorems in this section, we do not need as strong a condition as (viii), and we shall require only the following:

(ix) For every $U \in J(H)$ and every $x \in H$, $x \in U$ implies $x - (x - x) \in U$.

The notation of the proof of Theorem 5 will be adhered to throughout this section. From the foregoing definition, we see that each G_α is a (not necessarily T_1) topological group in its relative topology, and each S_α is a topological pseudo-group.

THEOREM 8. *Let H be a topological half-group satisfying* (ix) *and $K \supset E$ an invariant sub-half-group of H. Then the quotient mapping from H onto H/K becomes an open, continuous mapping when H/K is given the quotient topology, and H/K becomes a topological group.*

Proof. We first consider the case $K = F$ and show that, for any open $U \subset H$, $U^* = h^{-1}(h(U))$ is open. Let $V = \cup_{\alpha \in A} \psi_\alpha^{-1}(U)$. Since ψ_α is continuous on H and $\psi_\alpha(x) \equiv x$ for all $x \in H$, V is open and $V \subset U^*$. If $x \in U^*$, then $x \equiv u$ for some $u \in U$. Now $u \in S_\alpha$ for some $\alpha \in A$, and $u \equiv u - e_\alpha = u - (u - u) \in G_\alpha \cap U$ in view of (ix). Thus, we can assume that $u \in G_\alpha$. Then $\psi_\alpha(x) \equiv \psi_\alpha(u) = u$, and since $\psi_\alpha(x) \in G_\alpha$, we have $\psi_\alpha(x) = u$. Hence, $x \in V$ and $U^* \subset V$, so that $U^* = V$ is open. It follows immediately that h is continuous and open, and G is a topological group in the quotient topology.

Now let $K \supset E$ be an arbitrary invariant sub-half-group of H, k the quotient mapping from H onto H/K, and θ the quotient mapping from G onto G/N, where $N = h(K)$. From the corollary to Theorem 5, it follows that, for each $x \in H$, $x^* = k^{-1}(k(x)) = h^{-1}(\theta^{-1}(\theta(h(x))))$. Since h and θ are each open, continuous mappings, we conclude that if $U \subset H$ is open, then $U^* = k^{-1}(k(U))$ is open. Hence, k is open and continuous, and H/K is a topological group in its quotient topology.

A topological half-group H will be said to be T_i $(i = 1, 2)$, provided $\Gamma = \cup G_\alpha$ is T_i in its relative topology. If H satisfies (ix), then each h_α is a topological, as well as an algebraic, isomorphism between G_α in its relative topology and G. The continuity of h_α follows directly from Theorem 8. Let $U \subset H$ be open. Since the range of ψ_α is G_α and $\psi_\alpha(x) \equiv x$ for all $x \in H$, each $x \in \psi_\alpha^{-1}(U)$ is congruent (mod F) to some element in $U \cap G_\alpha$. Moreover, $\psi_\alpha(x) = x$ for each $x \in G_\alpha$, so that $\psi_\alpha^{-1}(U) \supset U \cap G_\alpha$. Hence, $h(U \cap G_\alpha) = h(\psi_\alpha^{-1}(U))$ is open, and h_α is open. Thus, if H is T_i, then G is also T_i.

Consider now the set $G \times E$ and the operation defined in it as follows:

$$(g_1, e_\alpha) \circ (g_2, e_\beta) = (g_1 - g_2, e_\alpha).$$

Under this operation, $G \times E$ evidently becomes a half-group satisfying (vii) (cf. the discussion preceding Theorem 6). If we introduce the product topology in $G \times E$ derived from the quotient topology in G and the relative topology from H in E, then $G \times E$ becomes a topological half-group. In fact, if $U_1 \times V_1$ and $U_2 \times V_2$ are open in $G \times E$, then $(U_1 \times V_1) \circ (U_2 \times V_2) = (U_1 - U_2, V_1)$.

Let $\phi: H \to G \times E$ be defined as follows: if $x \in S_\alpha$, then $\phi(x) = (h(x), e_\alpha)$. Since, for any $y \in H$, $x - y \in S_\alpha$, we have

$$\phi(x - y) = (h(x) - h(y), e_\alpha) = \phi(x) \circ \phi(y).$$

The restriction of ϕ to Γ is clearly one-to-one and onto, but, in general, $x - y \neq \phi^{-1}(\phi(x) \circ \phi(y))$ for $x, y \in \Gamma$.

THEOREM 9. *If H is a topological half-group satisfying* (ix), *then ϕ is a continuous, open homomorphism of H onto $G \times E$.*

Proof. It remains to prove that ϕ is continuous and open. Let $\omega(x) = x - x$ for all $x \in H$. Then ω is continuous on H, $\omega(S_\alpha) = e_\alpha$ for each $\alpha \in A$, and $\phi(x) = (h(x), \omega(x))$. If $U \times V \subset G \times E$ is open, then $\phi^{-1}(U \times V) = h^{-1}(U) \cap \omega^{-1}(V)$ is open in H, and ϕ is continuous. Now let $U \subset H$ be open and $y \in \phi(U)$. By an argument used in the proof of Theorem 8, we can assume that $y = \phi(x)$, where $x \in U \cap G_\alpha$ for some $\alpha \in A$. Since $x = e_\alpha - (e_\alpha - x)$, there exist neighbourhoods V, W, of x, e_α, respectively, such that $W - (W - V) \subset U$. Let $V' = h(V)$ and $W' = W \cap E$. Then $V' \times W'$ is open in $G \times E$, and clearly $y \in V' \times W'$. If $v \in V$ and $e_\beta \in W$, then $e_\beta - (e_\beta - v) \in U$, $h(e_\beta - (e_\beta - v)) = h(v)$, and $\omega(e_\beta - (e_\beta - v)) = e_\beta$. Hence,

$$(h(v), e_\beta) = \phi(e_\beta - (e_\beta - v)) \in \phi(U),$$

and $V' \times W' \subset \phi(U)$. Therefore, ϕ is open.

A further consequence of the continuity of ω is that, in a T_1 topological half-group H, each S_α is closed in H.

THEOREM 10. *If H is a connected, T_1 topological half-group with a finite number of idempotents, then H is a* (*topological*) *pseudo-group.*

Proof. We have $\omega(H) = E$, whence E is connected and T_1 in its relative topology. Since a finite T_1 space is discrete, E reduces to a single point.

COROLLARY. *If, in addition, the right cancellation law holds in H, then H is a* (*topological*) *group.*

To show the necessity for the assumption that E be finite, let $H = [0, 1]$ with the relative Euclidean topology and the structure of the simple half-group mentioned in § 2. Evidently H is a compact, connected, T_1 topological half-group in which every element is an idempotent.

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