BULL. AUSTRAL. MATH. SOC. MOS 1043
VOL. 1 (1969), 307-314

Some arithmetical identities for Ramanujan's and divisor functions

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A new linear expression in $\sigma(\nu)$, $\nu = 1,2,\ldots,n$, which vanishes identically is established. A linear expression in $\sigma(\nu)$'s has been found for $\sigma_3(n)$. A similar expression in $\sigma_3(\nu)$'s has been proved for $\sigma_7(n)$ also. Ramanujan's $\tau(n) = p_{24}(n-1)$ is given in three different ways as linear expressions in $\sigma_{2k+1}(n)$ and $\sigma_k(\nu)$'s with $k = 1$, 3 , 5 respectively. Again, the coefficient $p_{48}(n-2)$ is expressed as a linear expression in $\sigma_{11}(\nu)$'s and $\sigma_5(\nu)$'s . In establishing these results advantage is taken of the general theorem, also established, that the coefficients of the square of a power series whose coefficients satisfy a certain functional equation are expressible as linear functions of the latter coefficients.

1. Introduction and final results

There is a beautiful classical identity [7, p. 212] involving the arithmetical function $\sigma(n)$, the sum of the divisors of n . This may be stated as

$$\sum_{\nu} \pm \sigma(n-\nu) = 0 \tag{1}$$

where the summation is over the pentagonal numbers

$$\nu = \frac{1}{2}m(3m+1) \ , \ m = 0 \ , \pm 1 \ , \pm 2 \ , \pm 3 \ , \ \ldots\ldots \ ,$$

with the understanding that the sign to be prefixed to the term $\sigma(n-\nu)$ is

Received 21 April 1969. Received by J. Austral. Math. Soc. 12 February 1969. Communicated by E.S. Barnes.

positive or negative according as

$$v = (2m+1)(3m+1) \quad \text{or} \quad v = m(6m+1)$$

respectively, and further $\sigma(0) = n$ and $\sigma(m) = 0$ if $m < 0$. An interesting problem is to find other identities in which, as is in the relation (1) above, the only arithmetical function involved is $\sigma(\cdot)$ with arguments which are all functions of the integral variable n. An answer is provided by the following theorem which will be established later on.

THEOREM 1. $$(n-1)\sigma(n) - 6n \sum_{d|n} \left[\frac{1}{d^3} \sum_r{}' \, (d^2-5r^2)\sigma\left(\frac{d^2-r^2}{4}\right)\right] = 0 .$$

The arguments in the above theorem unlike those in (1) involve squares instead of pentagonal numbers. Also unlike that in (1), $\sigma(0) = 0$. In fact we shall adopt the convention throughout the paper, - with the single exception stated for (1), - that all terms involving negative, zero or fractional arguments which might appear in a function which is defined strictly for positive integral values only are really vanishing. The sum $\sum_{d|n}$ extends over the divisors of n. We shall use $\sum_r$ to denote a sum extended over all positive integers r, and $\sum_r{}' F(r)$ would be used to mean $\frac{1}{2} F(0) + \sum_r F(r)$.

Another problem which the author has considered is to express $\sigma_k(n)$, the elementary divisor function of degree k, in terms of divisor functions of lower degree. The answers for $k = 3$ and 7 are shown in the following theorems. For the sake of uniformity we have used $\sum_r{}'$ in Theorem 2 even though $\sum_r$ could have been used as well.

THEOREM 2. $$\sigma_3(n) = (2n-1)\sigma(n) + 24n \sum_{d|n} \left[\frac{1}{d^3} \sum_r{}' \, r^2\sigma\left(\frac{d^2-r^2}{4}\right)\right] ;$$

THEOREM 3. $$\sigma_7(n) = \sigma_3(n) + 240n^3 \sum_{d|n} \left[\frac{1}{d^3} \sum_r{}' \, \sigma_3\left(\frac{d^2-r^2}{4}\right)\right] .$$

In connection with Theorems 2 and 3 it is relevant to point out that the author has in an earlier paper [6] given expressions for

$\sigma_k(n)$, $k = 1$, 3 , 5 , 7 , 9 in terms of an arithmetical function of a different category, namely, the unrestricted partition function $p(n)$. As illustrations we give below the cases corresponding to $k = 3$ and 7 .

(2) $$5\,\sigma_3(n) = \sum_{v} \pm\,[18n-d_1(v)].vp(n-v) ,$$

(3) $$5\,\sigma_7(n) = \sum_{v} \pm\,[1512n^3-504d_1(v).n^2+42d_2(v).n-d_3(v)].vp(n-v) ,$$

where the polynomials $d_i(v)$, $i = 1$, 2 , 3 are as follows:

$$\begin{aligned} d_1(v) &= 1 + 12v , \\ d_2(v) &= 1 + 24v + 192v^2 , \\ d_3(v) &= 1 + 36v + 576v^2 + 3456v^3 . \end{aligned}$$

The method followed in establishing the above theorems leads us naturally to expressions for Ramanujan's $\tau(n)$ in terms of $\sigma_k(n)$'s where $\tau(n)$, as is well known, is defined by

$$x\left[\prod_{n=1}^{\infty}(1-x^n)\right]^{24} = \sum_{n=1}^{\infty} \tau(n)x^n .$$

These expressions are given below.

THEOREM 4. *For all positive integral values of* n

$$\begin{aligned} \tau(n) &= \frac{15}{8}\,n^4\sigma_3(n) - \frac{7}{8}(6n^5-5n^4)\sigma(n) - 105n^5.\sum_{d|n}\left[\frac{1}{d^5}\sum_{r}{}' \,r^4\sigma\left(\frac{d^2-r^2}{4}\right)\right] , \\ &= -\frac{1}{8}n^2\sigma_7(n) + \frac{9}{8}n^2\sigma_3(n) + 270n^5.\sum_{d|n}\left[\frac{1}{d^5}\sum_{r}{}' \,r^2\sigma_3\left(\frac{d^2-r^2}{4}\right)\right] , \\ &= \frac{65}{756}\,\sigma_{11}(n) + \frac{691}{756}\,\sigma_5(n) - \frac{1382}{3}\,n^5.\sum_{d|n}\left[\frac{1}{d^5}\sum_{r}{}' \,\sigma_5\left(\frac{d^2-r^2}{4}\right)\right] . \end{aligned}$$

Incidentally a pair of expressions of a different type are given for $\tau(n)$ in the paper [6]. There not only $\sigma_k(n)$'s are involved as in Theorem 4 but also $p(n)$.

Another function to which the method is applicable is $p_{48}(n)$. The functions $p_k(n)$ defined for different values of k by

https://doi.org/10.1017/S0004972700042179 Published online by Cambridge University Press

$$(4) \qquad \left[\prod_{1}^{\infty} (1-x^n)\right]^k = \sum_{0}^{\infty} p_k(n) x^n$$

have received considerable attention. Explicit expressions for the values of $p_k(n)$ for the cases $k = 1$ and 3 are given respectively by the famous identities of Euler and Jacobi. It is of interest to find explicit values of k. We have already referred to such expressions for $k = 24$, – we recall $p_{24}(n-1) = \tau(n)$. We shall now give for $p_{48}(n)$ a finite series involving the divisor functions only as in the previous theorems, in addition to one involving Ramanujan's function.

THEOREM 5. $p_{48}(n-2) = 2n^{11} \sum_{d|n} \left[\frac{1}{d^{11}} \cdot \sum_{r}' \tau\left(\frac{d^2-r^2}{4}\right)\right]$

$$= 2n^{11} \sum_{d|n} \left[d^{-11} \sum_{r}' \left\{65\sigma_{11}(\beta) + 691\sigma_5(\beta) - 348264\beta^5 . \sum_{t|\beta} t^{-5} \sum_{r}' \sigma_5(\beta)\right\}\right]$$

where

$$\beta = \tfrac{1}{4}(d^2-r^2) .$$

2. Some lemmas

LEMMA 1. $D(pn) = D(p).D(n) - p^{k+2\alpha}.D(n/p)$ *where* p *is any prime and*

$$D(n) = n^{\alpha} \sigma_k(n) , \qquad \alpha , k \geqq 0 .$$

The above lemma can be proved from the following identities which can be established successively without any difficulty.

$$\sigma_k\left(p^{\lambda+1}\right) = \sigma_k(p).\sigma_k\left(p^{\lambda}\right) - p^k.\sigma_k\left(p^{\lambda-1}\right) , \quad \lambda \geqq 0 ,$$

$$\sigma_k(pn) = \sigma_k(p).\sigma_k(n) - p^k.\sigma_k(n/p) ,$$

$$(pn)^{\alpha} \sigma_k(pn) = p^{\alpha} \sigma_k(p).n^{\alpha} \sigma_k(n) - p^{k+2\alpha}.\left(\frac{n}{p}\right)^{\alpha} \sigma_k\left(\frac{n}{p}\right) .$$

LEMMA 2. $12\left[\sum_{1}^{\infty} \sigma(n)x^n\right]^2 = \sum_{1}^{\infty} [5\sigma_3(n) - (6n-1)\sigma(n)]x^n ,$

$$12\left[\sum_{1}^{\infty} n\sigma(n)x^n\right]^2 = \sum_{1}^{\infty} [n^2\sigma_3(n) - n^3\sigma(n)]x^n ,$$

$$120\left[\sum_{1}^{\infty} \sigma_3(n)x^n\right]^2 = \sum_{1}^{\infty} [\sigma_7(n) - \sigma_3(n)]x^n .$$

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The above identities are already known; they are respectively mere restatements of the identities (3.1), (3.3) and (7.1) of Table B(1) given by the author in [4] where we have written

$$\sum_{1}^{\infty} n^r\sigma_{s-r}(n)x^n = (r,s)$$

for the sake of simplicity.

LEMMA 3. $$840\left[\sum_{1}^{\infty} n^2\sigma(n)x^n\right]^2 = \sum_{1}^{\infty} [15n^4\sigma_3(n)-14n^5\sigma(n)-\tau(n)]x^n ,$$

$$540\left[\sum_{1}^{\infty} n\sigma_3(n)x^n\right]^2 = \sum_{1}^{\infty} [n^2\sigma_7(n)-\tau(n)]x^n ,$$

$$174132\left[\sum_{1}^{\infty} \sigma_5(n)x^n\right]^2 = \sum_{1}^{\infty} [65\sigma_{11}(n)+691\sigma_5(n)-756\tau(n)]x^n .$$

The above identities are also known; they are respectively mere restatements of the identities (3.2), (7.1) and (11.1) of Table B(2) given in [5] where we have written

$$\sum_{1}^{\infty} \tau(n)x^n = \{0\} .$$

3. The basic theorem

All the theorems stated in the introductory section require for their proof the following basic theorem.

THEOREM 0. *If* $N(n)$ *is a function of the integral variable* $n > 0$ *such that for any prime* p

$$N(pn) = N(p)N(n) - \lambda(p)N(n/p)$$

where $\lambda(x)$ *is completely multiplicative, that is,*

$$\lambda(uv) = \lambda(u)\lambda(v) , \quad \lambda(u) \neq 0 ,$$

u *and* v *being any arbitrary pair of positive numbers, then*

$$\sum_{d|n} \left[\frac{2}{\lambda(d)} \sum_{r}{}' N\left(\frac{d^2-r^2}{4}\right)\right] = \frac{1}{\lambda(n)} \cdot M(n)$$

where

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$$\left[\sum_1^{\infty} N(n)x^n\right]^2 = \sum_1^{\infty} M(n)x^n .$$

The validity of the above theorem can be seen from the following observations. Formulas have been given by Hurwitz [3] for the number of ways a square can be expressed as the sum of 3 and 5 squares. While extending the results to 7 squares [8] and to 9 , 11 and 13 squares [9], Sandham pointed out that Hurwitz's arguments applies to more general numbers, and he gave in [9] a theorem in three parts. The part (or rather an important particular case of it) which is relevant for our purpose is that the coefficient of q^{m^2} in the product

$$(1+2q^{1^2}+2q^{2^2}+2q^{3^2}+\ldots\ldots) \; [N(1)q^4+N(2)q^8+N(3)q^{12}+\ldots\ldots]$$

is equal to

$$\sum_r M\left(\frac{m}{r}\right)\lambda(r)\mu(r) ,$$

where $\mu(n)$ is the Möbius function, and $N(n)$ and $M(n)$ are the same as those stated in Theorem 0. We can restate this fact as

$$(5) \qquad 2\sum_r{}' \; N\left(\frac{m^2-r^2}{4}\right) = \sum_r M\left(\frac{m}{r}\right)\lambda(r)\mu(r) .$$

Now remembering the multiplicative property of $\lambda(x)$ we have

$$\lambda(r).\lambda\left(\frac{m}{r}\right) = \lambda(m) .$$

In virtue of the above relation and the fact that $\lambda(u) \neq 0$ we obtain from (5) the following

$$(6) \qquad \frac{2}{\lambda(m)}\sum_r{}' \; N\left(\frac{m^2-r^2}{4}\right) = \sum_r \mu(r)M\left(\frac{m}{r}\right)/\lambda\left(\frac{m}{r}\right) = \sum_{r|m} \mu\left(\frac{m}{r}\right)M(r)/\lambda(r) .$$

Theorem 0 is now easily established from (6) by the use of the Möbius inversion formula [2].

4. Proof of the theorems

Remembering Lemmas 1 and 2 and writing successively $\sigma(n)$, $n\sigma(n)$ and $\sigma_3(n)$ in place of $N(n)$, and n , n^3 and n^3 for $\lambda(n)$ in the basic theorem we can show that

(7) $$\frac{1}{24}\left[5\sigma_3(n) - (6n-1)\sigma(n)\right] = \sum_{d|n}\left[\frac{1}{d}\sum_r{}' \sigma\left(\frac{d^2-r^2}{4}\right)\right] ,$$

(8) $$\frac{1}{6n}\left[\sigma_3(n) - n\sigma(n)\right] = \sum_{d|n}\left[\frac{1}{d^3}\sum_r{}' (d^2-r^2)\sigma\left(\frac{d^2-r^2}{4}\right)\right] ,$$

(9) $$\frac{1}{240n^3}\left[\sigma_7(n) - \sigma_3(n)\right] = \sum_{d|n}\left[\frac{1}{d^3}\sum_r{}' \sigma_3\left(\frac{d^2-r^2}{4}\right)\right] .$$

Theorem 1 follows easily from (7) and (8) by the elimination of $\sigma_3(n)$ between them. Theorem 2 is equally simple, - we subtract (8) from (7). Theorem 3 is an obvious consequence of the relation (9).

Again remembering Lemmas 1 and 3 and writing successively $n^2\sigma(n)$, $n\sigma_3(n)$ and $\sigma_5(n)$ in place of $N(n)$, and n^5 for $\lambda(n)$ in the basic theorem one can obtain without difficulty the following.

(10) $$\frac{1}{7n}.\sigma_3(n) - \frac{2}{15}.\sigma(n) - \frac{1}{105n^5}.\tau(n) = \sum_{d|n}\left[\frac{1}{d^5}\sum_r{}' (d^2-r^2)^2\sigma\left(\frac{d^2-r^2}{4}\right)\right] ,$$

(11) $$\frac{1}{270n^3}.\sigma_7(n) - \frac{1}{270n^5}.\tau(n) = \sum_{d|n}\left[\frac{1}{d^5}\sum_r{}' (d^2-r^2)\sigma_3\left(\frac{d^2-r^2}{4}\right)\right] ,$$

(12) $$65\sigma_{11}(n) + 691\sigma_5(n) - 756\tau(n) = 348264n^5.\sum_{d|n}\left[\frac{1}{d^5}\sum_r{}' \sigma_5\left(\frac{d^2-r^2}{4}\right)\right] .$$

The first part of Theorem 4 is obtained easily by subtracting twice the relation (8) from the sum of the relations (7) and (10). The second part is obtained by subtracting (11) from (9). The last part is a direct consequence of (12).

To prove Theorem 5 we note that in virtue of the relation [1]

$$\tau(p^\lambda n) = \tau(p)\tau(p^{\lambda-1}n) - p^{11}\tau(p^{\lambda-2}n)$$

one is justified in substituting $\tau(n) = p_{24}(n-1)$ for $N(n)$, and 11 for α in Theorem 0. The first part of Theorem 5 is an immediate consequence when one notes that

$$\left[\sum_1^\infty \tau(n)x^n\right]^2 = x^2\left[\prod_1^\infty (1-x^n)\right]^{48} = \sum_2^\infty p_{48}(n-2)x^n .$$

The second part of the theorem follows easily on a joint consideration of the first part of Theorem 5 and the last part of Theorem 4.

https://doi.org/10.1017/S0004972700042179 Published online by Cambridge University Press

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