Bull. Aust. Math. Soc. **95** (2017), 199–208
doi:10.1017/S000497271600071X

ON PAIRS OF GOLDBACH–LINNIK EQUATIONS

YAFANG KONG and ZHIXIN LIU✉

(Received 29 June 2016; accepted 21 July 2016; first published online 19 October 2016)

Abstract

In this paper, we show that every pair of large positive even integers can be represented in the form of a pair of Goldbach–Linnik equations, that is, linear equations in two primes and k powers of two. In particular, $k = 34$ powers of two suffice, in general, and $k = 18$ under the generalised Riemann hypothesis. Our result sharpens the number of powers of two in previous results, which gave $k = 62$, in general, and $k = 31$ under the generalised Riemann hypothesis.

2010 *Mathematics subject classification*: primary 11P32; secondary 11P05, 11P55.

Keywords and phrases: Goldbach–Linnik problem, circle method, pairs of equations.

1. Introduction

The Goldbach conjecture asks whether every even integer greater than two can be represented as a sum of two primes. There are many variations on the original conjecture. The Goldbach–Linnik problem was first considered by Linnik [3, 4], who proved that every large even integer N is a sum of two primes and a bounded number of powers of two: that is,

$$N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_k}, \tag{1.1}$$

where p and ν, with or without subscripts, denote a prime number and a positive integer, respectively. In 2002, Heath-Brown and Puchta [1] showed that $k = 7$ is acceptable under the generalised Riemann hypothesis (GRH). In 2011, Liu and Lü [6] showed that $k = 12$ is acceptable, in general.

We study a simultaneous version of the Goldbach–Linnik problem. Instead of considering representations of a single even integer, we attempt simultaneous representations of pairs of positive even integers as sums of two primes and powers of two, given by

$$\begin{cases} N_1 = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \\ N_2 = p_3 + p_4 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}. \end{cases} \tag{1.2}$$

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11426048 and 11301372), Specialised Research Fund for the Doctoral Program of Higher Education (Grant No. 20130032120073) and Independent Innovation Foundation of Tianjin University (Grant Nos 190-0903061029 and 190-0903062072).

In 2013, Kong [2] proved that the simultaneous equations (1.2) are solvable for every pair of sufficiently large positive even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$ for $k = 63$, in general, and for $k = 31$ assuming the (GRH). Very recently, Platt and Trudgian [7] computed some parameters in the proof of [2] carefully and gave a slight improvement of [2]. They proved that $k = 62$ in (1.2), unconditionally, but could not give any improvement on the value of k in (1.2) under the GRH. For further background and the details of the progress on the problems (1.1) and (1.2), we refer the reader to [2].

In this paper, by using a different method to treat the minor arcs in the circle method, we improve the value of k in (1.2). Only improving the major arc estimate as in Lemma 2.1 below may lead to an improvement in the value of k, but, presumably, this is relatively small compared with the minor arc improvement. In applying the Hardy–Littlewood circle method, we divide $[0, 1]^2$ into three arcs, whereas Kong [2] divided $[0, 1]^2$ into nine arcs. By using the method of integral transforms, we avoid the restrictions of two arcs in Kong's method (see Lemma 2.2 below), which leads to the improvement in Theorem 1.1.

Theorem 1.1. *For $k = 34$, the simultaneous equations (1.2) are solvable for every pair of sufficiently large positive even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$. Furthermore, $k = 18$ is admissible under the GRH.*

2. The proof of Theorem 1.1

We use the same notation as in [2]. Let ω be a small positive constant. Set

$$S(\alpha, N) = \sum_{\omega N < p \le N} e(p\alpha)$$

and

$$T(\alpha) = \sum_{1 \le \nu \le L} e(2^\nu \alpha),$$

where $e(x) := \exp(2\pi i x)$ and $L = \log_2 N_1$.

Let $R(N_1, N_2)$ be the number of solutions of (1.2) in $(p_1, p_2, p_3, p_4, \nu_1, \nu_2, \dots, \nu_k)$ with

$$\omega N_1 < p_1, p_2 \le N_1, \quad \omega N_2 \le p_3, p_4 \le N_2, \quad 1 \le \nu_j \le L \quad \text{for } j = 1, 2, \dots, k.$$

We begin with

$$R(N_1, N_2) = \iint_{(\alpha_1,\alpha_2)\in[0,1]^2} S^2(\alpha_1, N_1) S^2(\alpha_2, N_2) T^k(\alpha_1 + \alpha_2) e(-N_1\alpha_1 - N_2\alpha_2)\, d\alpha_1\, d\alpha_2. \tag{2.1}$$

In order to apply the Hardy–Littlewood method, following the same choice as Heath-Brown and Puchta [1], we choose $P_i = N_i^{45/154}$ with $i = 1, 2$. For $i = 1, 2$ and any integers a_i, q_i satisfying

$$1 \le a_i \le q_i \le P_i \quad \text{and} \quad (a_i, q_i) = 1, \tag{2.2}$$

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we define

$$\mathfrak{M}_i(a_i, q_i) = \left\{\alpha \in [0,1] : \left|\alpha - \frac{a_i}{q_i}\right| \le \frac{P_i}{qN_i}\right\},$$
$$\mathfrak{M}_i = \bigcup \mathfrak{M}_i(a_i, q_i), \qquad \mathfrak{m}_i = [0,1] \setminus \mathfrak{M}_i,$$

where the union $\bigcup$ is over all a_i, q_i satisfying (2.2). We further define

$$\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{M}_2 = \{(\alpha_1, \alpha_2) \in [0,1]^2 : \alpha_1 \in \mathfrak{M}_1, \alpha_2 \in \mathfrak{M}_2\},$$
$$\mathfrak{m} = [0,1]^2 \setminus \mathfrak{M}.$$

In addition, we set

$$\mathcal{E}_\lambda = \{(\alpha_1, \alpha_2) \in [0,1]^2 : |T(\alpha_1 + \alpha_2)| \ge \lambda L\}.$$

With this notation, we can dissect the integral representation (2.1) for $R(N_1, N_2)$ as

$$\begin{aligned} &R(N_1, N_2) \\ &= \Big(\iint_{\mathfrak{M}} + \iint_{\mathfrak{m} \cap \mathcal{E}_\lambda} + \iint_{\mathfrak{m} \setminus \mathcal{E}_\lambda}\Big) S^2(\alpha_1, N_1) S^2(\alpha_2, N_2) T^k(\alpha_1 + \alpha_2) e(-N_1\alpha_1 - N_2\alpha_2)\, d\alpha_1 d\alpha_2 \\ &=: R_1(N_1, N_2) + R_2(N_1, N_2) + R_3(N_1, N_2). \end{aligned}$$

We will establish Theorem 1.1 by estimating $R_1(N_1, N_2)$, $R_2(N_1, N_2)$ and $R_3(N_1, N_2)$.

LEMMA 2.1. *For every pair of sufficiently large positive even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$,*

$$R_1(N_1, N_2) \ge 3.535(1 - 4\omega) N_1 N_2 (\log N_1 \log N_2)^{-2} L^k.$$

PROOF. This lemma is actually [2, Proposition 2.1], only with the coefficient 1.74293 instead of 3.535. Thus we only give the sketch of the proof here.

Our proof begins with [2, (2.2)]. Define a multiplicative function $k(d)$ by taking

$$k(p^e) = \begin{cases} 0 & p = 2 \text{ or } e \ge 2, \\ 1/(p-2) & \text{otherwise.} \end{cases}$$

Then

$$R_1(N_1, N_2) \ge 4C_0^2 (1 - 2\omega)^2 N_1 N_2 (\log N_1 \log N_2)^{-2} \cdot \sum,$$

where

$$\begin{aligned} \sum &= \sum_{1 \le v_1, \dots, v_k \le L} \ \sum_{d | N_1 - 2^{v_1} - \dots - 2^{v_k}} k(d) \sum_{l | N_2 - 2^{v_1} - \dots - 2^{v_k}} k(l) \\ &= \sum_d k(d) \sum_l k(l) \sum_{(v_1, \dots, v_k)} 1 \end{aligned} \tag{2.3}$$

and

$$C_0 := \prod_{p > 2} \Big(1 - \frac{1}{(p-1)^2}\Big).$$

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According to Wrench [8], the value of C_0 satisfies

$$0.6601618158 < C_0 < 0.6601618159. \tag{2.4}$$

The conditions $(v_1, \ldots, v_k)$ in the term $\sum_{(v_1,\ldots,v_k)}$ in (2.3) are

$$1 \le v_1, \ldots, v_k \le L, \quad 2^{v_1} + \cdots + 2^{v_k} \equiv N_1 \pmod{d}$$

and

$$2^{v_1} + \cdots + 2^{v_k} \equiv N_2 \pmod{l}.$$

For any odd integer d we define $\varepsilon(d)$ to be the order of two in the multiplicative group modulo d, and we set

$$H(d; N, k) = \#\Big\{(v_1, \ldots, v_k) : 1 \le v_i \le \varepsilon(d), d | N - \sum 2^{v_i}\Big\}.$$

Then

$$2 \sum_d H(d; N, k)\varepsilon(d)^{-k} \ge 3.02858417.$$

The sum was first computed by Heath-Brown and Puchta [1] and the value of 3.02858417 was given by Platt and Trudgian [7].

Following the same argument as in Heath-Brown and Puchta [1, Section 4], in particular, just before [1, equation (25)],

$$\sum_{\substack{1 \le v_1,\ldots,v_k \le L \\ 2^{v_1}+\cdots+2^{v_k} \equiv N_1 \pmod{d}}} 1 \ge H(d; N_1, k)\varepsilon(d)^{-k} L^k.$$

Since $k(1) = 1$,

$$\begin{aligned} \sum &> k(1) \sum_l k(l) \sum_{\substack{1 \le v_1,\ldots,v_k \le L \\ 2^{v_1}+\cdots+2^{v_k} \equiv N_2 \pmod{l}}} 1 + k(1) \sum_d k(d) \sum_{\substack{1 \le v_1,\ldots,v_k \le L \\ 2^{v_1}+\cdots+2^{v_k} \equiv N_1 \pmod{d}}} 1 \\ &\qquad - k(1)k(1) \sum_{1 \le v_1,\ldots,v_k \le L} 1 \\ &\ge \Big(\sum_l k(l)H(l; N_2, k)\varepsilon(l)^{-k} + \sum_d k(d)H(d; N_2, k)\varepsilon(d)^{-k} - 1\Big)L^k \\ &\ge 2.028L^k. \end{aligned}$$

The estimate in Lemma 2.1 follows by an easy computation. □

Lemma 2.2. *For every pair of sufficiently large positive even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$,*

$$R_2(N_1, N_2) \ll N_1 N_2 (\log N_1 \log N_2)^{-2} L^{k-1},$$

provided

$$\lambda = \begin{cases} 0.8594000 & \text{in general,} \\ 0.7163436 & \text{under the GRH.} \end{cases}$$

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Proof. From the definition of $\mathfrak{m}$,

$$\mathfrak{m} = \{(\alpha_1,\alpha_2) : \alpha_1 \in \mathfrak{m}_1, \alpha_2 \in [0,1]\} \bigcup \{(\alpha_1,\alpha_2) : \alpha_1 \in \mathfrak{M}_1, \alpha_2 \in \mathfrak{m}_2\}$$
$$\subset \{(\alpha_1,\alpha_2) : \alpha_1 \in \mathfrak{m}_1, \alpha_2 \in [0,1]\} \bigcup \{(\alpha_1,\alpha_2) : \alpha_1 \in [0,1], \alpha_2 \in \mathfrak{m}_2\}.$$

Thus

$$\begin{aligned}
R_2(N_1,N_2) &= \iint_{\mathfrak{m}\bigcap\mathcal{E}_\lambda} S^2(\alpha_1,N_1)S^2(\alpha_2,N_2)T^k(\alpha_1+\alpha_2)e(-N_1\alpha_1 - N_2\alpha_2)\,d\alpha_1\,d\alpha_2 \\
&\ll L^k\Big(\iint_{\substack{(\alpha_1,\alpha_2)\in\mathfrak{m}_1\times[0,1]\\ |T(\alpha_1+\alpha_2)|\ge\lambda L}} + \iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]\times\mathfrak{m}_2\\ |T(\alpha_1+\alpha_2)|\ge\lambda L}}\Big)|S^2(\alpha_1,N_1)S^2(\alpha_2,N_2)|\,d\alpha_1\,d\alpha_2 \\
&\ll N_1^{2\theta+\varepsilon} \iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |T(\alpha_1+\alpha_2)|\ge\lambda L}} |S^2(\alpha_2,N_2)|\,d\alpha_1\,d\alpha_2 + N_2^{2\theta+\varepsilon} \\
&\quad\times \iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |T(\alpha_1+\alpha_2)|\ge\lambda L}} |S^2(\alpha_1,N_1)|\,d\alpha_1\,d\alpha_2,
\end{aligned}$$

where we used the trivial bound of $T(\alpha)$ and the bounds

$$\max_{\alpha_1\in\mathfrak{m}_1} |S(\alpha_1)| \ll N_1^{\theta+\varepsilon} \quad \text{and} \quad \max_{\alpha_2\in\mathfrak{m}_2} |S(\alpha_2)| \ll N_2^{\theta+\varepsilon}$$

with

$$\theta = \begin{cases} 263/308 & \text{in general,} \\ 3/4 & \text{under the GRH,} \end{cases}$$

which can be found on page 561 in Heath-Brown and Puchta [1]. Moreover,

$$\begin{aligned}
\iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |T(\alpha_1+\alpha_2)|\ge\lambda L}} |S^2(\alpha_2,N_2)|\,d\alpha_1\,d\alpha_2 &= \int_0^1 |S^2(\alpha_2,N_2)|\Big(\int_{\substack{\beta\in[\alpha_2,1+\alpha_2]\\ T(\beta)\ge\lambda L}} d\beta\Big)\,d\alpha_2 \\
&\ll N_2 \int_{\substack{\beta\in[0,1]\\ T(\beta)\ge\lambda L}} d\beta \ll N_2 N_1^{-E(\lambda)},
\end{aligned}$$

where we set $\beta = \alpha_1 + \alpha_2$ to give the integral transformation and we used the prime number theorem and the periodicity of $T(\beta)$. Similarly,

$$\iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ T(\alpha_1+\alpha_2)\ge\lambda L}} |S^2(\alpha_1,N_1)|\,d\alpha_1 d\alpha_2 \ll N_1^{1-E(\lambda)}.$$

Since $N_2 \gg N_1 > N_2$, this yields

$$R_2(N_1,N_2) \ll N_1N_2(\log N_1 \log N_2)^{-2}L^{k-1},$$

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provided that $E(\lambda) > 2\theta - 1$: that is,

$$\lambda = \begin{cases} 0.8594000 & \text{in general,} \\ 0.7163436 & \text{under the GRH,} \end{cases}$$

using the values computed by Platt and Trudgian [7]. □

Lemma 2.3. *For every pair of sufficiently large positive even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$,*

$$R_3(N_1, N_2) \leq 305.716\lambda^{k-4} N_1 N_2 (\log N_1 \log N_2)^{-2} L^k.$$

Proof. We begin by estimating the mean square

$$J = \iint\limits_{(\alpha_1,\alpha_2)\in[0,1]^2} |S^2(\alpha_1)S^2(\alpha_2)T^4(\alpha_1+\alpha_2)|\,d\alpha_1 d\alpha_2.$$

Observe that

$$J = \sum_{1\leq m_1,m_2,m_3,m_4\leq L} r_1(2^{m_1}+2^{m_2}-2^{m_3}-2^{m_4})r_2(2^{m_1}+2^{m_2}-2^{m_3}-2^{m_4}),$$

where

$$r_i(n) = \#\{\omega N_i < p_i \leq N_i : n = p_1 - p_2\}.$$

We distinguish between two cases, and write

$$J = \sum_{\substack{1\leq m_1,m_2,m_3,m_4\leq L \\ 2^{m_1}+2^{m_2}-2^{m_3}-2^{m_4}\neq 0}} + \sum_{\substack{1\leq m_1,m_2,m_3,m_4\leq L \\ 2^{m_1}+2^{m_2}-2^{m_3}-2^{m_4}=0}} =: J_1 + J_2.$$

Case 1. In this case, we treat the contribution from those (m_1, m_2, m_3, m_4) such that

$$2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4} \neq 0.$$

Let

$$h(n) = \prod_{p|n,p>2} \left(\frac{p-1}{p-2}\right).$$

Then

$$r_i(n) \leq C_0 C_1 h(n) \frac{N_i}{(\log N_i)^2}$$

for $n \neq 0$ and N sufficiently large, where C_0 is given by (2.4) and

$$C_1 = 7.8209,$$

as proved by Wu [9]. Thus

$$J_1 \leq C_0^2 C_1^2 \frac{N_1 N_2}{(\log N_1 \log N_2)^2} \sum_{1\leq m_1,m_2,m_3,m_4\leq L} h^2(2^{m_1}+2^{m_2}-2^{m_3}-2^{m_4}).$$

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Denote the sum above by $\sum$. Noting that $h^2(n) = h^2(-n)$ for $n \neq 0$ and $h(2^\nu m) = h(m)$,

$$\sum = 4 \sum_{\substack{1\le m_1,m_2,m_3,m_4\le L\\ m_4=\min\{m_1,m_2,m_3,m_4\}}} h^2(2^{m_1-m_4} + 2^{m_2-m_4} - 2^{m_3-m_4} - 1).$$

For a fixed integral vector (h_1, h_2, h_3) with $1 \le h_j \le L$,

$$|\{(m_1, m_2, m_3, m_4) : 1 \le m_j \le L, m_1 - m_4 = h_1, m_2 - m_4 = h_2, m_3 - m_4 = h_3\}| \\ \le \min(L - h_1, L - h_2, L - h_3).$$

Since the positions of h_1, h_2 and h_3 are symmetrical, one deduces further that

$$\sum \le 4 \sum_{0\le h_1,h_2,h_3\le L-1} \min(L - h_1, L - h_2, L - h_3) h^2(2^{h_1} + 2^{h_2} - 2^{h_3} - 1) \\ = 12 \sum_{0\le h_1\le L-1} (L - h_1) \sum_{0\le h_2\le h_1} \sum_{0\le h_3\le h_1} h^2(2^{h_1} + 2^{h_2} - 2^{h_3} - 1).$$

Following the treatment of Case 2, we will show that, for $H \gg 1$,

$$\sum_{1\le j\le H} h^2(2^j - t) \le C_2 H$$

uniformly for all positive odd numbers t with $|t| \le N$. Thus we obtain

$$\sum \le 12 \sum_{0\le h_1\le L-1} (L - h_1) h_1 C_2 h_1.$$

Since

$$\sum_{0\le h_1\le L-1} (L h_1^2 - h_1^3) \le \int_0^L (Lx^2 - x^3)\, dx = \frac{L^4}{12},$$

we get

$$\sum \le C_2 L^4$$

and, consequently,

$$J_1 \le C_0^2 C_1^2 C_2 \frac{N_1 N_2 L^4}{(\log N_1 \log N_2)^2}.$$

Case 2. It remains to estimate the contribution from those (m_1, m_2, m_3, m_4) with

$$2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4} = 0.$$

Clearly, J_2 is the number of solutions of

$$p_1 = p_2, \quad p_3 = p_4 \tag{2.5}$$

multiplied by the number of solutions of

$$2^{m_1} + 2^{m_2} = 2^{m_3} + 2^{m_4}, \tag{2.6}$$

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where $\omega N_i < p_i \le N_i$ and $1 \le m_j \le L$. It is easy to see that the total number of solutions of (2.5) is $(1+\varepsilon)N_1N_2/\log N_1 \log N_2$. For (2.6), if m_1, m_3 are fixed arbitrarily, there is at most one choice for m_2, m_4. It follows that (2.6) has at most L^2 solutions and, consequently,

$$J_2 \le (1+\varepsilon)\frac{N_1N_2L^2}{\log N_1 \log N_2}.$$

Thus we reach the following result.

$$J \le \{C_0^2C_1^2C_2 + (1+\varepsilon)\log^2 2\}\frac{N_1N_2L^4}{(\log N_1 \log N_2)^2}.$$

Now we will give the estimation of C_2. Observe that

$$\sum_{1\le j\le H} h^2(2^j - t) = \sum_{1\le j\le H}\prod_{p|2^j-t,p>2}\Big(\frac{p-1}{p-2}\Big)^2 = \sum_{1\le j\le H}\prod_{p|2^j-t,p>2}\Big(1+\frac{2p-3}{p^2-4p+4}\Big).$$

Let

$$a(p^e) = \begin{cases} 0 & p = 2 \text{ or } e \ge 2,\\ \dfrac{2p-3}{p^2-4p+4} & \text{otherwise.}\end{cases}$$

Thus

$$\sum_{1\le j\le H} h^2(2^j - t) = \sum_{1\le j\le H}\sum_{d|2^j-1} a(d) = \sum_{d\le 2N} a(d)\sum_{1\le j\le H, d|2^j-t} 1.$$

It follows, from $d|2^j - t$, that $t \equiv 2^j \pmod d$. Let j_0 be the least positive integer such that $t \equiv 2^{j_0} \pmod d$. Then $2^j \equiv 2^{j_0} \pmod d$ or $2^{j-j_0} \equiv 1 \pmod d$ and, consequently, $\varepsilon(d)|j - j_0$. Hence

$$\sum_{1\le j\le H} h^2(2^j - t) = \sum_{d\le 2N} a(d)\sum_{1\le j\le H, \varepsilon(d)|j-j_0} 1 \le H\sum_{d=1}^{\infty}\frac{a(d)}{\varepsilon(d)} =: C_2H.$$

Now we want to compute C_2. We set

$$m = \prod_{e\le x}(2^e - 1) \quad\text{and}\quad s(x) = \sum_{\varepsilon(d)\le x} a(d),$$

and hence

$$\begin{aligned} s(x) &\le \sum_{d|m} a(d) = h^2(m) = \prod_{p|m,p>2}\Big(\frac{p-1}{p-2}\Big)^2 \\ &= \prod\Big(\frac{(p-1)^2}{p(p-2)}\Big)^2\prod_{p|m}\Big(\frac{p}{p-1}\Big)^2 = C_0^{-2}\Big(\frac{m}{\varphi(m)}\Big)^2. \end{aligned}$$

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Liu, Liu and Wang [5] showed that $m/\varphi(m) \le e^{\gamma} \log x$ for $x \ge 9$, and hence

$$\begin{aligned} C_2 = \int_0^{\infty} s(x)\frac{dx}{x^2} &= \int_1^{M} s(x)\frac{dx}{x^2} + \int_M^{\infty} s(x)\frac{dx}{x^2} \\ &\le \sum_{\varepsilon(d)\le M} \int_{\varepsilon(d)}^{M} a(d)\frac{dx}{x^2} + C_0^{-2}e^{2\gamma}\int_M^{\infty} \log^2 x\frac{dx}{x^2} \\ &\le \sum_{\varepsilon(d)<M} a(d)\Big(\frac{1}{\varepsilon(d)} - \frac{1}{M}\Big) + C_0^{-2}e^{2\gamma}\Big(\frac{2+2\log M+(\log M)^2}{M}\Big) \end{aligned}$$

for any integer $M \ge 9$. We now set

$$\sum_{\varepsilon(d)=e} a(d) = A(e)$$

so that

$$\sum_{e|d} A(e) = \sum_{\varepsilon(e)|d} a(d).$$

However, $\varepsilon(e)|d$ if and only if $e|2^d - 1$. Thus

$$\sum_{e|d} A(e) = \sum_{e|2^d-1} a(e) = h^2(2^d - 1).$$

We therefore deduce that

$$A(e) = \sum_{d|e} \mu\Big(\frac{e}{d}\Big)h^2(2^d - 1).$$

This enables us to compute

$$\sum_{\varepsilon(d)<M} a(d)\Big(\frac{1}{\varepsilon(d)} - \frac{1}{M}\Big) = \sum_{m<M}\Big(\frac{1}{m} - \frac{1}{M}\Big)$$

by using information on the prime factorisation of $2^d - 1$ for $d < M$. In particular, taking $M = 10$, we find that

$$C_2 \le \sum_{m<10} A(m)\Big(\frac{1}{m} - \frac{1}{10}\Big) + C_0^{-2}e^{2\gamma}\Big(\frac{2+2\log 10+(\log 10)^2}{10}\Big) = 11.4569\ldots.$$

So we reach the bound

$$J \le 305.8869\frac{N_1N_2L^4}{(\log N_1 \log N_2)^2}$$

and the estimate

$$\begin{aligned} R_3(N_1,N_2) &\le \lambda^{k-4}L^{k-4} \iint_{(\alpha_1,\alpha_2)\in[0,1]^2} |S^2(\alpha_1)S^2(\alpha_2)T^4(\alpha_1+\alpha_2)|\,d\alpha_1\,d\alpha_2 \\ &\le 305.8869\lambda^{k-4}N_1N_2(\log N_1 \log N_2)^{-2}L^k. \end{aligned}$$

□

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Finally, by comparing the estimate for the major arc integral, $R_1(N_1, N_2)$, with those for $R_2(N_1, N_2)$ and $R_3(N_1, N_2)$, we conclude that

$$R(N_1, N_2) > 0,$$

provided that N_1 and N_2 are large enough, ω is small enough and

$$305.8869\lambda^{k-4} < 3.535(1 - 4\omega). \tag{2.7}$$

Using

$$\lambda = \begin{cases} 0.8594000 & \text{in general,} \\ 0.7163436 & \text{under the GRH,} \end{cases}$$

we see that (2.7) is satisfied for $k \geq 33.4382$ and $k \geq 17.371$ in the respective cases. This completes the proof of Theorem 1.1.

Acknowledgement

The authors would like to express their thanks to the referee for many useful suggestions and comments on the manuscript.

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YAFANG KONG, College of Mathematics and Statistics,
Chongqing Jiaotong University, Chongqing 400074, PR China
e-mail: yfkong@cqjtu.edu.cn

ZHIXIN LIU, Department of Mathematics,
School of Science, Tianjin University, Tianjin 300072, PR China
e-mail: zhixinliu@tju.edu.cn

https://doi.org/10.1017/S000497271600071X Published online by Cambridge University Press