HYPERBOLIC MIXED PROBLEMS FOR HARMONIC TENSORS

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This paper may be regarded as a sequel to **(1)**, where the initial value or Cauchy problem for harmonic tensors on a normal hyperbolic Riemann space was treated. The mixed problems to be studied here involve boundary conditions on a timelike boundary surface in addition to the Cauchy data on a spacelike initial manifold. The components of a harmonic tensor satisfy a system of wave equations with similar principal part, and we assign two initial conditions and one boundary condition for each component. Our results will for the most part be deduced from the analogue of the mixed boundary value theorem of generalized potential theory, in which some components and some normal derivatives of components are specified on the boundary surface.

To establish this result in the hyperbolic case, we study linear systems of hyperbolic equations with similar principal part under corresponding boundary conditions of the two types mentioned. For this purpose we rely on **(7)** where the case of assigned boundary values is treated, and on the modifications **(5)** necessary for the normal derivative problem. We then apply the general existence theorem so found to a number of the special problems of harmonic tensors.

The analogy between mixed problems for hyperbolic equations and potential theory for elliptic equations has led some authors **(6; 9)** to study surface layer potentials in the hyperbolic case. However, this method has not yet been successfully used to find existence theorems for hyperbolic equations with variable coefficients. In this paper we shall encounter a situation familiar from potential theory—the occurrence of orthogonality conditions on the data in cases where the solution of the given problems is not unique. Again, however, indirect methods of constructing the solution when these conditions are satisfied seem to be necessary. This aspect of hyperbolic mixed problems arises only for systems of second order equations as the solutions of all such problems for a single equation are unique **(5)**.

The Maxwell equations can be conveniently written in the notation of harmonic tensors, and Theorem V below can be interpreted as an existence theorem for boundary problems concerning time-variable electromagnetic fields.

1. Mixed problems for systems of hyperbolic equations. Let a normal hyperbolic Riemannian space V_N be defined by the metric

$$ds^2 = a_{ik}\, dx^i\, dx^k, \tag{1.1}$$

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with Lorentz signature $(1, N-1)$ and containing one positive term. The associate tensor a^{ik} exists and is contravariant symmetric. We study systems of M normal hyperbolic linear partial differential equations in M dependent variables $u_r (r = 1, \ldots, M)$:

$$(1.2) \qquad L_r(u_r) = a^{ik}\frac{\partial^2 u_r}{\partial x^i \partial x^k} + b^i \frac{\partial u_r}{\partial x^i} + \sum_{s=1}^{M} c_{rs}\, u_s = e_r.$$

Here the coefficients a^{ik} and b^i of the partial derivatives are the same in all equations of the system so that the interrelationship of the dependent variables is effected only by the coefficients c_{rs} $(r, s = 1, \ldots, M)$. Conditions of regularity for the coefficients will be discussed below.

The Cauchy initial value problem for (1.2) involves giving values for the u_r and their first normal derivatives on a surface S: $\phi(x^i) = 0$ which is spacelike:

$$a(\phi) = a^{ik}\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^k} > 0.$$

This type of problem has been treated by many authors **(6, 8, 9, 10)** and can be regarded as solved. The solution is defined in the domain D_S of exclusive dependence on S which is bounded by a characteristic surface G passing through the rim C of the spacelike surface S. We note that the geometry of the characteristic surfaces of (1.2) is the same as for a single equation.

Let T: $\psi(x^i) = 0$ be a timelike surface $(a(\psi) < 0)$ which meets S in the rim C. We shall consider only that region D on the positive side of S in a chosen orientation of the timelike coordinate. The normal n to T has contravariant components $a^{ik}\, \partial\psi/\partial x^k$. We now divide the equations (1.2) into two sets. For $r = 1, \ldots, p$, we assign on T values of u_r:

$$(1.3) \qquad u_r = f_r, \qquad r = 1, \ldots, p, \text{ on } T,$$

while for $r = p + 1, \ldots, N$ we assign values of the normal derivatives

$$(1.4) \qquad \frac{\partial u_r}{\partial n} = g_r, \qquad r = p + 1, \ldots, M, \text{ on } T.$$

Together with Cauchy data on S, these conditions determine our mixed problem.

There are certain compatibility conditions to be satisfied by the datum functions f_r and g_r; these conditions ensure that u_r and its derivatives up to a certain order k are continuous across the characteristic surface G which is the locus of all discontinuities originating on the rim $C = S \cap T$. In practice it is convenient to subtract out the Cauchy data by first solving the Cauchy problem, and then subtracting the solutions from the u_r. The new problem so found has zero Cauchy data and the compatibility condition of order k is now the vanishing on C of the derivatives

$$\frac{\partial^k f_r}{\partial t^k}, \quad \frac{\partial^{k-1} g_r}{\partial t^{k-1}},$$

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where t, a timelike variable, can be taken as equal to $\phi(x^i)$. Thus, the condition of order zero applies only to f_r. Consider first the case where the system (1.2), the data, and the surfaces S, T are analytic. The Cauchy problem can be solved and subtracted out. The domain R between G and T remains. To define the solutions in R we note that their values are now determined on G and these assigned values will be zero since the Cauchy data of the new problem are zero.

A new coordinate system in R is now introduced. Let G_t be a family of characteristic surfaces meeting T in hypercurves C_t, with $G_0 = G$, $C_0 = C$; and suppose that G_t is analytic. Since we can only secure a local existence theorem in the analytic case, it is enough to suppose that the G_t are defined in a neighbourhood of S and T. We now take G_t as coordinate hypersurface of a new variable $t = x'^N$, and also we set $x'^{N-1} = x' = \psi(x^i)$ so that the timelike surface T becomes a coordinate hyperplane. The reduction given by Hadamard (**6**, p. 76), applied to the system (1.2), shows that in the new coordinate system the differential equations can be written

$$(1.5) \qquad \frac{\partial^2 u_r}{\partial x \partial t} = L_1(u_r) + \sum_{S=0}^{M} c_{rs} u_s + e_r,$$

where we have dropped the primes on coordinates and coefficients. Here the new dependent variables u_r are equal to

$$u_r \exp[\textstyle\int b^N dx^{N-1}]$$

as in Hadamard's reduction, and the operator $L_1(u_r)$ contains no differentiations with respect to t.

The auxiliary conditions are now

$$(1.6) \qquad u_r = 0, \qquad r = 1, \ldots, M;\ t = 0,$$

and

$$(1.7) \qquad u_r = f_r, \qquad r = 1, \ldots, p;\ x = 0,$$

with

$$(1.8) \qquad \frac{\partial u_r}{\partial n} = b^N u_r + g_r, \qquad r = p + 1, \ldots, M; x = 0.$$

We shall suppose that the compatibility conditions of order up to $k \geqslant N + 2$ are satisfied, and it follows as in (**5**) that we can subtract from the u_r certain functions, analytic in R and C^k across G, which satisfy the above boundary conditions. This means that in (1.7) and (1.8) we can assume $f_r = 0$ and $g_r = 0$, while the non-homogeneous terms e_r in (1.5) absorb the extra terms appearing in the differential equations.

From (**5**, Lemma I) we see that the normal n to T is never tangent to the characteristic surfaces G_t. Consequently we may now write the homogeneous boundary conditions in the form

$$(1.9)\qquad \begin{aligned} u_r &= 0; && r = 1, \ldots, p,\\ \frac{\partial u_r}{\partial t} &= \sum_{k=1}^{N-1} \beta^k \frac{\partial u_r}{\partial x^k} + h u_r, && r = p+1, \ldots, M.\end{aligned}$$

Here we have solved for the derivative with respect to t in the second group. These conditions apply for $x = 0$.

Let the unknowns u_r and all other functions in (1.5) and (1.9) be expanded in powers of t: e.g.

$$u_r = \sum_{n=0}^{\infty} u_{rn} t^n, \quad L_1(u_r) = \sum_{n=0}^{\infty} L_{1n}(u)\, t^n.$$

By a subscript n attached to a function we shall mean the coefficient of t^n in such an expansion. We now determine the u_{rn} recursively for $n = 1, 2, \ldots$. From (1.5) we have

$$(1.10)\qquad n\frac{du_n}{dx} = L_{01}(u_{n-1}) + \ldots + \sum_s c_{rso}\, u_{s,n-1} + \ldots + e_{n-1},$$

where the terms omitted contain the $u_{rl}, \ldots, u_{rk}, \ldots$ up to the $u_{r\,n-2}$, and their derivatives with respect to $x^1, \ldots, x^{N-2}$, and $x = x^{N-1}$. The boundary conditions (1.9) now read

$$(1.11)\qquad \begin{aligned} u_{rn} &= 0, && r = 1, \ldots, p;\\ n\, u_{rn} &= \sum_{k=1}^{N-1} \beta^k{}_0 \frac{\partial u_{r,n-1}}{\partial x^k} + h_0\, u_{r,n-1} + \ldots, && r = p+1, \ldots, M.\end{aligned}$$

To satisfy (1.6) we take $u_{r0} = 0$. Thus the u_{rn} are determined by integration with respect to x. The expressions so found contain additions, multiplications and integrations, all of which operations preserve the relation of dominance between power series in the variables $x^1, \ldots, x^{N-1}, t$. Thus by dominating the coefficients in (1.5) and (1.9) we would find a series solution dominating the original one.

By choosing a single function U to dominate all of the u_r, we can reduce the proof that the dominating series converges to the case considered in **(5)**. This can be done by choosing a single dominating coefficient for each of the combinations

$$\sum_s c_{rs} \qquad (s = 1, \ldots, M)$$

which would appear in (1.5) if all u_r were equal, and a common dominant for the e_r. The dominating series in **(5)** has a positive radius of convergence in each variable, and has positive coefficients, so that (1.6) and the first of (1.9) are dominated. This series also satisfies for all $x^1, \ldots, x^{N-1}, t$ a relation **(5**, 2.27**)** which is equivalent to the domination of the second group of conditions (1.9). We may therefore conclude that a local analytic solution of (1.5), (1.6), (1.7), (1.8) exists, with a radius of convergence independent of the datum functions. Referring back to the original problem, we see that the solution here is analytic except possibly on G where it is of class C^k.

To extend the domain of the solution, or to lighten the hypothesis of analyticity, certain estimates of the square integrals of the solutions and their derivatives are needed. Except for one point the procedure follows that in (**5**; **7**), so we shall not give details. The procedure involves integrations by parts carried out over (1.2) after multiplication by a first derivative of u in a certain timelike direction. A different choice of this direction is necessary for the two groups $r = 1, \ldots, p$ and $r = p + 1, \ldots, M$. For $r = 1, \ldots, p$ with the Dirichlet-type condition as in (**7**), this direction must cross T inwards with increasing time, while for $r = p + 1, \ldots, M$ and the normal derivative condition as in (**5**), the direction must cross T outwards with increasing time. The result of these calculations is estimates of the k-norms $||u_r||_k$ as defined in (**7**) in terms of the functions e_r, the Cauchy data, and the given data in (1.3) and (1.4). Applying the lemmas of (**7**) to secure convergence we are able to extend the domain of definition of the solution in the direction of the future, and to show the existence of solutions in the non-analytic case.

Finally, we shall remove the compatibility conditions on the rim C, with the exception of the first of these conditions for the group $r = 1, \ldots, p$. This is done, as in (**5**, §6), by writing $u_r = u_{1r} + v_r$, and requiring that $L_r(v_r)$ be of class C^k in the entire domain while v_r satisfies boundary conditions such that the conditions for u_{1r} are compatible of order k. We assume that the differential equation and data are of class C^{2k}, and construct v_r as the first k terms of the series expansion in the analytic case. The problem for u_{1r} is now seen to be C^k and so is reduced to the preceding case.

We state the general result as an existence theorem; the following sections contain the applications to harmonic tensors.

EXISTENCE THEOREM. *Let the coefficients in* (1.2) *be of class* C^{2k} $(k \geqslant N + 2)$ *and let the Cauchy data together with the data in* (1.3) *and* (1.4) *be of class* C^{2k+N}. *Let the data in* (1.3) *and* (1.4) *satisfy the compatibility conditions up to order* q *inclusive,* $q \leqslant k - N$. *Then there exists a unique solution* u_r *of* (1.2), (1.3) *and* (1.4), *together with the given Cauchy data, and this solution is of class* C^{k-N} *except across* G *where it is of class* C^q.

The hypotheses stated in the above theorem have been simplified and will be applied with $q = 1$ or 2 to the theory of harmonic tensors. However less restrictive hypotheses, and a slightly stronger conclusion, are possible as the situation is effectively the same as in (**5**, §6).

We shall use the uniqueness of the solution several times below, and so we state here the conditions of regularity which a solution u_r, satisfying the homogeneous equations and auxiliary conditions, must satisfy for this purpose. Let u_r be of class C^2, except possibly across G, where continuity only is assumed. Since the homogeneous problem is compatible of every order k, it follows that the first and second derivatives of u_r across G are also continuous (**5**, §1). Thus u_r is C^2 and the estimates referred to above show that $u_r = 0$.

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The statements of the theorems below will specify the value of q in each case. For brevity, we now assume once for all that $k \geqslant N + 2$ and all solutions are of class C^{k-N} except possibly across G.

2. Application to harmonic tensors. We recall here the main points of the necessary notation **(1)**. A differential form

$$\phi = \phi_{i_1,\dots,i_p} dx^{i_1} \wedge \dots \wedge dx^{i}, \tag{2.1}$$

where summation with $i_1 < \dots < i_p$ is understood, is based on a totally skew symmetric tensor $\phi_{i_1,\dots,i_p}$ of rank p. The differentials dx^i are combined by exterior multiplication which has the anti-commuting property $dx^i \wedge dx^j = -dx^j \wedge dx^i$. The differential of ϕ is a form of degree $p + 1$:

$$d\phi = (d\phi_{i_1,\dots,i_p}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}. \tag{2.2}$$

and

$$(d\phi)_{i_1\dots i_{p+1}} = \Gamma_{i_1\dots i_{p+1}}{}^{j(j_1\dots j_p)} \frac{\partial}{\partial x^j} \phi_{(j_1\dots j_p)}, \tag{2.3}$$

where Γ is the skew symmetrized Kronecker delta of rank $p + 1$, and the brackets indicate $j_1 < \dots < j_p$. The dual $*\phi$ is an $(N - p)$-form with components defined by contraction with the permutation symbol of order N. Then the coderivative $\delta\phi$, of degree $p - 1$, is defined by

$$\delta\phi = (-1)^{Np+p+1} {*d*\phi}.$$

The identities

$$d.d\phi \equiv 0, \qquad **\phi \equiv (-1)^{Np+p}\phi, \qquad \delta.\delta\phi \equiv 0, \tag{2.4}$$

hold.

If $d\phi = 0$, ϕ is said to be closed, and if $\phi = d\chi$, derived; and dually $\delta\phi = 0$, $\phi = \delta\chi$ describe coclosed and coderived forms.

The Laplacian operator Δ is given by

$$\Delta = \delta d + d\delta, \tag{2.5}$$

and if the equation $\Delta\phi = 0$ of harmonic tensors is written out in component form it is of the type of (1.2), with $M = \binom{N}{p}$ equations. The coefficients c_{rs} then consist of contractions of the Riemann curvature tensor of the metric (1.1).

On any hypersurface S with equation, say $x^m = 0$, a p-form ϕ induces a p-form $t\phi$, which contains those components free of the index m, and is known as the tangential part of ϕ. The residual or normal part, is denoted by $n\phi$, and has the factor dx^m. Thus we shall write

$$\phi = t\phi + n\phi = t\phi + \bar{n}\phi \wedge dx^m, \tag{2.6}$$

which decomposition is invariant only on S. The dual interchanges t and n; thus $*t = n*$, $*n = t*$. If $t\phi$ and $n\phi$ are assigned on S, then $td\phi = d_S t\phi$ and $n\delta\phi$ are determined, but $nd\phi$ and $t\delta\phi$ can be assigned independently.

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From Stokes formula

$$\int_C d\phi = \int_{bC} \phi, \tag{2.7}$$

which is valid for any $(p + 1)$-chain C with boundary bC, we observe that the period integral $\int_Z \phi$ of a closed form ϕ depends only on the homology class of Z. A closed form with vanishing absolute periods on a manifold with boundary is derived (**2**, Theorem 4).

To distinguish among the various cases which follow, we shall give to each problem the name of the problem in elliptic generalized potential theory with the corresponding boundary conditions. The Cauchy data $t\phi$, $n\phi$, $t\delta\phi$, $nd\phi$ on an initial surface S will be assigned in each case. We may allow the initial surface S to be a multiply-connected manifold, bounded by a manifold $C = S \cap T$. We shall, however, assume that any non-bounding cycle of the region enclosed by S and T is homologous to a cycle of S, and that any cycle of T is homologous to a cycle of C; and vice versa. In effect, we take the domain to be topologically uniform in the time dimension. The data will always be assumed to satisfy the conditions of the preceding existence theorem.

The Dirichlet problem involves values of $t\phi$ and $n\phi$ on T; thus we shall invoke the existence theorem with $p = M$ so that all equations belong to the first group. Then we have

THEOREM I. *There exists a unique harmonic form ϕ with given Cauchy data on S and given values of $t\phi$ and $n\phi$ on T.*

Here the compatibility condition of the general existence theorem implies that $t\phi$ and $n\phi$ given on T should have values on C determined by the Cauchy data $t\phi$ and $n\phi$ on C.

In the mixed problem of generalized potential theory the boundary data are values of $t\phi$ and $t\delta\phi$, or alternatively and dually (**3**), of $n\phi$ and $nd\phi$. From (2.3) we deduce the formula

$$\bar{n}d\phi = (-1)^{p+1}\frac{\partial t\phi}{\partial x^m} + d_S\,\bar{n}\phi, \tag{2.8}$$

the component form of which is found by setting $i_{p+1} = m$ and separating the term $j = m$ from the others. Thus, if $n\phi$ and $nd\phi$ are given, the first normal derivative of $t\phi$ is determined and so the data in this case amount to values of $n\phi$ and normal derivatives of components of $t\phi$. We therefore choose the two groups of the system (1.2) accordingly, and conclude with the

THEOREM II. *There exists a unique harmonic form ϕ with given Cauchy data on S and given values of $n\phi$ and $nd\phi$ (or $t\phi$ and $t\delta\phi$) on T.*

Here the compatibility condition on C requires that $n\phi$ (or $t\phi$, as the case may be) should have values on C in accordance with the Cauchy data. We shall later need solutions of class C^1 or C^2 and so will assume the compatibility conditions of order 2 and 3 as necessary. These restrict both $n\phi$ and $nd\phi$ (or $t\phi$ and $t\delta\phi$) in the neighbourhood of the rim C.

The uniqueness of this solution in the hyperbolic problem is in sharp contrast to the elliptic case where there exist eigenforms in number determined by the p-dimensional connectivity (Betti number) of the domain.

In both of these theorems we could state results for the non-homogeneous differential equations

$$\Delta\phi = \rho, \tag{2.9}$$

by solving a Cauchy problem for (2.9) in a larger domain and subtracting out the solution.

3. Particular cases of the mixed problem. The result of Theorem II admits of a number of special cases in which the governing differential equation $\Delta\phi = \rho$ is replaced by a pair of equations. The boundary conditions also take various forms; these have been studied in the elliptic case **(4)**. We begin with co-closed harmonic forms which satisfy $\delta\phi = 0$, $\delta d\phi = 0$. The corresponding non-homogeneous equations will be treated.

For our first theorem relative to coclosed harmonic forms, we assign initial data, indicated by the subscript 0, and values for the tangential part of ϕ on T. The differential equations are

$$\delta d\phi = \rho, \quad \delta\phi = \sigma, \tag{3.1}$$

where $\rho = \rho_p$ and $\sigma = \sigma_{p-1}$, are assigned coderived forms. The initial values to be given are such that together with (3.1) Cauchy data for the solution can be obtained. Thus we prescribe $t\phi_0$, $t{*}\phi_0$ and $t{*}d\phi_0$, indicating initial values with the subscript 0. In order that these data should be compatible with (3.1) we must require

$$\begin{aligned} d_S t{*}\phi_0 &= (-1)^N t{*}\sigma_0, \\ d_S t{*}d\phi_0 &= (-1)^N t{*}\rho_0. \end{aligned} \tag{3.2}$$

We then set $t\delta\phi_0 = t\sigma_0$ and so obtain complete Cauchy data.

In the boundary condition

$$t\phi = \theta, \qquad \text{on } T, \tag{3.3}$$

where θ is a given p-form, the first two compatibility conditions must be satisfied. That is, θ and its first timelike derivative on C shall have values given by the Cauchy values there.

THEOREM III. *Let $t\phi_0$, $t{*}\phi_0$ and $t{*}d\phi_0$ be assigned on S subject to* (3.2). *Let the boundary condition* (3.3) *satisfy the first two compatibility conditions on C. Then there exists a unique solution of* (3.1) *satisfying these conditions, and this solution is of class C^1.*

To prove this we set up the mixed problem

$$\Delta\phi = \rho + d\sigma, \tag{3.4}$$

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with boundary conditions

$$t\phi = \theta, \; t\delta\phi = t\sigma, \quad \text{on } T, \tag{3.5}$$

and the Cauchy data introduced above. The values of $t\phi$ are compatible for $k = 0$ and 1, while the values of $t\delta\phi$ are compatible for $k = 1$ since $\delta\phi_0 = \sigma_0$ in view of the first of (3.2) and the definition $t\delta\phi_0 = t\sigma_0$. The general existence theorem now asserts that ϕ exists and is of class C^1.

We now define

$$\psi = \delta\phi - \sigma \tag{3.6}$$

and note that ψ is continuous across G and is elsewhere of class C^{k-N-1}. Also

$$d\psi = d\delta\phi - d\sigma = \rho - \delta d\phi$$

is coderived and so $\delta d\psi = 0$. Since $\delta\psi = \delta.\delta\phi - \delta\sigma = 0$ it follows that $\Delta\psi = 0$, We now show that ψ has zero Cauchy data. From $t\delta\phi_0 = t\sigma_0$, we get $t\psi_0 = 0$. and likewise

$$t{*}\psi_0 = t{*}\delta\phi_0 - t{*}\sigma_0 = (-1)^N d_S t{*}\phi_0 - t{*}\sigma_0 = 0,$$

from the first of (3.2). Also

$$\begin{aligned} t{*}d\psi_0 &= t{*}(\rho - \delta d\phi) = t{*}\rho_0 - (-1)^N td{*}d\phi_0 \\ &= t{*}\rho_0 - (-1)^N d_S t{*}d\phi_0 = 0, \end{aligned}$$

from the second of (3.2). Finally $t\delta\psi_0 = 0$ since ψ is coclosed, and thus the Cauchy data for ψ are zero. Turning to the boundary conditions, we find $t\psi = 0$ on T from the second of (3.5) while $t\delta\psi \equiv 0$. Thus ψ satisfies homogeneous boundary conditions of the mixed type. From the uniqueness of the solution in Theorem II we conclude that ψ vanishes identically. Together with (3.4) and (3.6) this establishes (3.1) and concludes the proof of the theorem.

Another set of boundary conditions applicable to the differential equations (3.1) are found from the dual mixed problem: thus

$$t{*}\phi = \xi, \quad t{*}d\phi = \eta, \quad \text{on } T \tag{3.7}$$

are specified. The restrictions

$$d_T\xi = d_T t{*}\phi = (-1)^N t{*}\delta\phi = (-1)^N t{*}\sigma \tag{3.8}$$

and

$$d_T\eta = d_T t{*}d\phi = (-1)^N t{*}\delta d\phi = (-1)^N t{*}\rho \tag{3.9}$$

are necessary in view of (3.1). The initial values are as in the preceding theorem. Again, we require that the compatibility conditions of orders $k = 0$ and 1 for ξ and order $k = 1$ for η should hold. Thus the value and first derivative of ξ and the value of η on C must agree with those calculated from the Cauchy data.

THEOREM IV. *Let* $t\phi_0$, $t*\phi_0$ *and* $t*d\phi_0$ *be assigned on* S *subject to* (3.2). *Let the data in* (3.7) *satisfy* (3.8) *and* (3.9). *Then there exists a unique solution of* (3.1) *satisfying these conditions, and this solution is of class* C^1.

For the proof we again solve (3.4), this time with boundary conditions (3.7) and the above Cauchy data. The compatibility conditions of orders $k = 0$ and 1 are satisfied and a C^1 solution ϕ exists, by the dual of Theorem II. Again, let $\psi = \delta\phi - \sigma$, which is continuous, and let us show that ψ vanishes. As before, $\Delta\psi = 0$ and ψ has zero Cauchy data. On T we have

$$\begin{aligned} t*\psi &= t*(\delta\phi - \sigma) = (-1)^N d_T t*\phi - t*\sigma \\ &= (-1)^N d_T\xi - t*\sigma = 0 \end{aligned}$$

from (3.8). Also

$$\begin{aligned} t*d\psi &= t*(d\delta\phi - d\sigma) = t*(\rho - \delta d\phi) \\ &= t*\rho - (-1)^N d_T t*d\phi \\ &= t*\rho - (-1)^N d_T\eta = 0 \end{aligned}$$

from (3.9). Thus ψ satisfies the homogeneous dual mixed boundary conditions and so must vanish everywhere. This proves the theorem.

We turn now to the differential equations associated with harmonic fields—forms which are both closed and coclosed. The corresponding nonhomogeneous equations are

$$d\phi = \rho, \quad \delta\phi = \sigma, \tag{3.10}$$

where $\rho = \rho_{p+1}$ must be derived and $\sigma = \sigma_{p-1}$ coderived.

For initial values we now assign $t\phi_0$ and $t*\phi_0$ subject to the necessary conditions

$$d_s t\phi_0 = td\phi_0 = t\rho_0 \tag{3.11}$$

and

$$d_s t*\phi_0 = td*\phi_0 = (-1)^N t*\delta\phi_0 = (-1)^N t*\sigma_0. \tag{3.12}$$

We can now obtain Cauchy data for this problem by adjoining to these values

$$t*d\phi_0 = t*\rho_0 \text{ and } t\delta\phi_0 = t\sigma_0.$$

The appropriate boundary condition is

$$t\phi = \xi, \tag{3.13}$$

where the necessary condition

$$d_T\xi = td\phi = t\rho \tag{3.14}$$

holds. This restriction on ξ will be examined in more detail below; we remark that ξ shall also satisfy two compatibility conditions in relation to the above Cauchy data.

THEOREM V. *Let* $t\phi_0$ *and* $t*\phi_0$ *be assigned on* S *subject to* (3.11) *and* (3.12). *Let* ξ *in* (3.13) *satisfy* (3.14) *and the first two compatibility conditions. Then* *there exists a unique solution of* (3.10) *which satisfies these conditions and this solution is of class* C^1.

To prove this we establish the mixed problem

$$\Delta\phi = \delta\rho + d\sigma \tag{3.15}$$

with the Cauchy data given above, and with boundary conditions

$$t\phi = \xi, \quad t\delta\phi = t\sigma. \tag{3.16}$$

The data being compatible of orders $k = 0$ and 1, the solution ϕ is C^1. Let

$$\psi = \delta\phi - \sigma, \quad \chi = d\phi - \rho; \tag{3.17}$$

both of these forms are continuous across G.

We will show that both ψ and χ vanish identically. Taking first ψ, we have $\delta\psi = \delta.\delta\phi - \delta\sigma = 0$ and

$$d\psi = d\delta\phi - d\sigma = \delta\rho - \delta d\phi,$$

so that $\delta d\psi = 0$ and $\Delta\psi = 0$. The Cauchy data of ψ are now to be found. First, $t\psi_0 = t\delta\phi_0 - t\sigma_0 = 0$ from the construction of $t\delta\phi_0$. Next

$$t{*}\psi_0 = t{*}\delta\phi_0 - t{*}\sigma_0 = (-1)^N d_S\, t{*}\phi_0 - t{*}\sigma_0 = 0,$$

from (3.12). Then $t\delta\psi_0 = 0$ since $\delta\psi \equiv 0$, and finally

$$t{*}d\psi_0 = t{*}(\delta\rho_0 - \delta d\phi_0) = (-1)^N d_S\, t{*}(\rho_0 - d\phi_0) = 0,$$

according to the construction of $t{*}d\phi_0$. Turning to the boundary conditions satisfied by ψ, we have $t\psi = t\delta\phi - t\sigma = 0$ from the second of (3.16), while $t\delta\phi = 0$. Thus ψ satisfies homogeneous mixed boundary conditions as well and so vanishes as required. This proves the second of (3.10).

To show that χ is zero, we have $d\chi = d.\, d\phi - d\rho = 0$, while

$$\delta\chi = \delta d\phi - \delta\rho = d\sigma - d\delta\phi = -\, d\psi = 0$$

so $\Delta\chi = 0$. The Cauchy data for χ are:

$$t\chi_0 = t\delta\phi - t\rho = d_S t\phi - t\rho = 0$$

by (3.11);

$$t{*}\chi_0 = t{*}d\phi_0 - t{*}\rho_0 = 0$$

by the way $t{*}d\phi_0$ was defined;

$$t\delta\chi_0 = 0, \quad t{*}d\chi_0 = 0.$$

The boundary conditions for χ are

$$t\chi = td\phi - t\rho = d_T t\phi - t\rho = d_T\xi - t\rho = 0$$

by (3.14); and $t\delta\chi = 0$ since χ is coclosed as demonstrated above. Since χ now satisfies all conditions of the homogeneous mixed problem it vanishes identically and this proves the first of (3.10). Thus ϕ satisfies both differential equations and the proof of the theorem is complete.

The restriction (3.14) on the datum form ξ will now be examined in more detail in order to find just what components can be freely assigned on T. For simplicity we shall suppose the differential equations homogeneous and the initial values zero. We assert that in these circumstances (3.14) together with the first compatibility condition imply that ξ is a derived form on T. Indeed if $d_T\xi = 0$ the periods of ξ on p-cycles of T are well defined, and by our general topological hypothesis of §2, any cycle of T is homologous in T to a cycle of C. On this cycle the period is zero since ξ vanishes there. By **(2**, Theorem 4**)**, it follows that ξ is derived on T, say $\xi_p = d_T\,\zeta_{p-1}$.

Let C_t be a family of spacelike $(N-2)$-dimensional surfaces covering T, with $C_0 = C$, and let ξ be decomposed into tangential and normal parts relative to C_t. We shall, for convenience, refer to these as the spacelike and timelike parts of ξ, respectively. We now show that due to (3.14) the spacelike part is determined by the timelike part. Indeed

$$(d_T\xi)_{ti_1\ldots i_p} = \frac{\partial\xi_{i_1\ldots i}}{\partial t} - \ldots + \ldots = 0$$

where the subscript t refers to the timelike parameter of the family C_t; and where the terms not shown are derivatives of components of the timelike part of ξ. Since ξ is given for $t = 0$ by the compatibility condition, we may determine

$$\xi_{i_1\ldots i_p}, \qquad\qquad i_1, \ldots, i_p \neq t$$

by integration over t. That is, the spacelike part, having $\binom{N-2}{p}$ components, is determined by the timelike part, which has $\binom{N-2}{p-1}$ components. That the components of the timelike part can be freely assigned is evident from the component form of $\xi_p = d_T\zeta_{p-1}$, namely

$$\xi_{ti_1\ldots i_{p-1}} = \frac{\partial\xi_{i_1\ldots i_{p-1}}}{\partial t} - \ldots + \ldots,$$

since the time-derivatives can be chosen at will.

Maxwell's equations in empty space have the form

$$dF_2 = 0, \quad \delta F_2 = J_1,$$

where, in locally Cartesian coordinates with Lorentz metric,

$$F_2 = H_1 dy \wedge dz + H_2 dz \wedge dx + H_3 dx \wedge dy + (E_1 dx + E_2 dy + E_3 dz) \wedge dt,$$

and

$$J_1 = J_x dx + J_y dy + J_z dz + \rho dt.$$

Theorem V applies to determine F_2 if J_1 is known and the auxiliary conditions are specified as follows. For $t = 0$, assign tF_2 and $t{*}F_2$, that is, values of the components of $\underline{E}$ and $\underline{H}$. These are subject to (3.11) and (3.12) which become

$$\operatorname{div} \underline{H} = 0, \quad \operatorname{div} \underline{E} = \rho$$

(ρ denoting charge density), respectively. Thus two components of each vector can be freely specified. For the boundary conditions on a surface with local equation $x = 0$, we have

$$tF_2 = H_1 dy \wedge dz + E_2 dy \wedge dt + E_3 dz \wedge dt,$$

the first term on the right being the spacelike part. Thus the two tangential components of $\underline{E}$ are to be assigned, and determine the field uniquely. Applying the theorem to the dual of F_2, we should take the tangential components of $\underline{H}$.

As another direct application of Theorem II, we consider the equations

(3.18) $$\delta d\phi = \rho, \quad d\delta\phi = \sigma,$$

associated with the class of harmonic tensors called biharmonic fields **(4)**. Here $\rho = \rho_p$ is coderived and $\sigma = \sigma_p$ derived. For initial values we now assign the four Cauchy data $t\phi_0$, $t{*}\phi_0$, $t{*}d\phi_0$ and $t\delta\phi_0$. However the latter two are subject to the conditions

(3.19) $$d_S t\delta\phi_0 = t\sigma_0; \quad d_S t{*}d\phi_0 = (-1)^N t{*}\rho_0.$$

The boundary values

(3.20) $$t\phi = \xi, \quad t\delta\phi = \eta, \qquad \text{on } T,$$

shall be required, where η satisfies

(3.21) $$d_T\eta = t\sigma, \qquad \text{on } T.$$

In the proof we shall need a uniqueness theorem for second differentials of ϕ and so we assume compatibility conditions for ξ and η of orders $k = 0, 1$ and 2 on the rim C.

THEOREM VI. *Let Cauchy data on S be assigned subject to* (3.19). *Let ξ and η satisfy* (3.21) *and the first three compatibility conditions on C. Then there exists a unique solution of* (3.18) *satisfying the initial and boundary conditions, and this solution is of class C^2.*

The theorem is proved by constructing ϕ as a solution of

(3.22) $$\Delta\phi = \rho + \sigma,$$

with mixed boundary conditions (3.20). In view of the compatibility conditions assumed for ξ and η, the solution is of class C^2 across G. Now let

$$\psi = \delta d\phi - \rho = \sigma - d\delta\phi;$$

this form is continuous, and being both derived and coderived is also harmonic. Now the Cauchy data for ψ vanish since

$$t\psi_0 = t\sigma_0 - d_T t\delta\phi_0 = 0,$$
$$t{*}\psi_0 = t{*}\delta d\phi_0 - t{*}\rho_0 = (-1)^N d_S t{*}d\phi_0 - t{*}\rho_0,$$

by (3.19); while $t{*}d\psi_0$ and $t\delta\psi_0$ are clearly zero. On the boundary T, we have

$$t\psi = t\sigma - d_T t\delta\phi = t\sigma - d_T\eta = 0,$$

from (3.21), while $t\delta\psi$ is zero identically. Thus ψ vanishes and (3.18) hold, which concludes the proof.

4. Orthogonality conditions. The problems studied in this section do not have unique solutions and moreover the data must satisfy certain integral conditions of the orthogonal type familiar from potential theory or integral equations. As above, we consider a spacelike surface S bounded by its intersection C with a timelike surface T. We denote by Σ any spacelike surface lying in the region of definition of the solutions, such that S, T, and Σ bound a portion of the region. More precisely, we denote by S_Σ and T_Σ the parts of these surfaces which, together with Σ, have this property.

The Neumann problem of generalized potential theory concerns solutions of $\Delta\phi = 0$ with given values of $t{*}d\phi$ and $t\delta\phi$ on the boundary. We assume these conditions to hold on T and take zero Cauchy data for simplicity. Now any harmonic field ($d\tau = 0$, $\delta\tau = 0$) with zero Cauchy data is an eigenform of this problem and from Theorem V we see that an infinity of linearly independent eigenforms τ exist. In the elliptic case these furnish the orthogonality conditions directly. If we write down Green's theorem for the domain bounded by S_Σ, T_Σ and Σ in the form

$$(4.1)\quad (d\tau, d\phi) + (\delta\tau, \delta\phi) - (\tau, \Delta\phi) = \int_{S_\Sigma + T_\Sigma + \Sigma} (\tau \wedge {*}d\phi - \delta\phi \wedge {*}\tau),$$

where (α, β) denotes the volume integral of $\alpha \wedge {*}\beta$, **(1)**, we see that the terms on the left vanish, and that the integral over S is zero since ϕ has zero Cauchy data. Thus, if τ is a harmonic field, with zero tangential and normal part on Σ, defined in the region bounded by S_Σ, T_Σ and Σ, the integral over Σ disappears and we have

$$(4.2)\quad \int_{T_\Sigma} (\tau \wedge {*}d\phi - \delta\phi \wedge {*}\tau) = 0.$$

These are the orthogonality conditions of the Neumann problem and they arise from the eigenforms of the backward problem for harmonic fields with "final" surface Σ. We remark that in (4.2) $t{*}\tau$ and $t\tau$ are not independent since, by Theorem V, values of $t\tau$ alone determine τ as a harmonic field.

The problem to be solved is then to find a harmonic form ϕ with zero Cauchy data on S and

$$(4.2)\quad t{*}d\phi = \xi, \quad t\delta\phi = \eta \qquad \text{on } T,$$

where

$$(4.3)\quad \int_{T_\Sigma} (\tau \wedge \xi - \eta \wedge {*}\tau) = 0$$

for every backward harmonic field τ vanishing on Σ, for every such surface Σ. Let ξ and η vanish on C.

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We begin by solving the mixed problem with $t\delta\phi = \eta$; thus let $\Delta\phi_1 = 0$, let ϕ_1 have zero Cauchy data, let $t\phi_1 = 0$ on T and $t\delta\phi_1 = \eta$. By Theorem II there exists a C^1 form ϕ_1 satisfying these conditions. Now also

$$\int_{T_\Sigma} (\tau \wedge *d\phi_1 - \delta\phi_1 \wedge *\tau) = \int_{T_\Sigma} (\tau \wedge *d\phi_1 - \eta \wedge *\tau) = 0,$$

and if we subtract these from (4.3) we get

$$\int_{T_\Sigma} \tau \wedge (\xi - t*d\phi_1) = 0. \tag{4.4}$$

The following lemma is now necessary.

LEMMA. *Let ζ be a form of degree $N - p + 1$ defined on T, and zero on $C = S \cap T$. Let*

$$\int_{T_\Sigma} \tau \wedge \zeta = 0 \tag{4.5}$$

for all closed forms τ defined on T, and vanishing on $C_\Sigma = T \cap \Sigma$. Then ζ is closed on T: $d_T\zeta = 0$.

To prove this we let ψ be a solution of

$$\delta d\psi = 0, \quad \delta\psi = 0,$$

having zero Cauchy data on Σ, and defined in our region. According to Theorem III, we may choose $t\psi$ on T arbitrarily (of class C^k) except that $t\psi$ and its first timelike derivative must vanish on C_Σ. Then $\tau = d\psi$ is a harmonic field, with zero Cauchy data on Σ and so vanishing on C_Σ. Also $t\tau$ is closed on T so we may insert $t\tau = td\psi$ in (4.5). Since $S_\Sigma + T_\Sigma + \Sigma$ bounds a region, we have by Stokes formula

$$0 = \int_{S_\Sigma + T_\Sigma + \Sigma} d(t\psi \wedge \zeta) = \int_{S_\Sigma + T_\Sigma + \Sigma} t\tau \wedge \zeta \pm \int_{S_\Sigma + T_\Sigma + \Sigma} t\psi \wedge d_T\zeta.$$

Here we have extended the definition of ζ to a form vanishing on S and defined throughout the region. By (4.5) and in view of our hypotheses, the first integral on the right vanishes. The second reduces to an integral over T_Σ and so we have

$$\int_{T_\Sigma} t\psi \wedge d_T\zeta = 0.$$

But $t\psi$ is arbitrary except for its values on C_Σ; we conclude that $d_T\zeta = 0$ as required.From (4.4) we now infer

$$d_T(\xi - t*d\phi_1) = 0, \tag{4.6}$$

since we can construct harmonic fields τ satisfying the condition of the lemma, and since $\xi - t*d\phi_1$ vanishes on C. By Theorem IV we construct a second form χ, with zero Cauchy data on S, such that

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(4.7) $$\delta d\chi = 0, \quad \delta\chi = 0,$$
and
(4.8) $$t{*}\chi = 0, \quad t{*}d\chi = \xi - t{*}d\phi_1, \qquad \text{on } T.$$

Here the necessary conditions (3.8) and (3.9) hold, the latter in view of (4.6). The data in (4.8) are also compatible of the order stated in Theorem IV since $\xi - t{*}d\phi_1$ vanishes on S. Therefore such a C^1 form χ exists.

Now let
(4.9) $$\phi = \phi_1 + \chi;$$

we have $\Delta\phi = 0$ while ϕ has zero Cauchy data on S. Also

(4.10) $$t\delta\phi = t\delta\phi_1 + t\delta\chi = \eta + 0 = \eta,$$
while
(4.11) $$\begin{aligned} t{*}d\phi &= t{*}d\phi_1 + t{*}d\chi \\ &= t{*}d\phi_1 + \xi - t{*}d\phi_1 = \xi. \end{aligned}$$

This proves

THEOREM VII. *There exists a harmonic form ϕ with zero Cauchy data satisfying* (4.2) *subject to the orthogonality conditions* (4.3); *and this form is C^1.*

The above proof also shows that the orthogonality conditions are sufficient to determine $d_T\xi$ when η is given. Therefore the global conditions (4.3) are equivalent to this pointwise condition, which however cannot be stated explicitly without introducing as we have done a solution ϕ_1 of the mixed problem.

When the data ξ and η satisfy q compatibility conditions, which we shall not need to write out explicitly, then the solution constructed above is of class C^q. We shall take $q \geqslant 2$ in the following second theorem for the biharmonic field equations. The differential equations shall be (3.18) and the Cauchy data are again subject to (3.19). The boundary conditions are (4.2) where ξ and η each satisfy two compatibility conditions with respect to the given equations and initial values. In addition the conditions

(4.12) $$d_T\xi = (-1)^N t{*}\rho, \quad d_T\eta = t\sigma$$

are clearly necessary.

THEOREM VIII. *Let Cauchy data on S be assigned subject to* (3.19). *Let ξ and η satisfy* (4.12) *and compatibility conditions of orders $k = 0$ and 1 on C. Then there exists a solution of* (3.18) *satisfying the initial and boundary conditions, and this solution is of class C^2.*

To prove this we again consider the equation $\Delta\phi = \rho + \sigma$ with the given Cauchy data and the boundary conditions (4.2). For this nonhomogeneous equation the orthogonality conditions take the form

(4.13) $$(\rho + \sigma, \tau) + \int_{T_\Sigma + S_\Sigma + \Sigma} (\tau \wedge {*}d\phi - \delta\phi \wedge {*}\tau) = 0.$$

As in (3.18) we must have ρ coderived and σ derived; from (3.19) we see that we can write $\rho = \delta\theta$ and $\sigma = d\zeta$ where

$$t\zeta = t\delta\phi_0, \quad (-1)^N t{*}\theta = t{*}d\phi_0. \tag{4.14}$$

Making use of Stokes formula we transform the volume integral and obtain for the left hand side in (4.13)

$$\int_{T_\Sigma + S_\Sigma} (\tau \wedge [t{*}d\phi - t{*}\theta] - [t\delta\phi - t\zeta] \wedge {*}\tau) = 0. \tag{4.15}$$

According to (4.14) the integrand is zero over S_Σ, and from (4.12) it follows (as a converse of the lemma) that the integrals over T_Σ vanish. Thus the orthogonality conditions are satisfied and a solution, which will be of class C^2, exists. We can now set

$$\psi = \delta.d\phi - \rho = \sigma - d\delta\phi$$

and show that ψ is harmonic with zero Cauchy data. From the second of (4.12) we find $t\psi = 0$ on T while $\delta\psi \equiv 0$ implies $t\delta\psi = 0$ on T. Thus ψ vanishes identically and this proves the stated result. Though these differential equations are very little stronger than the single Laplace or Poisson equation, the pointwise conditions on the data can be stated directly and no orthogonality condition intervenes. The eigenforms of this problem are again the harmonic fields with vanishing Cauchy data.

As a final example we discuss another problem wherein orthogonality conditions appear, and for simplicity we consider the homogeneous equation

$$\Delta\phi = 0 \tag{4.16}$$

with zero Cauchy data. Here we specify on T the values of

$$t\phi = \xi_p, \quad t{*}d\phi = \eta_{N-p-1}. \tag{4.17}$$

The number of components so specified is not in general equal to the number of components of ϕ, and indeed if $p = N$ (4.17) are empty conditions. However, we shall prove that the appropriate conditions of orthogonality are necessary and sufficient for the existence of a solution. The eigenforms satisfy

$$d\delta\psi = 0, \quad d\psi = 0,$$

with $t\psi = 0$ on T and any such ψ is an eigenform. This is demonstrated by showing that an eigenform is necessarily closed, as follows: $d\psi$ is harmonic, with zero Cauchy data, vanishing normal part on T, and with $t{*}d.d\psi = 0$. Hence $d\psi$ vanishes by the uniqueness in Theorem II.

The orthogonality conditions arise from the eigenforms ρ of the dual backward problem based on Σ. These satisfy

$$\delta d\rho = 0, \quad \delta\rho = 0, \tag{4.18}$$

with $t{*}\rho = 0$ on T, and with zero Cauchy data on Σ. The backward form of Theorem IV shows that infinitely many linearly independent eigenforms ρ exist for each surface Σ. Thus the orthogonality conditions

$$\int_{T_\Sigma} (\xi \wedge *d\rho - \rho \wedge \eta) = 0 \tag{4.19}$$

take effect.

THEOREM IX. *Let Cauchy data zero be assigned on S. Let the data in* (4.17) *satisfy compatibility conditions of orders* $k = 0$ *and* 1. *Then there exists a harmonic form satisfying* (4.17) *if and only if* (4.19) *holds.*

We have shown that (4.19) is necessary. To prove that it is sufficient, we begin by constructing a harmonic form ϕ_1 with zero Cauchy data and with $t{*}d\phi_1 = \eta$. By Theorem II such a form exists (with $t{*}\phi_1 = 0$) and is of class C^1. The orthogonality conditions apply to ϕ_1 so that

$$\int_{T_\Sigma} (\phi_1 \wedge *d\rho - \rho \wedge *d\phi_1) = 0,$$

and on subtraction we get

$$\int_{T_\Sigma} (\xi - t\phi_1) \wedge *d\rho = 0. \tag{4.20}$$

By Theorem IV, or its dual, we can, however, choose $t{*}d\rho$ to be an arbitrary closed form which vanishes on C_Σ. The lemma now shows that

$$d_T(\xi - t\phi_1) = 0. \tag{4.21}$$

Again making use of the dual of Theorem IV, we see that it is possible to construct a form χ with

$$d\delta\chi = 0, \quad d\chi = 0,$$

having zero Cauchy data and satisfying the boundary conditions

$$t\chi = \xi - t\phi_1, \quad t\delta\chi = 0.$$

The necessary condition dual to (3.8) is satisfied here in view of (4.21), while the remaining conditions are clearly fulfilled. Now we take the C^1 form

$$\phi = \phi_1 + \chi$$

and observe that $\Delta\phi = 0$, while

$$t\phi = t\phi_1 + \xi - t\phi_1 = \xi$$

and

$$t{*}d\phi = t{*}d\phi_1 + 0 = \eta.$$

This completes the proof.

As remarked above, these boundary conditions are empty if $p = N$. For $p = 0$, on the other hand, the orthogonality conditions are highly restrictive and determine either one of ξ and η in terms of the other. This set of boundary conditions has the property that its adjoint set, in the sense of the theory of differential operators, is the dual set in the notation of generalized potential theory.

References

1. G. F. D. Duff, *Harmonic p-tensors on normal hyperbolic Riemannian spaces*, Can. J. Math., *5* (1953), 57–80.
2. ———, *Differential forms in manifolds with boundary*, Ann. Math., *56* (1952), 115–127.
3. ———, *A tensor boundary value problem of mixed type*, Can. J. Math., *6* (1954), 427–440.
4. ———, *The potential theory of coclosed harmonic forms*, Can. J. Math., *7* (1955), 126–137.
5. ———, *A mixed problem for normal hyperbolic equations*, Can. J. Math., *9* (1957), 141–160.
6. J. Hadamard, *Lectures on Cauchy's Problem* (New York, 1952).
7. M. Krzyzanski and J. Schauder, *Quasilineare Differentialgleichungen zweiter ordnung vom hyperbolischen Typus. Gemischte Randwertaufgaben*, Studia Math., *6* (1936), 162–189.
8. J. Leray, *Hyperbolic Differential Equations* (Princeton, 1953).
9. M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math., *81* (1949), 1–223.
10. J. Schauder, *Das Anfangswertproblem einer quasi-linearen hyperbolischen Differentialgleichungen*, Fund. Math., *24* (1935), 213–246.

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